PHYSICAL REVIEW D VOLUME 27, NUMBER 6 15 MARCH 1983

Rigid-motion conditions in special relativity

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(Received 25 May 1982)

A rigid-motion condition is proposed as a generalization of the well-known Born rigidity in order to include rotational degrees of freedom. The explicit solution is obtained in the general case and comparison is made with Born and other conditions. Classical rigidity is recovered in full in the nonrelativistic limit.

INTRODUCTION

As rigid motions is the topic of this work, let us state that throughout this paper a congruence of timelike lines is said to be rigid if the infinitesimal space distance between lines is conserved along them. A rigid body would be a body which can only move in a rigid way.

It is clear that the concept of a rigid body is at odds with relativity as, if such a material would exist, it could be used to send signals at an infinite speed. The concept of rigid motions, however, does not need that of rigid bodies to be valid; falling drops of rain or defiling troops provide examples of rigid motions without recourse to rigid bodies even in nonrelativistic mechanics. Moreover, the pioneering work of Born[1] formulated in a mathematical way the rigidity condition (as stated above) by assigning a precise sense to the notion of "infinitesimal space distance" and gave a respectable status to rigid motions in special relativity.

Nevertheless, there is an argument against Born rigidity to be considered as a relativistic generalization of classical rigidity. Herglotz and Noether[2] showed that Born rigid motions have, roughly speaking, three degrees of freedom only,[3] not the six exhibited by a classical rigid motion. This is reflected in the fact that in every application considered other generalizations have been used.

First of all, in classical continuum mechanics the rigid body is useful to study deformations by comparison with it. The concept of incompressible material associated with Born rigidity is formulated by the condition $\partial_\alpha u^\alpha=0$, which implies an infinite speed of sound. Actually, other incompressibility conditions are used to make the speed of sound equal to that of light.[4]

In addition, classical rigidity is used to define noninertial reference frames. The Born motions lead in this way to Fermi coordinates[5] in Minkowski case. It is clear that in order to get a generic noninertial frame one must generalize the Fermi coordinates in such a way that an arbitrary rotation may be allowed for.

Also, classical rigid motions are useful in the modeling of extended bodies as its use may avoid the need to discuss the details of the internal structure. The most widely known relativistic substitutes of this are the Dixon skeletons.[6] This approach allows also the construction of models for spinning particles; at this level there are many alternative methods, as that of Finkelstein,[7] based on group-theoretical considerations. In any case, Born rigidity has not been used.

This list, which does not pretend to be exhaustive, reflects the failure of the Born condition to reach at the relativistic level the same generality and wideness of applications that the rigid motions have at the classical one.

In this work an alternative (more general) formulation of the rigidity condition is proposed. The conceptual framework is associated in an essential way with the notion of systems of synchronized observers[8] which is in the basis of the so-called "3 + 1" or evolution formalism of relativity.[9] The resulting rigidity condition is completely integrated in the special-relativistic case so that its general solution is explicitly presented. A careful analysis of the results shows that both Born and (generalized) Fermi congruences fit in the proposed framework. It is also explicitly shown that one recovers the whole classical formalism when going to the nonrelativistic limiting case. There is no conceptual difficulty in extending the proposed framework to general relativity, although all the work is done in the special-relativistic case for the sake of obtaining the general solution in an explicit way.

I. SYSTEMS OF SYNCHRONIZED OBSERVERS

Let us fix some notations. A "system of observers" will be a timelike vector field $\xi = \xi^\alpha \partial_\alpha$, $\eta_{\alpha\beta}\xi^\beta\xi^\beta < 0$ [$\eta = \text{diag}(- + + +)$]. The solutions of the differential equations

$$\frac{dX^\alpha}{dt} = \xi^\alpha[X(t)] \ , \tag{1.1}$$

corresponding to the initial conditions $X(t_0) = X_0$, will be "world lines" of the observers and the parameter t, which is the standard one corresponding to the local transformations group generated by ξ, will be called the "time" associated with ξ.

The space metric usually associated with ξ is the quotient metric g_Q:

$$(g_Q)_{\alpha\beta} = \eta_{\alpha\beta} + u_\alpha u_\beta \ , \tag{1.2}$$

where u is the unit vector field ($\eta_{\alpha\beta}u^\alpha u^\beta = -1$) corresponding to ξ.

