On the detection of cosmological neutrinos by coherent scattering

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There have been several proposals that the sea of cosmological relic neutrinos could be detected by the energy, momentum, or angular momentum transferred during their coherent interaction with matter. We show that all but one of these proposals is incorrect and that the one exception leads to an effect that is probably immeasurably small. We first review the existing limits on the cosmological neutrinos and describe the expectations from the standard hot big-bang model. We then prove a general theorem that if the time average of the neutrino flux is spatially homogeneous (which is expected for cosmological neutrinos), then to first order in the weak coupling the energy and momentum transfer to any microscopic or macroscopic target is zero. Similarly, the angular momentum transfer is zero unless the target has a nonzero polarization or current density (but in this case the effect is probably immeasurably small). No assumption is made concerning the isotropy of the neutrino flux. Finally, we reexamine the individual proposals using the language of geometrical optics and show where all but one of them was incorrect. In particular, we prove that the momentum and angular momentum transferred by refraction from a homogeneous incoming beam to a (microscopically) isotropic target vanish to order $n-1$, with n the index of refraction.

I. INTRODUCTION

A. Cosmological neutrinos

The standard hot big-bang cosmological model[1] is now widely accepted theoretically. In addition to explaining the observed expansion of the universe, it successfully predicts the abundances of primordial helium and deuterium,[2] explains the observed background microwave radiation (which is characterized by a blackbody temperature $T_\gamma \simeq 2.7\,°\text{K}$, and a corresponding number density $n_\gamma \simeq 400/\text{cm}^3$), and—when combined with grand unified theories[3]—may give a dynamical explanation of the small baryon asymmetry $(N_B - N_{\bar{B}})/N_\gamma \simeq N_B/N_\gamma \simeq 10^{-10\pm 1}$ observed in the present universe.

The big-bang model also predicts the existence of a sea of cosmological relic neutrinos, analogous to the sea of microwave photons, left over from the very early universe. The idea is that soon after the big bang the neutrinos would have been kept in thermodynamic equilibrium by charged- and neutral-current scattering processes. When the temperature fell to the decoupling temperature $T_D \simeq 1$ MeV (at a time $t_\nu \simeq 1$ sec after the big bang), however, the weak-reaction rates would have become slow compared to the expansion rate of the universe and the neutrinos would have essentially decoupled from all other particles (and from each other). Except for a red-shifting of their momenta due to the expansion of the universe, these neutrinos would have remained undisturbed until the present time. If these neutrinos could be detected they would provide a direct experimental probe of the period 1 sec after the big bang (by way of contrast, the microwave photons provide information on the much later period $T \sim 4000\,°\text{K}$, $t \sim 4\times 10^5$ yr, when the photons decoupled after electrons and protons recombined into neutral hydrogen). Unfortunately, our conclusions in this paper are that the relic neutrinos predicted by the standard model are essentially impossible to detect directly.

Let us now consider the expected properties of the relic neutrinos in more detail.[4] At the decoupling temperature T_D the neutrinos would still have been in equilibrium, so that the phase-space density of the ith type of neutrino ν_i and antineutrino $\bar{\nu}_i$ would be $F_{\nu_i}(p)d\tau$ and $F_{\bar{\nu}_i}(p)d\tau$, respectively, where

$d\tau=d^3x\,d^3p/h^3$ is the volume element in phase space and F_{ν_i} and $F_{\bar\nu_i}$ are given by the equilibrium forms

$$F_{\nu_i}(p)=F_{\rm eq}(p,m_i,\mu_{D_i},T_D) \equiv \frac{1}{\exp\left[\dfrac{(p^2+m_i^2)^{1/2}-\mu_{D_i}}{T_D}\right]+1}\ ,$$

$$F_{\bar\nu_i}(p)=F_{\rm eq}(p,m_i,-\mu_{D_i},T_D)\ . \tag{1.1}$$

Of course, m_i is the mass of ν_i and μ_{D_i} is a possible chemical potential that occurs if there is an asymmetry between ν_i and $\bar\nu_i$.

