Finiteness of the vacuum energy density in quantum electrodynamics

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Recent interest in the finiteness problem of the vacuum energy density (VED) in finite QED has motivated us to reexamine this problem in the light of an analysis we have carried out earlier. By a loopwise summation procedure, supplemented by a renormalization-group analysis, we study the finiteness of the VED with α, the renormalized fine-structure constant, fixed in the process as the (infinite order) zero of the eigenvalue condition $F^{[1]}(x)\,|_{x=\alpha}=0^{\infty}$, and with the electron mass totally dynamical of origin. We propose a possible finite solution for the VED in QED which may require only one additional eigenvalue condition for α.

I. INTRODUCTION

Recent interest[1] in the finiteness of the vacuum energy density (VED) in finite quantum electrodynamics (QED), first studied in Ref. 2, has motivated us to reexamine this problem in the light of our earlier analysis. By finite QED it is meant here that all photon self-energy subgraphs in renormalized QED are summed loopwise with α, the renormalized fine-structure constant, fixed in the process as the (infinite order) zero of the eigenvalue condition[3]: $F^{[1]}(\alpha)=0^{\infty}[\beta(\alpha)=0^{\infty}]$, and with the electron mass totally dynamical of origin. That is, for the latter, the anomalous mass dimension[3,4] $\delta(\alpha)$ is assumed to be strictly positive—a result well supported by low-order perturbative calculations. Here $F^{[1]}(\alpha)$ denotes the coefficient of the "single" logarithm of the single-closed-fermion-loop contribution to the renormalized photon self-energy part, and $\beta(\alpha)$ denotes the Callan-Symanzik[5,3] function. Unfortunately, the analysis in Ref. 1 did not lead to constructive suggestions for the finiteness of VED. Our earlier paper,[2] however, contains basic ingredients for a more complete discussion of this problem. We propose a possible solution which is completely finite and may require only one additional eigenvalue condition for α. A significance of the finiteness of the VED on the vanishing of the (electro-) magnetic form factors at large momentum transfer is also discussed. Some repetition of our earlier work[2] is unavoidable. However, to make any repetition minimal, we urge the reader to consult the just-mentioned reference while reading this paper.

II. VACUUM ENERGY DENSITY

Let $\langle 0\text{out}\,|\,0\text{in}\rangle$ denote the vacuum-to-vacuum transition amplitude in the sense of Schwinger.[6] We write the Lagrangian density in the form $\mathscr{L}=\mathscr{L}_0+\lambda\mathscr{L}_I$, where the mass term $m_0\bar{\psi}\psi$ is included in $\mathscr{L}_I$, and $\mathscr{L}_0$ is the free Lagrangian density of QED with massless particles. From Schwinger's dynamical principle one then obtains the well-known expression

$$\exp(-i\Omega\mathscr{E})=\langle 0\text{out}\,|\,0\text{in}\rangle^{\lambda=1}/\langle 0\text{out}\,|\,0\text{in}\rangle^{\lambda=0}\,, \tag{1}$$

where Ω denotes the extension of space-time to be ultimately taken to be infinite ($\Omega\to\infty$). $\mathscr{E}$ is the VED due to the *dynamics* with the mass of the electron totally dynamical of origin. Let m denote the renormalized mass of the electron. The anomalous dimension $\delta(\alpha)$ may be defined through[3]

$$m\,[1+\delta(\alpha)]^{-1}=m_0\left[\frac{\partial}{\partial m_0}m\right],$$

and $\delta(\alpha)$ is cutoff independent, and at least low-order perturbation theory results show that[3,4] $\delta(\alpha)>0$. Schwinger's dynamical principle also yields

$$m_0\frac{\partial}{\partial m_0}\mathscr{E}=i\int\frac{dp}{(2\pi)^4}\mathrm{Tr}[m_0S(p)]\,, \tag{2}$$

where $S(p)$ is the full unrenormalized electron propagator. A standard renormalization-group analysis, at the eigenvalue, then shows[2] directly from (2) by formally cutting off the integral in (2) at $p^2\le\Lambda^2$ that the general structure of $\mathscr{E}$ is

$$\mathscr{E}\underset{\Lambda^2\to\infty}{\sim}a_1(\alpha)\Lambda^4+a_2(\alpha)m^2\Lambda^2\left[\frac{m^2}{\Lambda^2}\right]^{\delta(\alpha)}+\cdots\,, \tag{3}$$

where the dots denote less singular terms of the form $O(m^2\Lambda^2(m^2/\Lambda^2)^{2\delta})$. Clearly if $\delta(\alpha)>1$, then (3) implies that $\mathscr{E}$ diverges at worst like

$$\mathscr{E} \underset{\Lambda^2\to\infty}{\sim} a_1(\alpha)\Lambda^4 . \tag{4}$$