A system of synchronized observers is a system of observers such that the locus of the points corresponding to $t = t_0$ is a given spacelike hypersurface. The family of hypersurfaces formed by the loci of points corresponding to the same time t is then given and it will be called "synchronization." One local coordinate expression for that family being $\phi(X) = t$, one has

$$L_\xi(\phi) = 1 \ , \tag{1.3}$$

where L_ξ stands for the Lie derivative along ξ. Note that a synchronization orthogonal to the world lines, that is, the existence of a function $\phi(X)$ such that

$$\partial_\alpha \phi = \eta_{\alpha\beta}\xi^\beta(\eta_{\mu\nu}\xi^\mu\xi^\nu)^{-1} \ ,$$

is possible if and only if ξ is irrotational.

The metric on the Minkowski space induces a metric structure on every surface of the synchronization: the "induced metric" g_I. This is a three-dimensional object, as it acts only on vectors in the tangent space to every hypersurface. Nevertheless, one can give a covariant four-dimensional expression of this object by constructing a projection operator Π ($\Pi_\alpha{}^\beta \Pi_\beta{}^\gamma = \Pi_\alpha{}^\gamma$), which projects vectors into tangent vectors and this for every hypersurface of the family, that is,

$$\Pi_\alpha{}^\rho \partial_\rho \phi = 0, \quad L_\xi(\Pi) = 0 \ . \tag{1.4}$$

One expression for Π which is suggested by (1.3) is

$$\Pi_\alpha{}^\beta = \delta_\alpha{}^\beta - (\partial_\alpha \phi)\xi^\beta \ , \tag{1.5}$$

where (1.4) follows from

$$L_\xi(\partial_\alpha \phi) = 0 \ ,$$

which is obtained by taking the Lie derivative of (1.3). This gives for the induced metric

$$(g_I)_{\alpha\beta} = \Pi_\alpha{}^\mu \Pi_\beta{}^\nu \eta_{\mu\nu} \tag{1.6}$$

and for its (spatial) inverse

$$(g_I{}^{-1})^{\alpha\beta} = \eta^{\alpha\beta} + n^\alpha n^\beta \ , \tag{1.7}$$

$$g_I g_I{}^{-1} = \Pi \ , \tag{1.8}$$

where n is the vector field of unit normals to the surfaces ($\eta_{\alpha\beta}n^\alpha n^\beta = -1$).

In the same way, one obtains for the second fundamental form on the surfaces

$$(\chi_I)_{\alpha\beta} = \Pi_\alpha{}^\mu \Pi_\beta{}^\nu (\partial_\mu n_\nu) \ . \tag{1.9}$$

II. RIGIDITY CONDITIONS

As pointed out in the Introduction, a congruence of world lines is said to be rigid if the (infinitesimal) spatial distances among lines are conserved along them. It is clear that this requirement depends in an essential way on the concept of spatial distance. If, for instance, one defines the spatial distance between a pair of infinitely near world lines as the distance along a path orthogonal to one of them, then the condition of rigidity is

$$L_\xi(g_Q) = 0 \ , \tag{2.1}$$

which leads to the well-known Born rigidity.[1]

A different condition is obtained if one defines the spatial distance as the interval between two simultaneous (in the sense of the synchronization) events, that is,

$$L_\xi(g_I) = 0 \ , \tag{2.2}$$

which is equivalent to

$$L_\xi(g_I{}^{-1}) = 0 \tag{2.3}$$

as it follows easily from (1.8) and (1.4). The expression for (2.3) in local inertial coordinates is

$$D_\alpha \xi_\beta + D_\beta \xi_\alpha = n'_\alpha n_\beta + n_\alpha n'_\beta \ , \tag{2.4}$$

where the following notations are used:

$$D_\alpha \equiv \Delta_\alpha{}^\rho \partial_\rho, \quad (\ )' \equiv \xi^\rho \partial_\rho(\ ) \ , \tag{2.5}$$

Δ being the orthogonal projector

$$\Delta_\alpha{}^\rho \equiv \delta_\alpha{}^\rho + n_\alpha n^\rho \ .$$

There is in addition a requirement on the synchronization which may be imposed on physical grounds. The hypersurfaces must maintain the same form as one moves along the world lines. If (2.2) is verified, this amounts to the demand

$$L_\xi(\chi_I)=0 . \qquad (2.6)$$

Examples of such hypersurfaces are

$$\chi_I=Kg_I ,$$

where K is a numerical constant that corresponds to hypersurfaces of constant curvature (hyperplanes are obtained when $K=0$).