A neutrino which had momentum $\vec p\,'$ at decoupling would have its momentum red-shifted to $\vec p=\vec p\,'/\eta$ at a later time t, where $\eta(t)\equiv R(t)/R(t_D)$ represents the expansion of the universe between t_D and t (R is the scale parameter). Since the neutrinos are noninteracting, the phase-space density $F_{\nu_i}(p)d\tau$ is still given by the equilibrium form for $t>t_D$, except that $F_{\rm eq}$ must be evaluated at the original momentum p' (the volume element $d\tau$ is left invariant by the expansion, since $d^3x=\eta^3 d^3x'$, $d^3p=d^3p'/\eta^3$),

$$\begin{aligned}F_{\nu_i}(p)&=F_{\rm eq}(p',m_i,\mu_{D_i},T_D)\\&=F_{\rm eq}\left[p,\frac{m_i}{\eta},\frac{\mu_{D_i}}{\eta},\frac{T_D}{\eta}\right]\\&\equiv F_{\rm eq}(p,m_{{\rm eff},i},\mu_i,T)\\&=\frac{1}{\exp\left[\dfrac{(p^2+m^2_{{\rm eff},i})^{1/2}-\mu_i}{T}\right]+1}\end{aligned} \tag{1.2}$$

with $\mu_i\to-\mu_i$ for $F_{\bar\nu_i}$. We see that the distribution function of the decoupled neutrinos maintains an equilibrium form, but is characterized by a red-shifted effective temperature $T\equiv T_D/\eta=T_DR_D/R$, an effective chemical potential $\mu_i\equiv\mu_{D_i}/\eta$, and an effective mass $m_{{\rm eff},i}\equiv m_i/\eta$. Note that if the neutrinos were relativistic at decoupling ($m_i\ll T_D$), then F_{ν_i} is still of relativistic form even for $T\ll m_i$, because it is $m_{{\rm eff},i}\ll T$ and not m_i that appears in F_{ν_i}. This occurs because the neutrinos are not truly in equilibrium. The interactions needed to reduce F to a nonrelativistic distribution ceased being important before the temperature fell to m_i. In this paper we will always assume that $m_{{\rm eff},i}$ is negligible (see below).

The present neutrino temperature is presumably $T_\nu\simeq(4/11)^{1/3}T_\gamma\simeq1.9\,°$K, the factor $(4/11)^{1/3}\approx1.401$ being due to the annihilation of e^+e^- pairs at $T\sim4\times10^9\,°$K ($t\sim4$ sec) which reheated the photon (but not the neutrino) gas.

In the standard model it is usually assumed that the neutrino asymmetries are small [i.e., $(N_{\nu_i}-N_{\bar\nu_i})/N_\gamma\ll1$]. This is suggested[5] but not rigorously proved[6] by grand unified theories. In that case one has $|\mu_i|/T\ll1$ so that the number densities are

$$N_{\nu_i}\simeq N_{\bar\nu_i}\simeq\int\frac{d^3p}{h^3}\frac{1}{\exp(p/T)+1}\simeq50/{\rm cm}^3 \tag{1.3}$$

and the average momentum is

$$\langle p\rangle=\int\frac{d^3p}{h^3}\frac{p}{e^{p/T}+1}\simeq3.2T_\nu\simeq6.0\,°{\rm K}\simeq5.2\times10^{-4}\ {\rm eV}\ . \tag{1.4}$$

For three families, the total number density is then $N_{\rm tot}\simeq300/{\rm cm}^3$. If the neutrinos are massless, the corresponding flux $j=j_\nu+j_{\bar\nu}\simeq O(N_{\rm tot}c)\simeq10^{13}/{\rm cm}^2$ sec is very large: It is comparable to the flux of antineutrinos from nuclear reactors. By way of comparison, the expected flux of solar neutrinos is[7] $j_\nu^{\rm solar}\simeq6\times10^{10}/{\rm cm}^2$ sec, while the estimated flux of high-energy ^{8}B neutrinos presumably measured in the Davis experiment[7] is only $j_\nu^{^8{\rm B}}\simeq3\times10^6/{\rm cm}^2$ sec. If ν_i has a mass $m_i\gg T_\nu$, on the other hand, then $\langle p\rangle\simeq3.2T_\nu$ and $N_{\nu_i}=N_{\bar\nu_i}\simeq50/{\rm cm}^3$ are unchanged, but $\langle E_{\nu_i}\rangle\simeq m_i$. It is curious that the neutrinos are highly nonrelativistic in this case [though $F_\nu(p)$ is of relativistic form], with $\langle v_i\rangle/c\simeq\langle p\rangle/m_i\simeq2\times10^{-5}$ for[8] $m_i\simeq30$ eV. Of course, the flux $j_i=j_{\nu_i}+j_{\bar\nu_i}$ is then reduced by a factor $\langle v_i\rangle/c$.