The condition $\delta(\alpha)>1$ is not ruled out and it will be assumed in our analysis. The implication of this condition on the vanishing of the (electro-) magnetic form factor at large momentum transfer will be discussed later. The result in (3) could have been written down by inspection from the following analysis. To this end we note that a well-known "second Legendre transform" method shows that $\mathscr{E}$ may be expressed as a functional solely in the exact electron and photon propagators (see, e.g., Appendix B in Ref. 2). We also make the following observations. Since $\mathscr{E}$ is gauge invariant [see, e.g., Eq. (5)], we may work in particular in the so-called generalized Landau gauge[4,3] in which the electron propagator is finite at the eigenvalue. We then note that (at the eigenvalue) any mass corrections to the photon propagator [see Eq. (23) in Ref. 7] and to light-light scattering graphs [see Eq. (23) in Ref. 2], and the electron propagator (multiplied by m)[3,4] vanish, at worst, like $(m^2)^{1+\delta(\alpha)}$ for $m\to 0$. Accordingly if $\delta(\alpha)>1$ then by dimensional analysis we learn that any mass contribution to $\mathscr{E}$ should vanish. (Note that, from power counting, the possible quartic divergence appears as a result of the overall integration in $\mathscr{E}$.) This explains the origin of the second (third, ...) term(s) in (3). Accordingly we may assume that, at the eigenvalue, to obtain the expression for $\mathscr{E}$, we may formally set $m=0$ in its expression. This leads to the study of $\mathscr{E}$ in massless QED and leads to the result stated in (4).

To study the nature of the coefficient $a_1(\alpha)$, we have divided the subgraphs contributing to $\mathscr{E}$ into two classes (a) and (b). Class (b) contains those graphs which are four-or more-photon line irreducible. That is, class (b) graphs in $\mathscr{E}$, expressed as a functional of the exact photon and electron propagators, contain light-light scattering subgraphs and may be broken into two or more disconnected graphs by cutting four or more photon lines. Class (a) contains all of the remaining graphs. In particular class (a) contains no two-or more-photon line irreducible subgraphs since the photon lines are exact. Suppose that one replaces all the (exact) photon propagators multiplied by α_0 $(\equiv e_0{}^2/4\pi)$, the unrenormalized fine-structure constant, by x/q^2, and one omits all closed fermion loops contributing to $\mathscr{E}$ (except of the overall one defining $\mathscr{E}$) and one replaces the electron propagator by its free massless counterpart. The resulting expression $\mathscr{E}_a^{[1]}$ for $\mathscr{E}$ would then correspond to the single-closed-fermion-loop contribution to $\mathscr{E}$ in massless QED with a coupling (renormalized = unrenormalized) equal to x. According to the above discussion class (b) graphs cannot possibly contribute to the coefficient $a_1(\alpha)$, at the eigenvalue, due to the vanishing (and rapid damping) of light-light scattering graphs for $m\to 0$ [see Eq. (23) in Ref. 2]. Some graphs contributing to class (a) and class (b) are shown in Fig. 1 and Fig. 2, respectively.

According to the above discussion we may restrict our study to the graphs in class (a) to extract the net coefficient $a_1(\alpha)$. The following summation pro-

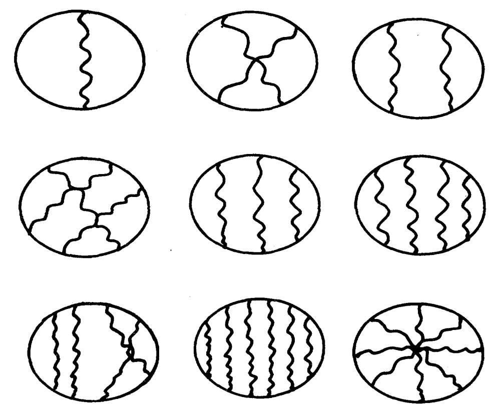

FIG. 1. Some graphs contributing to class (a) of graphs. It is argued that at the eigenvalue and with an anomalous mass dimension $\delta(\alpha)>1$, only graphs within this class may possibly give a contribution to the coefficient $a_1(\alpha)$ in Eq. (4).