In the sequel, the Eqs. (2.4) and (2.6) will be studied in detail. For an analogous study of (2.1) see, for instance, Ref. 3.

III. INTEGRABILITY CONDITIONS

The commutators of the differential operators (2.5) are

$$[D_\alpha,D_\beta]=(D_\alpha\Delta_\beta{}^\rho-D_\beta\Delta_\alpha{}^\rho)D_\rho , \qquad (3.1)$$

where the term in $n^\rho\partial_\rho$ does not appear because n is irrotational. That is,

$$D_\alpha n_\beta=D_\beta n_\alpha\equiv\sigma_{\alpha\beta} . \qquad (3.2)$$

One has also

$$[D_\alpha,\xi^\rho\partial_\rho]=[D_\alpha\xi^\rho-(n_\alpha n^\rho)']\partial_\rho$$
$$=(\eta^{\rho\lambda}D_\lambda\xi_\alpha)D_\rho , \qquad (3.3)$$

where we have taken (2.4) into account.

Therefore, the resulting conditions of integrability for an equation of the kind $D_\alpha A=\Delta_\alpha{}^\rho B_\rho$ are

$$\Delta_\alpha{}^\rho D_\beta B_\rho=\Delta_\beta{}^\rho D_\alpha B_\rho , \qquad (3.4)$$

where A and B may have additional tensor indices. In the particular case

$$D_\alpha A=0 \Longleftrightarrow \Pi_\alpha{}^\rho\partial_\rho=0 , \qquad (3.5)$$

the general solution is $A=f(\phi)$, where f is an arbitrary function of $\phi(x)$.

IV. INTEGRATION OF THE RIGIDITY CONDITION (2.4)

Equation (2.4) may be put in the form

$$D_\alpha\xi_\beta=\omega_{\alpha\beta}+n'_\alpha n_\beta=\Delta_\alpha{}^\rho\Omega_{\rho\beta} , \qquad (4.1)$$

where

$$\omega_{\alpha\beta}=-\omega_{\beta\alpha},\quad \omega_{\alpha\rho}n^\rho=0 ,$$
$$\Omega_{\alpha\beta}\equiv\omega_{\alpha\beta}+n'_\alpha n_\beta-n'_\beta n_\alpha . \qquad (4.2)$$

The necessary integrability conditions (3.4) for ξ in (4.1) are

$$\Delta_\alpha{}^\rho D_\beta\Omega_{\rho\gamma}=\Delta_\beta{}^\rho D_\alpha\Omega_{\rho\gamma} , \qquad (4.3)$$

which can be decomposed in two parts

$$\Delta_\alpha{}^\rho\Delta_\beta{}^\mu\Delta_\gamma{}^\nu(\partial_\mu\Omega_{\rho\nu}-\partial_\rho\Omega_{\mu\nu})=0 , \qquad (4.3a)$$

$$n^\gamma(D_\alpha\Omega_{\beta\gamma}-D_\beta\Omega_{\alpha\gamma})=0 . \qquad (4.3b)$$

By combining cyclic index permutations of (4.3a), one gets

$$\Delta_\alpha{}^\rho\Delta_\beta{}^\mu\Delta_\gamma{}^\nu\partial_\mu\Omega_{\rho\nu}=0 ,$$

that is,

$$D_\alpha\Omega_{\beta\gamma}=S_{\alpha\beta}n_\gamma-S_{\alpha\gamma}n_\beta , \qquad (4.4)$$

where, taking (4.3b) into account, $S_{\alpha\beta}=S_{\beta\alpha}$, $S_{\alpha\rho}n^\rho=0$.