If the neutrinos have a nonzero chemical potential, their number density and average momentum are enhanced. One has an asymmetry[9]

$$N_{\nu_i}-N_{\bar\nu_i}=\frac{T_\nu^{\,3}}{6}\left[\xi_i+\frac{1}{\pi^2}\xi_i^{\,3}\right]\ , \tag{1.5}$$

where $\xi_i=\mu_i/T=(\mu_{D_i}/\eta)/(T_D/\eta)$ is constant for decoupled neutrinos. For $\xi_i\gg1$, the neutrinos form a degenerate Fermi gas with

$$N_{\nu_i}\simeq\mu_i^{\,3}/6\pi^2,\ \ N_{\bar\nu_i}\simeq0,\ \ \langle p_i\rangle\simeq\tfrac{3}{4}\mu_i \tag{1.6}$$

(the results are of course interchanged for ξ_i large and negative).

B. Detection of the cosmological neutrinos

We have seen that the expected density and flux of cosmological neutrinos is relatively large. However, the average energy is so small that the charged- or neutral-current cross sections $\sigma_\nu \simeq G_F{}^2 E_\nu{}^2$ for scattering from ordinary matter are negligibly small for $|\xi_i| \lesssim O(1)$: One has $\sigma_\nu \sim 10^{-62}$ cm^2 for massless neutrinos ($E_\nu \simeq p_\nu \sim 3T_\nu$) and $\sigma_\nu \sim 10^{-53}$ cm^2 for $E_\nu \sim m_\nu \sim 30$ eV. Similarly, the effects of the neutrino sea on the electron spectrum near the end point in β decay[10] and the implications of cosmic-ray scattering from the neutrinos[11] (either in distorting the primary cosmic-ray spectrum or in producing high-energy secondary neutrinos) are probably immeasurable for any reasonable chemical potentials.

The most stringent existing limits on the relic neutrinos are therefore indirect. One limit comes from the requirement that the neutrino energy density not exceed the observed upper limit[12] $\rho_0 < 8\times 10^{-29}$ g/cm^3 on the total energy density of the present universe. For massless nondegenerate neutrinos this is easily satisfied. One has

$$\rho_\nu = \frac{7\pi^2}{15\times 16} T_\nu{}^4 g \underset{g=6}{\simeq} 4\times 10^{-6}\rho_0 , \tag{1.7}$$

where g is the number of neutrino helicity states. For massive nondegenerate neutrinos,[13] on the other hand,

$$\rho_\nu \simeq 2N_{\nu_i} \sum m_{\nu_i} \simeq \frac{\rho_0}{400 \text{ eV}} \sum m_{\nu_i} , \tag{1.8}$$

so that the sum of the masses of the light stable neutrinos cannot exceed 400 eV. Similarly, for massless degenerate neutrinos, one has[10,6]

$$\left[\sum_i \xi_i{}^4\right]^{1/4} < 80 , \tag{1.9}$$

while[4] for $m_{\nu_i} \simeq (20–30)$ eV,

$$|\xi_i| < 6 . \tag{1.10}$$

More stringent (but less direct) constraints on the chemical potentials are obtained from nucleosynthesis.[14] The observed element abundances are compatible with $\xi_i = 0$, and simple perturbations around this standard model allow[14] $|\xi_{\nu_e}| \lesssim 0.2$, $|\xi_{\nu_\mu},\nu_\tau| \lesssim 2$. However, David and Reeves[15] have found a continuum of new solutions to nucleosynthesis in which $|\xi_{\nu_e}|$ increases from 0 to 1.2, with its effect on the n/p ratio balanced by an increased expansion rate, which could be due to large asymmetries in ν_μ or ν_τ ($|\xi_{\nu_\mu,\nu_\tau}| \lesssim 20$), additional neutrino species, magnetic monopoles, anisotropic shear, etc.

If the relic neutrinos are massive, they may account for the missing mass in galactic clusters.[16] It is also possible that massive neutrinos are clustered in the galactic halo, but the prevalent opinion is that such clustering does not occur.[16] Even if it did, the local enhancement of neutrino density would be no more than a factor $\lesssim 10^5$.

We finally mention the theoretical prejudice[5] that lepton-number-violating interactions in grand unified theories would probably have reduced any initial large neutrino chemical potentials to negligibly small values ($|\xi_i| \ll 1$) long before nucleosynthesis. Harvey and Kolb[6] have recently shown, however, that such an erasure could have been evaded if the theory involved an approximately conserved global quantum number. Asymmetries as large as $|\xi_i| = O(1)$ may then be possible.[17] It has also been recently shown[17] that for a large class of models with (lepton-number violating) Majorana neutrino masses in the 10-eV range, an arbitrary large initial asymmetry would have been reduced to the interesting range $|\xi| = O(1)$ prior to nucleosynthesis.