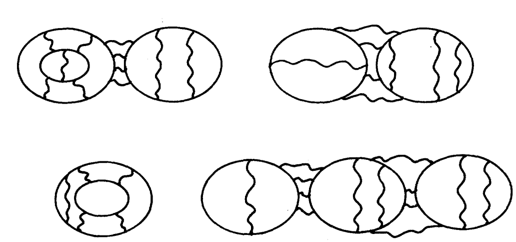

FIG. 2. Some graphs contributing to class (b) of graphs. This class consists of all graphs which are four-or-more-photon-line irreducible. The graphs in this class together with the graphs in class (a) consist of all the graphs appearing in the definition of the vacuum energy density $\mathscr{E}$. At the eigenvalue and with an anomalous mass dimension $\delta(\alpha)>1$, it is argued, due to the rapid damping of light-light scattering graphs for $m\to 0$, that the graphs in class (b) cannot give a contribution to the coefficient $a_1(\alpha)$ in Eq. (4).

cedure is *assumed* to study the nature of the coefficient $a_1(\alpha)$. We sum the diagrams in class (a) loopwise,[3] fix α as the infinite-order zero $F^{[1]}(x)|_{x=\alpha}=0^{\infty}$ in the process, and set $m=0$ *in* the expression for $\mathscr{E}$ when summing over the closed fermion loops to extract the net coefficient $a_1(\alpha)$.

Schwinger's dyamical principle leads to[2]

$$\alpha_0 \frac{\partial}{\partial \alpha_0}\mathscr{E} = -\frac{3i}{2}\int \frac{(dq)}{(2\pi)^4}[1+\alpha_0\pi(q^2)]^{-1}[\alpha_0\pi(q^2)] , \quad (5)$$

where $\pi(q^2)$ is defined through the vacuum polarization tensor

$$\pi_{\mu\nu}(q)=(g_{\mu\nu}q^2-q_\mu q_\nu)\pi(q^2) . \quad (6)$$

In the single-closed-fermion-loop contribution to (5) we have [with $\mathscr{E}\to\mathscr{E}_a^{[1]}$, $\pi(q^2)\to\pi^{[1]}(q^2)$, $\alpha_0\to\alpha$]

$$\frac{\partial}{\partial\alpha}\mathscr{E}_a^{[1]}=-\frac{3i}{2}\int\frac{(dq)}{(2\pi)^4}\pi^{[1]}(q^2) . \quad (7)$$

By making an expansion $\pi^{[1]}(q^2)=\sum_{n\geq 0}\alpha^n a_n(q^2)$, we then obtain from (7)

$$\mathscr{E}_a^{[1]}=\frac{3\pi^2}{2(2\pi)^4}\sum_{n\geq 0}\frac{\alpha^{n+1}}{n+1}\int_{q^2\leq\Lambda^2}q^2dq^2a_n(q^2) , \quad (8)$$

by formally cutting off the integral at Λ^2. We use the well-known fact[4,3]

$$a_n(q^2)\underset{m\to 0}{\sim}c_n+b_n\ln(q^2/\Lambda'^2) , \quad (9)$$

where c_n and b_n are constants, and Λ' is some ultraviolet cutoff. We note that the logarithmic factor in (9) may give ambiguities in the evaluation of the integral in (8). We argue below, however, that this logarithmic factor will not contribute to $\mathscr{E}_a^{[1]}$ (or to $\mathscr{E}$). We rewrite (8) as

$$\mathscr{E}_a^{[1]}=\frac{3\pi^2}{2(2\pi)^4}\sum_{n\geq 0}\frac{\alpha^{n+1}}{n+1}\int_{q^2\leq\Lambda^2}q^2dq^2\{[a_n(q^2)-c_n]+c_n\} , \quad (10)$$

or equivalently as ($m\to 0$)

$$\frac{3\pi^2}{2(2\pi)^4}\left[A_1^{[1]}(\alpha)\int_0^{\Lambda^2}q^2dq^2 - A_2^{[1]}(\alpha)\int_0^{\Lambda^2}q^2dq^2\int_0^{\Lambda'^2}\frac{dq'^2}{q'^2}\right] , \quad (11)$$

where

$$A_1^{[1]}(x)=\sum_{n\geq 0}\frac{x^{n+1}}{n+1}c_n , \qquad A_2^{[1]}(x)=\sum_{n\geq 0}\frac{x^{n+1}}{n+1}b_n . \quad (12)$$