The expression for S may be computed in terms of the σ appearing in (3.2) if one takes into account the definition (4.2) of Ω and the integrability conditions (3.3). This gives

$$S_{\alpha\beta}=\sigma'_{\alpha\beta}+\Omega_\alpha{}^\rho\sigma_{\rho\beta}+\Omega_\beta{}^\rho\sigma_{\alpha\rho} , \qquad (4.5)$$

where $S_{\alpha\rho}n^\rho=0$ due to (4.2), that is

$$(n')^\alpha-n^\rho\Omega_\rho{}^\alpha=0 . \qquad (4.6)$$

At this point, one may impose Eq. (2.6), which reads

$$\Pi_\alpha{}^\mu\Pi_\beta{}^\nu L_\xi(\sigma_{\mu\nu})=0 , \qquad (4.7)$$

where we have used (1.4) and the expression

$$(\chi_I)_{\alpha\beta}=\Pi_\alpha{}^\mu\Pi_\beta{}^\nu\sigma_{\mu\nu}$$

for χ_I in terms of σ. A simple calculation shows that (4.5) is equivalent to

$$S_{\alpha\beta}=0 \qquad (4.8)$$

and this implies, as stated in the preceding section, that Ω is an arbitrary function of $\phi(x)$ and, going to (4.1),

$$\xi^\beta=\Omega_\alpha{}^\beta(\phi)X^\alpha+v^\beta(\phi) , \qquad (4.9)$$

where v is another arbitrary function of $\phi(x)$.

The expression (4.9) is the general solution of the rigidity condition (2.4) for any synchronization $\phi(x)$ verifying (1.3) and (2.6).

V. COMPUTATION OF THE WORLD LINES

The differential equations (1.1) for the world lines are written in the case (4.9) as

$$\frac{dX^\beta}{dt}=\Omega_\alpha{}^\beta(t)X^\alpha+v^\beta(t) , \qquad (5.1)$$

where we have used (1.3). That is, $\phi(X[t])=t$. The freedom in the choice of v can be used to impose any given timelike curve $\mathscr{C}[t]$ to be a solution of (5.1),

$$v^{\beta}(t)\equiv\mathscr{C}'^{\beta}[t]-\Omega_{\alpha}{}^{\beta}(t)\mathscr{C}^{\alpha}[t] .$$

Then (5.1) reads

$$\frac{d}{dt}(X[t]-\mathscr{C}[t])^{\beta}=\Omega_{\alpha}{}^{\beta}(t)(X[t]-\mathscr{C}[t])^{\alpha} . \tag{5.2}$$

The structure of (5.2) is very close to that of the Fermi transport law.[5] The analogous structure of (4.5) and (4.6) suggests to ourselves the definition of a derivation operator δ, which will be called "rigid derivative," associated to any vector field ξ of the form (4.9). Its expression in local inertial coordinates is

$$\begin{aligned}\delta(T^{\alpha\beta\cdots}{}_{\mu\nu\cdots})\equiv(T^{\alpha\beta\cdots}{}_{\mu\nu\cdots})'-\Omega_{\rho}{}^{\alpha}T^{\rho\beta\cdots}{}_{\mu\nu\cdots}\\ -(\text{all upper indices})\\ +\Omega_{\mu}{}^{\rho}T^{\alpha\beta\cdots}{}_{\rho\nu\cdots}\\ +(\text{all lower indices}) .\end{aligned}$$

This operator preserves the contractions and it also verifies

$$\begin{aligned}&\delta(\eta_{\alpha\beta})=0,\ \ \delta(\sigma_{\alpha\beta})=S_{\alpha\beta}=0 ,\\ &\delta(n^{\alpha})=0,\ \ [D_{\alpha},\delta]=0 .\end{aligned} \tag{5.3}$$

The transport law of the kind (5.2) associated with δ will be called "rigid transport."