We therefore have the tantalizing and frustrating situation that the big-bang model predicts that the Universe is filled with the essentially undisturbed neutrino remnants of the very early universe. The detailed characteristics of this neutrino sea depend on such fundamental issues as the magnitude and character of neutrino masses, grand unification, and the initial conditions of the big bang. The neutrino sea may profoundly affect the structure and formation of galactic clusters and the detailed scenario of nucleosynthesis (and therefore the limits on the number of neutrino species and the precise determination of the baryon density[2] that are derived from it), yet these relic neutrinos are essentially impossible to detect by any conventional means. Clearly, any encouraging new approach would be very exciting.

C. Coherent detectors

There have been several[18–22] recent proposals to detect the relic neutrinos by their coherent interaction with matter. It is unfortunately our conclusion that all but one[18] of these proposals is wrong.

The basic idea is to think of a low-energy neutrino passing through matter as a wave with wavelength $\lambda = h/p \sim 2.4$ mm (for $p \sim 3.2T_\nu = 6$ °K). Since λ is large compared to the interatomic spacing, the effect of the medium can be described by introducing an index of refraction n in the free field equation for the propagation of the neutrino wave.[23] If λ is also small compared to the size of the scatterer, so that diffraction can be ignored, one can describe the propagation of a neutrino "ray" through matter by geometrical optics.

The index of refraction for $\nu(\bar{\nu})$ is given (for small $n-1$) by

$$n_{\nu,\bar{\nu}}-1=\frac{2\pi}{p^2}\sum_a N_a f^a_{\nu,\bar{\nu}}(0)\ , \tag{1.11}$$

where N_a is the number density of scatterers of type a and $f^a_\nu(0)$ $[f^a_{\bar{\nu}}(0)]$ is the forward-scattering amplitude for νa $[\bar{\nu}a]$ elastic scattering. f^a is easily computed to give (for a target at rest)

$$f^a_{\nu,\bar{\nu}}(0)=\mp\frac{1}{\pi}\frac{G_F E}{\sqrt{2}}K(p,m_\nu)(g^a_V+g^a_A\vec{\sigma}_a\cdot\hat{p})\ , \tag{1.12}$$

where the upper (lower) sign refers to neutrinos (antineutrinos),[24] $E=(p^2+m_\nu{}^2)^{1/2}$ is the neutrino energy,

$$K(p,m_\nu)\equiv\frac{1}{4E}(E+m_\nu)\left[1+\frac{p}{E+m_\nu}\right]^2$$

$$\rightarrow\begin{cases}1, & m_\nu=0\\ \frac{1}{2}, & p\ll m_\nu\ ,\end{cases} \tag{1.13}$$

and g^a_V and g^a_A are the vector and axial-vector couplings in the effective Lagrangian

$$-L=\frac{G_F}{\sqrt{2}}\bar{\nu}\gamma_\mu(1+\gamma_5)\nu\bar{\psi}_a\gamma^\mu(g^a_V+g^a_A\gamma^5)\psi_a \tag{1.14}$$

for $\nu a\rightarrow\nu a$. For massive neutrinos we have assumed helicity -1 $(+1)$ for ν $(\bar{\nu})$, as would be the case for neutrinos that decoupled while relativistic.[25] (The sign of the $\vec{\sigma}\cdot\hat{p}$ term changes for the opposite helicities.)

For a target with Z protons and electrons and $A-Z$ neutrons (assuming $\langle\vec{\sigma}_p\rangle=\langle\vec{\sigma}_n\rangle=0$, but allowing $\langle\vec{\sigma}_e\rangle\neq 0$, as in a ferromagnet) and the neutral-current parameters of the $SU_2\times U_1$ model one obtains[24]

$$n_{\nu_e,\bar{\nu}_e}-1=\mp\frac{G_F}{\sqrt{2}}\frac{EKN}{p^2}(3Z-A+Z\langle\vec{\sigma}_e\rangle\cdot\hat{p})\ , \tag{1.15}$$

where N is the number density of target atoms, and

$$n_{\nu_i,\bar{\nu}_i}-1=\mp\frac{G_F}{\sqrt{2}}\frac{EKN}{p^2}(Z-A-Z\langle\vec{\sigma}_e\rangle\cdot\hat{p}) \tag{1.16}$$

for $\nu_i=\nu_\mu$ or ν_τ. The difference between (1.15a) and (1.15b) is due to the charged-current contribution to $g^e_{V,A}$.