We also recall that, if $\delta(\alpha)>1$, then any "correction" due to mass insertions should make any corresponding contribution vanish (for $\Lambda^2\to\infty$) as discussed above. We write

$$\sum_{n\geq 0}x^nc_n=C^{[1]}(x) , \quad \sum_{n\geq 0}x^nb_n=F^{[1]}(x) , \quad (13)$$

where we have identified the second sum with the standard notation $F^{[1]}(x)$. We argue that the coefficient $A_2^{[1]}(x)$ is zero at the eigenvalue $x=\alpha$, and hence the ambiguous second term in (11) does not contribute. We note from (12) and (13) that $A_2^{[1]}(x)$ may be expressed as

$$A_2^{[1]}(x)=\int_0^x dy\, F^{[1]}(y) . \quad (14)$$

We conjecture that due to the infinite-order-zero nature of $F^{[1]}(x)$ at $x=\alpha$ [i.e., $(d/dx)^jF^{[1]}(x)=0$, $x=\alpha$, $j=0,1,2,\ldots$] the integral of $F^{[1]}(x)$ evaluated at $x=\alpha$ is also zero. For an interesting example[2,3] where this indeed does happen suppose that

$$F^{[1]}(x)=-(C/A)\frac{d}{dx}\{x\exp[-A/(\alpha-x)]\} = \left[-\frac{C}{A}+\frac{Cx}{(\alpha-x)^2}\right]\exp[-A/(\alpha-x)] , \quad (15)$$

where A (>0) and C are some constants. This exhibits the infinite-order-zero nature of $F^{[1]}(x)$ at $x=\alpha$ and leads to the expression

$$A_2^{[1]}(x)=-(C/A)x\exp[-A/(\alpha-x)] , \quad (16)$$

and the latter does indeed vanish (with an infinite-order zero) at $x=\alpha$. This may possibly lead to the only consistent solution as there are ambiguities in the evaluation of the second integral in (11) [unless one imposes an additional eigenvalue condition for α through $A_2^{[1]}(\alpha)$]. With such a conjecture we will see that only one additional eigenvalue condition for α is to be imposed. We may extract the single-closed-fermion-loop contribution $a_1^{[1]}(\alpha)$ to $a_1(\alpha)$ to be

$$a_1^{[1]}(\alpha)=\frac{3}{64\pi^2}\sum_{n\geq 0}\frac{\alpha^{n+1}}{n+1}c_n . \quad (17)$$

Now we generalize the above analysis to the multiloop contribution $a_1^{[l]}(\alpha)$ to $a_1(\alpha)$ and obtain the expressions for $A_1^{[l]}(\alpha)$ and $A_2^{[l]}(\alpha)$. The above conjecture will be consistent if we can show that $A_2^{[l]}(\alpha)$ is still zero for $l \geq 2$. This will indeed be the case. At the eigenvalue we may write

$$\alpha_0[1-\alpha_0\pi(q^2)]^{-1}\,|_{m=0}=q(\alpha) = \sum_{l\geq 0} q^{[l]}(\alpha)\,, \tag{18}$$

where $q(\alpha)$ is a finite constant depending on α, and $q^{[l]}(\alpha)$ contains exactly l closed fermion loops, with $q^{[0]}(\alpha)=\alpha$. We note that a photon line $1/q^2$ in the expression for $A_i^{[1]}$ appears in the form x/q^2. Accordingly, we may replace each of the x's in x^{n+1}, as appearing in (11) and (12) by $q^{[1]}(x), q^{[2]}(x), \ldots$, using elementary combinatorics, to obtain the exact l-closed-fermion-loop contribution to $a_1(\alpha)$. For example, in the two-closed-fermion-loop contribution, we have the following substitutions in (11) and (12):

$$x^{n+1}\to x^{n+1}(n+1)q^{[1]}(x)\,, \tag{19}$$

and we obtain

$$A_1^{[2]}(x)=\sum_{n\geq 0}\frac{x^n(n+1)}{n+1}q^{[1]}(x)c_n = q^{[1]}(x)C^{[1]}(x)\,, \tag{20}$$

$$A_2^{[2]}(x)=\sum_{n\geq 0}\frac{x^n(n+1)}{n+1}q^{[1]}(x)b_n = q^{[1]}(x)F^{[1]}(x)\,, \tag{21}$$

and at the eigenvalue $A_2^{[2]}(\alpha)=0$.