Let us consider now the unit vector

$$(X[t]-\mathscr{C}[t])^{\alpha}[\eta_{\mu\nu}(X-\mathscr{C})^{\mu}(X-\mathscr{C})^{\nu}]^{-1/2} ,$$

which is rigidly transported allowing for (5.2) and (5.3). Let us consider the case in which $X[t]$ approaches $\mathscr{C}[t]$ along some path lying over $\phi(X)=t$. One gets in the limit a rigidly transported unit vector in the tangent space at $\mathscr{C}[t]$. As it can be done along any path, one can construct an orthonormal triad in such a tangent space and complete it with the unit normal n at $\mathscr{C}[t]$ to get finally an orthonormal tetrad $e_a(t)$ $[a=0,\ldots,3; e_0(t)=n^{\alpha}\partial_{\alpha}]$ rigidly transported along $\mathscr{C}$. It follows that to transport rigidly a vector along $\mathscr{C}$ amounts to assigning it constant components in the moving tetrad frame $e_a=e_a{}^{\alpha}\partial_{\alpha}$.

The moving tetrad e_a defines in a natural way a parameter-dependent Lorentz transformation L

$$\begin{aligned}&e_a(t)\equiv L_a{}^{b}(t)e_b(t_0) ,\\ &L_a{}^{b}(t_0)=\delta_a{}^{b} .\end{aligned} \tag{5.4}$$

If one writes now the condition for rigid transport of e_a, one gets

$$e_a{}^{\rho}(t_0)\Omega_{\rho}{}^{\beta}(t)=L_a^{Tb}(t)L_b'{}^{c}(t)e_c{}^{\beta}(t_0) , \tag{5.5}$$

where L^T is the transposed matrix of L and use has been made of the identity

$$LL^T=L^TL=I ,$$

valid for Lorentz matrices. Let us consider now the inverse matrix of $e_a{}^{\rho}(t_0)$, denoted by $e_{\rho}{}^{a}(t_0)$,

$$e_a{}^{\rho}e_{\rho}{}^{b}=\delta_a{}^{b},\ \ e_{\alpha}{}^{c}e_c{}^{\beta}=\delta_{\alpha}{}^{\beta}$$

and construct the Lorentz matrices

$$L_{\alpha}{}^{\beta}(t)\equiv e_{\alpha}{}^{a}(t_0)L_a{}^{b}(t)e_b{}^{\beta}(t_0) .$$

Then one gets easily from (5.5)

$$\begin{aligned}\Omega_{\alpha}{}^{\beta}(t)&=L^{T}{}_{\alpha}{}^{\gamma}(t)L'_{\gamma}{}^{\beta}(t)\\ &=-[L_{\alpha}^{T\alpha}(t)]'L_{\gamma}{}^{\beta}(t) ,\end{aligned} \tag{5.6}$$

$$L_{\alpha}{}^{\beta}(t)=\exp\left[\int_{t_0}^{t}\Omega_{\alpha}{}^{\beta}(\lambda)d\lambda\right] , \tag{5.7}$$

which shows that L is just as arbitrary as Ω is.

The integration is completed easily if one substitutes (5.6) into (5.2) to get

$$\frac{d}{dt}\{L_{\alpha}^{T\beta}(t)(X[t]-\mathscr{C}[t])^{\alpha}\}=0 , \tag{5.8}$$

$$X^{\beta}[t]=\mathscr{C}^{\beta}[t]+L_{\alpha}{}^{\beta}(t)(X[t_0]-\mathscr{C}[t_0])^{\alpha} ,$$

which is the general solution of (5.2).

VI. ANALYSIS OF THE RESULTS

The condition of rigidity (2.4) contains as particular cases the two branches of solutions (Killing and irrotational vector fields[2]) of the Born condition (2.1). The Killing case is recovered when

$$\Omega_{\alpha}'{}^{\beta}=0,\ \ v'^{\beta}=0 , \tag{6.1}$$

the vector field ξ then being independent of the synchronization. The irrotational case is recovered when one demands the synchronization to be orthogonal to the world lines, that is

$$n^{\alpha}\equiv u^{\alpha}\Longrightarrow D_{\alpha}u_{\beta}=0 , \tag{6.2}$$

which is the Born equation in the irrotational case ($\omega=0$).