For iron ($Z=26, A=56, N\sim 0.85\times 10^{23}/\text{cm}^3$), for example,

$$\begin{aligned}n_{\nu_e,\bar{\nu}_e}-1&=\mp 2.3\times 10^{-10}(1+0.85\langle\vec{\sigma}_e\rangle\cdot\hat{p})\ ,\\ n_{\nu_i,\bar{\nu}_i}-1&=\pm 3.1\times 10^{-10}(1+1.2\langle\vec{\sigma}_e\rangle\cdot\hat{p})\\ &\qquad (i=\mu,\tau)\end{aligned} \tag{1.17}$$

for massless neutrinos with $E=p=5.2\times 10^{-4}$ eV, while for massive neutrinos the coefficients are increased by $m_i/2p$. For $m=E=30$ eV, $p=5.2\times 10^{-4}$ eV the coefficients in (1.16) become 6.6×10^{-6} and 8.9×10^{-6}, respectively. We therefore see that $n-1$ is very tiny for all reasonable parameters for the relic neutrinos.

Several experiments have been proposed to measure the energy momentum, or angular momentum transferred from the neutrino sea to a macroscopic target due to the refractive bending of a neutrino ray as it passes through the target or the total external reflection that might be expected for neutrinos incident on a surface with angle (with respect to the plane) $\theta<\theta_c\equiv\sqrt{2(1-n)}$ (for $n<1$). However, we will show that all but one of these proposals are incorrect. The one exception[18] is technically correct but, as noted by the author, is probably impractically difficult.

D. Outline

Rather than initially discussing each of the proposals separately, we will begin (in Sec. II) by giving a general analysis in the Born approximation of the energy, momentum, and angular momentum transferred from a quantum-mechanical wave (e.g., a neutrino wave) to a target. We assume that the unperturbed target (which may be either macroscopic, as in the coherent scattering proposals, or microscopic) is static. Furthermore, we assume that the time average of the incident wave is spatially homogeneous over the size of the detector (which is certainly expected to be true for cosmological neutrinos). We do not need to make any assumption concerning the isotropy[26] of the incident wave, however; our results would apply even to a unidirectional neutrino beam. We show that under these assumptions the total momentum and energy transfer to the target vanishes in the Born approximation (i.e., to first order in the interaction strength). Furthermore, the net angular momentum transfer vanishes to the same order unless the target is either polarized or carried an electric current.

We expect the Born approximation to be valid for sufficiently weak scattering. For the scattering of neutrinos from a microscopic target it certainly holds. For a macroscopic target the condition for validity is that the phase of the wave not be significantly shifted over the dimension a of the detector in the direction of the neutrino flux. This is true provided

$$\frac{|n-1|a}{\lambda} \ll 1 . \tag{1.18}$$

For massless neutrinos $(n-1)/\lambda \simeq 10^{-7}\ \mathrm{m}^{-1}$ [this is actually independent of wavelength; see (1.15)]. Hence, (1.17) is well satisfied for any conceivable laboratory detector. For massive neutrinos, $(n-1)/\lambda$ is enhanced by $m/2p$, but even for $m=30$ eV, $p=5.2\times 10^{-4}$ eV the condition (1.17) is satisfied for $a \ll 300$ m.

We therefore conclude that the proposals in Refs. 19–21, which claim effects for unpolarized targets to order $n-1$, are incorrect. In Sec. III, we reexamine these proposals using the language of geometrical optics. We show that the momentum or angular momentum transferred from neutrinos to the detector by refraction[20] vanishes to order $n-1$, provided that the time-averaged flux of neutrinos is homogeneous over the size of the detector. (The optics proof breaks down in the very special case of a plane surface parallel to a unidirectional flux of neutrinos, but holds for a continuous but anisotropic distribution of neutrino directions.) The proposal[19] to use total external reflection of neutrinos from a surface is shown to fail because total external reflection in fact only occurs if the thickness d of the reflector is greater than $\lambda/2\pi\theta_c = \lambda/2\pi\sqrt{2(1-n)}$ (which is very far from being satisfied by the proposed detector). Otherwise, there is an evanescent wave through the detector and most of the wave is transmitted. Finally, the calculated[12] transfer of energy from refracted neutrinos to electrons in a superconductor is shown to rely on an incorrect boundary condition.

The only exception to our null result of Sec. II is that angular momentum can be transferred to a polarized target or, in principle, to a target carrying a current, such as an individual electron or a ferromagnet, provided there is also an asymmetry between neutrinos and antineutrinos and a forward-backward anisotropy in the neutrino flux (e.g., due to the motion of the earth through the neutrino sea). This effect, which is probably too small to measure, was first proposed sometime ago by Stodolsky,[18] who used the different (but valid) language of an electron interacting coherently with the neutrino sea. In Sec. III, we reinterpret this effect in the language of geometrical optics.

Section IV presents our conclusions and a summary of our results.