In general we obtain from an analysis as in (19) and in Ref. 2

$$A_1^{[l]}(x)=\sum_{\substack{j_i=0\\ \epsilon_1\cdots\epsilon_t\\ j_1\cdots j_t}}^{l-1}{}' \frac{[q^{[j_1]}(x)]^{\epsilon_1}}{\epsilon_1!}\cdots\frac{[q^{[j_t]}(x)]^{\epsilon_t}}{\epsilon_t!}C^{[1]}(x)\,, \tag{22}$$

$$A_2^{[l]}(x)=\sum_{\substack{j_i=0\\ \epsilon_1\cdots\epsilon_t\\ j_1\cdots j_t}}^{l-1}{}' \frac{[q^{[j_1]}(x)]^{\epsilon_1}}{\epsilon_1!}\cdots\frac{[q^{[j_t]}(x)]^{\epsilon_t}}{\epsilon_t!}F^{[1]}(x)\,, \tag{23}$$

for $l\geq 2$, where the prime on the summation signs means a sum over all positive integers $\epsilon_1,\ldots,\epsilon_t, j_1,\ldots,j_t$ such that all the j_i are distinct and $\epsilon_1 j_1+\cdots+\epsilon_t j_t=l-1$. Using the infinite-order-zero nature of $F^{[1]}(\alpha)$ we obtain from (23) that $A_2^{[l]}(\alpha)=0$. In particular we note that irrespective of the conjecture we may write for all l

$$A_2^{[l]}(\alpha)=\delta^{l1}\int_0^\alpha dx\, F^{[1]}(x)\,, \tag{24}$$

due to the infinite-order-zero nature of $F^{[1]}(\alpha)$. The summation over l then may be carried out in a standard manner[2] from (17) and (22) and the definition of $C^{[1]}(x)$ in (13) to obtain

$$a_1(\alpha)=\sum_{l\geq 1}a^{[l]}(\alpha)=\frac{3}{64\pi^2}\int_0^\alpha dx\, C^{[1]}(x)q'(x) \equiv G(\alpha)\,. \tag{25}$$

Accordingly to have a completely finite VED it may be necessary to impose one additional eigenvalue condition for α:

$$G(\alpha)=0\,. \tag{26}$$

The constant $G(\alpha)$ may be computed from perturbation theory and by a formal integration [Eq. (25)], by using in the process the perturbative expressions for $C^{[1]}(x)$ and $q'(\alpha)$. To lowest order, for example, we have computed

$$G(\alpha)=-\frac{11}{384\pi^2}\left[\frac{\alpha}{\pi}\right]+O(\alpha^2)\,. \tag{27}$$

[The minus sign in (27) should be noted as it may lead to the physically undesirable property that $\mathscr{E}$ is unbounded below ($\Lambda^2\to\infty$).] In the next section we summarize our method of study and our findings and make some further comments.

III. DISCUSSION

We have shown that if the anomalous mass dimension $\delta(\alpha)>1$ then, at the eigenvalue, the VED may at worst diverge quartically, the coefficient of which we have denoted by $a_1(\alpha)$ [see Eq. (4)]. Due to the vanishing (and rapid damping) of light-light scattering graphs and mass corrections for the propagators for $m\to 0$, we have argued that to study the nature of the coefficient $a_1(\alpha)$, it is sufficient to carry out a study in massless QED at the eigenvalue [see also Eq. (3)]. Proceeding loopwise we have conjectured, due to the infinite-order-zero nature of $F^{[1]}(x)$, at $x=\alpha$, that the integral [Eq. (14)] of $F^{[1]}(x)$ at $x=\alpha$ is also zero, and an example of this has been given where this happens. We have seen that the VED is completely finite if we impose only one additional eigenvalue condition for α through $G(\alpha)=0$ [see (25) and (26)]. The constant $G(\alpha)$ may be in principle computed. This is unfortunately a formidable problem [as is the situation for $F^{[1]}(\alpha)$] as only low-order computations are possible at this stage [Eq. (27)]. This possible solution not only leads to a finite expression for VED but also gives the physically very desirable result that $\mathscr{E}\equiv 0$. Finally the assumption that $\delta(\alpha)>1$ has an interesting consequence on the vanishing of the (electro-) magnetic form factor $G_M(Q^2)$ at large momentum transfer ($Q^2\to\infty$). We have shown in an earlier investigation[7] [Eq. (36)] that at the eigenvalue $[\beta(\alpha)=0^\infty]$ we have with $\delta(\alpha)>0$ that

$$|G_M(Q^2)| \underset{Q^2\to\infty}{\leq} C(Q^2)^{-(1+\delta)/2} .$$

[The notation $\beta_0(\alpha)/2$ for $\delta(\alpha)$ was used in Ref. 7.] The condition $\delta(\alpha)>1$ then implies the interesting property

$$|G_M(Q^2)| \underset{Q^2\to\infty}{\leq} C(Q^2)^{-1} .$$

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