On the other hand, if one imposes the unit tangent vector W^{α} to a single world line $\mathscr{C}$ to be rigidly transported along the curve

$$\frac{d}{dt}W^{\beta}=\Omega_{\alpha}{}^{\beta}W^{\alpha} ,$$

then one can decompose Ω in the neighborhood of $\mathscr{C}$ in the following way:

$$\begin{aligned}&\Omega_{\alpha\beta}=\omega_{\alpha\beta}+W'_{\alpha}W_{B}-W'_{\beta}W_{\alpha} ,\\ &\omega_{\alpha\beta}=-\omega_{\beta\alpha},\ \ \omega_{\alpha\rho}W^{\rho}=0\end{aligned} \tag{6.3}$$

and the law of rigid transport (5.2) corresponds to a law of (generalized) Fermi transport[5] along the single integral curve $\mathscr{C}$. Note that if one demands (6.3) to be true for all the world lines, then one is left only with Born motions (6.1) and (6.2).

It is clear from the general solution (5.8) for the integral curves that a rigid congruence [in the sense of (2.4)] is determined by a single line $\mathscr{C}$ and a Lorentz transformation L. The configuration space at any surface $t=\text{const}$ is then isomorphous to the Poincaré group. If one is allowed to impose invariant conditions on L, one recovers the framework of homogeneous spaces of the Poincaré group which has been used by Finkelstein[7] to define what he called rigid structures with a view to describe spinning particles. As the nonrelativistic limit of the Poincaré group is the Galilei group, one would have in principle a classical nine-dimensional configuration space (there is one degree of freedom which changes only the parameter in the world lines) in contrast with the well-known six-dimensional one of the classical rigid body. Let us see in detail what happens.

It is clear that if the formalism proposed here has to reduce in the nonrelativisitic limit to the classical one, the synchronization surfaces must tend in this limit to the hypersurfaces of constant Newtonian (absolute) time, that is, by (3.5)

$$D_\alpha X^0 = O(1/c^2) \Longleftrightarrow X^0 = f(\phi) + O(1/c^2) . \qquad (6.4)$$

That is,

$$\delta_\alpha{}^0 + n_\alpha n^0 = O(1/c^2) ,$$
$$n^\alpha = (1, \vec{n}/c) + O(1/c^2) .$$

This amounts to saying that the vector field n can be interpreted as the field of velocities corresponding to a (irrotational) system of physical observers whose world lines are orthogonal to the synchronization. In what follows it will be shown that this interpretative assumption is enough to recover the right classical limit. To see this, let us consider the nonrelativistic limit of the expression (5.8) for the world lines

$$X^0[t] - \mathscr{C}^0[t] = X^0[t_0] - \mathscr{C}^0[t_0] = 0 , \qquad (6.5a)$$

$$X^j[t] - \mathscr{C}^j[t] = R_i{}^j(t)(X[t_0] - \mathscr{C}[t_0])^i , \qquad (6.5b)$$

where (6.4) has been taken into account and R is the spatial rotation associated to L. If one defines

$$\omega_i{}^j \equiv R_i^{Tk} R_k'{}^j = -(R_i^{Tk})' R_k{}^j ,$$
$$v^j \equiv \mathscr{C}'^j - \omega_i{}^j \mathscr{C}^i ,$$

then one gets from (6.5)

$$\frac{dX^j}{dt} = \omega_i{}^j(t) X^i + v^j(t) , \quad X^0 = \mathscr{C}^0(t) , \qquad (6.6)$$

which is the general expression of the classical rigid motions.

Let us see what has happened. As the Newtonian time X^0 is a function of the parameter t only, because of (6.4), the whole class of (parameter-dependent) Galilean boosts is contained in the class of (parameter-dependent) space displacements and, then, it does not have any effect on the vector $(X - \mathscr{C})$.

Note that the classical limit (6.6) is not fully recovered from the Born condition as the corresponding relativistic configuration space does not include arbitrary (parameter-dependent) rotations.[3]

These considerations suggest that Eq. (2.4) provides the adequate framework to deal with rigid motions, even if it may be suitable to put some restrictions on the arbitrary functions or the synchronization (or both) on the grounds of the physics of the applications considered, definition of noninertial frames, modeling of spinning particles, or others.

ACKNOWLEDGMENTS

The author is indebted to L. Bel and B. Coll for useful discussions and allowing comparison of this work with their own. He also acknowledges the hospitality of the Collège de France and the Institute Henri Poincaré, and the financial support of the Ministerio de Universidades e Investigación (M.U.I.) during the completion of this work.

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