II. THE BORN APPROXIMATION

In this section we consider a field $\psi(x)$ interacting with a static classical source $j_\mu(\vec{x})$. For concreteness we will think of a neutrino field but the derivation will be of general validity. We shall calculate the net momentum, energy, and angular momentum transfer $\Delta\vec{p}$, ΔE, and $\Delta\vec{J}$ imparted to the source by the field over a large time, in the Born approximation. Because the source is static[27] it does not create or absorb quanta of the field, but merely scatters them. The Lagrangian density for the system is

$$\mathscr{L} = \mathscr{L}_{\text{free}} - j_\mu(\vec{x})\bar{\psi}(x)\gamma^\mu\psi(x) , \tag{2.1}$$

where $\mathscr{L}_{\text{free}}$ is the free Lagrangian for the field ψ. We need to compute the time-averaged integrals of the energy-momentum tensor $\theta^{\mu\nu}$ and the angular momentum current $J^{\mu\nu}$ over a closed surface $S \equiv \partial V$ enclosing the entire source (for simplicity we choose the surface to lie just outside the source as illustrated in Fig. 1),

$$\frac{\Delta p^\nu}{T} \equiv -\left\langle \int_s da_j \theta^{j\nu} \right\rangle_T , \tag{2.2}$$

$$\frac{\Delta J^i}{T} \equiv -\left\langle \int_s da_j J^{ji} \right\rangle_T , \tag{2.3}$$

where $\langle\ \rangle_T$ denotes time averaging (from $-T/2$ to $T/2$) and $d\vec{a}$ is the surface differential element with outward-pointing normal.

From (2.1) we find the usual expressions

$$\partial_\mu \theta^{\mu\nu} = \bar{\psi}\gamma^\alpha\psi\partial^\nu j_\alpha , \tag{2.4}$$

$$\partial_\mu J^{\mu i} = -\bar{\psi}\gamma^\alpha\psi(\vec{r}\times\vec{\nabla})^i j_\alpha + \bar{\psi}\gamma^j\psi\epsilon^{ijk}j^k . \tag{2.5}$$

The right-hand side of Eq. (2.4) expresses the fact

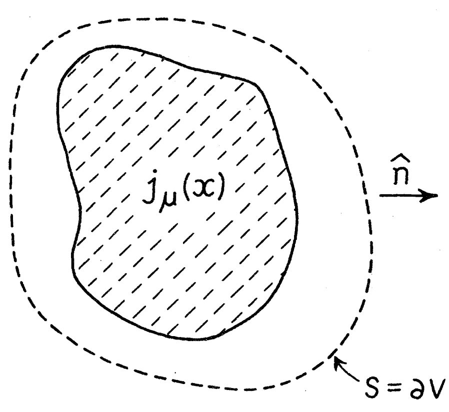

FIG. 1. A static classical source $j_\mu(\vec{x})$ confined to a volume V with surface S and outward normal $\hat{n}$.

that the presence of the source breaks translational invariance. Similarly, the right-hand side of Eq. (2.5) shows that rotational invariance is broken by the rotational noninvariance of the source. Because the source is assumed to be static the total field energy is conserved, and the time average of $\partial_0\theta^{0\nu}$ and $\partial_0 J^{0i}$ must vanish identically. Consequently, Eqs. (2.4) and (2.5) yield

$$\langle \partial_i\theta^{i\nu}\rangle_T=\langle \bar\psi\gamma^\alpha\psi\partial^\nu j_\alpha\rangle_T \ , \tag{2.6}$$

$$\langle \partial_j J^{ji}\rangle_T=-\langle \bar\psi\gamma^\alpha\psi(\vec r\times\vec\nabla)^i j_\alpha\rangle_T + \epsilon^{ijk}\langle\bar\psi\gamma^j\psi j^k\rangle_T \ . \tag{2.7}$$

Equations (2.6) and (2.7) represent the local changes in the densities. We integrate them over the volume V enclosing the scatterer to find

$$\frac{-\Delta p^\nu}{T}\equiv\left\langle \int_{\partial V} da_i\theta^{i\nu}\right\rangle_T=\int_V d^3x\langle\partial_i\theta^{i\nu}\rangle_T=\int_V d^3x\langle\bar\psi\gamma^\alpha\psi\partial^\nu j_\alpha\rangle_T \ , \tag{2.8}$$

$$\frac{-\Delta J^i}{T}\equiv\left\langle \int_{\partial V} da_j J^{ji}\right\rangle_T=\int_V d^3x\langle\partial_j J^{ji}\rangle_T=-\int_V d^3x\langle\bar\psi\gamma^\alpha\psi(\vec r\times\vec\nabla)^i j_\alpha\rangle_T+\int_V d^3x\,\epsilon^{ijk}\langle\bar\psi\gamma^j\psi j^k\rangle_T \ . \tag{2.9}$$

So far Eqs. (2.8)—(2.9) are exact. At this point we can show that the energy transfer to the scatterer vanishes. Equation (2.8) for $\nu=0$ involves $\partial^0 j_\alpha$ which is zero for a static source. We have neglected only the effects of inelastic scattering and the recoil of the scatterer; these are the assumptions which are implicit in our Lagrangian, Eq. (2.1). To evaluate these expressions we will now use the Born approximation. Since the expressions already contain the source term linearly, we will neglect the functional dependence of ψ on j and use the free fields $\psi^{(0)}(x)$ in evaluating the averages. Therefore the current $\bar\psi\gamma^\alpha\psi$ is to be replaced by the free incident current $\bar\psi^0\gamma^\alpha\psi^0$. Furthermore, the source is static so the time average becomes the time averaging of the incident unscattered current. At this stage we make the physically reasonable assumption that the time-averaged current $\langle\bar\psi^0\gamma^\alpha\psi^0\rangle_T$ is homogeneous, i.e., a constant in space. Note that this does not mean an isotropic flux. The assumption of homogeneity of the incident current can only be made in this lowest-order Born approximation. Indeed the full current would contain the scattered waves which break the homogeneity. With these assumptions, Eq. (2.8) becomes

$$\frac{\Delta p^\nu}{T}=-\langle\bar\psi^0\gamma^\alpha\psi^0\rangle_T\int_V d^3x\,\partial^\nu j_\alpha \ . \tag{2.10}$$

For $\nu=0$, Δp^0 vanishes by the assumption of a static unperturbed source. If $\nu=i$, the use of Gauss's theorem and the fact that j vanishes on S by assumption implies $\Delta\vec p=0$. Hence

$$\Delta p^\nu=0 \ . \tag{2.11}$$

Similarly, Eq. (2.9) yields

$$\frac{\Delta J^i}{T}=+\langle\bar\psi^0\gamma^\alpha\psi^0\rangle_T\int_V d^3x(\vec r\times\vec\nabla)^i j_\alpha -\langle\bar\psi^0\gamma^j\psi^0\rangle_T\epsilon^{ijk}\int_V d^3x\, j^k \ . \tag{2.12}$$

Using $(\vec r\times\vec\nabla)j_\alpha=-(\vec\nabla\times\vec r)j_\alpha$ and Gauss's theorem one easily sees that the first integral vanishes as before. The second term may be nonzero if the target has a directionality at the level of the individual scatterers. This could be due either to a net polarization of the electrons or a nonzero current density. The latter effect is expected to be much smaller (suppressed by v_s/c, where v_s is the drift velocity), so we will concentrate on a polarized medium. From (2.12) we have

$$\frac{\Delta\vec J}{T}=-\vec F\times\vec M \ , \tag{2.13}$$

where $\vec F\equiv\langle\bar\psi^0\vec\gamma\psi^0\rangle_T$ is the incident "flux" while $\vec M\equiv\int_V d^3x\ \vec j(\vec x)$ is the net "magnetization" of the source. If the incident flux is isotropic, then the net $\Delta\vec J$ would vanish identically when we average over directions. In the nonisotropic case, then $\Delta\vec J$ can be nonzero. Let us note, however, that even if $\vec M$ does not vanish identically, $\Delta\vec J$ is proportional to the net fermion number density $\vec F$, i.e., the fermion flux *minus* the antifermion flux. Hence there must be an asymmetry in the flux, otherwise no net effect will result.

This special case was considered previously by Stodolsky.[18] He argued that if there is an asymmetry between neutrinos and antineutrinos, then the interaction of a single electron or the polarized electrons in a ferromagnet with the neutrino sea will induce a net torque on the electron or on the ferromagnet.

Our results agree precisely with Stodolsky's in the appropriate limit. As we shall discuss briefly in Sec. IV, however, the effect is probably immeasurably small.

In conclusion, we have shown that in the Born approximation, the field cannot impart a net momentum to a static source, although it can give it an angular momentum if specific conditions are met. When should one expect this approximation to be valid? Since we have assumed that the full current can be replaced by the unperturbed current we cannot tolerate scatterers which violently disturb the incident flux. A measure of the scattering strength is the *optical mean free path* $\lambda/(n-1)$, with n the index of refraction, over which a wave develops a significant phase shift relative to its propagation in vacuum. Hence we expect the Born approximation to be valid provided the scatterer is small enough compared to the optical mean free path

$$a \ll \frac{\lambda}{n-1} \tag{2.14}$$

with a the size of the scatterer in the direction of propagation of the incident wave. As we already discussed in Sec. I, (2.14) is well satisfied for cosmological neutrinos for any reasonable laboratory detector.

We now extend our Born approximation to include the case of time-varying sources. This is of some interest as the proposal of Ref. 21 actually involved a time-varying current of electrons in the superconducting medium [the analysis there calculated the energy transfer to the electrons using a quasistatic approximation; in which case the result should have been zero by Eq. (2.10)].

The Born approximation consists of replacing $\bar\psi\gamma^\alpha\psi$ in Eqs. (2.8) and (2.9) with its zeroth-order value. The result is then linear in j. Therefore we consider the effects of a sinusoidally varying source

$$j^\alpha(\vec{x},t)=j^\alpha(\vec{x})e^{-i\omega t} \tag{2.15}$$

with the general result obtained by a superposition of frequencies. One can also take the time component of J^α to be a constant. Then Eqs. (2.8) and (2.9) become[28]

$$\begin{aligned}
\frac{\Delta p^0}{T} &= +i\omega \int_V d^3x \left[\frac{1}{T}\int_0^T dt\, e^{-i\omega t}\bar\psi^{(0)}\gamma^\alpha\psi^{(0)}\right] j_\alpha(\vec{x})\ ,\\
\frac{\Delta p^i}{T} &= -\int_V d^3x \left[\frac{1}{T}\int_0^T dt e^{-i\omega t}\bar\psi^{(0)}\gamma^\alpha\psi^{(0)}\right]\partial^i j_\alpha(x)\ ,\\
\frac{\Delta J^i}{T} &= -\int_V d^3x \left[\frac{1}{T}\int_0^T dt\, e^{-i\omega t}\bar\psi^{(0)}\gamma^\alpha\psi^{(0)}\right](\vec{r}\times\vec{\nabla})^i j_\alpha(x) + \int_V d^3x\left[\frac{1}{T}\int_0^T dt\, e^{-i\omega t}\bar\psi^{(0)}\gamma^j\psi^{(0)}\right]\epsilon^{ijk}j^k(\vec{x})\ .
\end{aligned} \tag{2.16}$$

Now if we make a somewhat stronger assumption about homogeneity, i.e., that

$$\begin{aligned}
J^\alpha(\omega,\vec{x},T) &\equiv N^\alpha(\omega,\vec{x},T)/T\\
&\equiv \frac{1}{T}\int_0^T dt\, e^{-i\omega t}\bar\psi^{(0)}\gamma^\alpha\psi^{(0)}
\end{aligned} \tag{2.17}$$

is independent of $\vec{x}$ over the volume of the target, then our previous theorems about $\Delta\vec{p}$ and $\Delta\vec{J}$ go through with the replacement

$$\langle\bar\psi\gamma^\alpha\psi\rangle_T \to J^\alpha(\omega,T)\ . \tag{2.18}$$

For energy, we find

$$\frac{\Delta p^0}{T} = i\frac{\omega}{T}\int d^3x\, N^\alpha(\omega,\vec{x},T)j_\alpha(\vec{x})\ . \tag{2.19}$$

This does not vanish in general. However, for large T, $N^\alpha(\omega,\vec{x},T)$ is just the Fourier component of the number or current density of incident particles at $\vec{x}$, for $\alpha=0$ or $\alpha=i$, respectively. Hence, N^α is independent of T and the rate of energy transfer vanishes as $1/T$.

III. THE GEOMETRICAL OPTICS APPROACH

In this section, we will analyze in some detail the proposals for detecting neutrinos using coherent-scattering geometrical optics. The conditions for the validity of geometrical optics are the following:

(1) The wavelength must be much larger than the interatomic spacing.

(2) The wavelength must be much smaller than the size a of the scatterer; i.e., $\lambda \ll a$.

The first condition is necessary for the medium to be treated as a smooth continuum. In our case, since $\lambda \gtrsim 1$ mm, this condition is very well satisfied and we will not mention it further. Condition 2 is necessary if we wish to neglect the effects of diffraction.

Note that in general, the currents appearing in the "mean field" Lagrangian are not simply the smoothed microscopic currents because the field acting on each scatterer is not the mean field. However, for weak scattering $|n-1| \ll 1$ and this distinction vanishes.

In particular, one can derive the index of refraction from our Lagrangian (2.1) (treated as a mean-field Lagrangian) and reproduce to lowest order in j the results of the usual forward-scattering amplitude analysis. Higher orders in j complicate the formulas for the index of refraction in terms of j, but do not invalidate geometrical optics.

The analyses in the literature also take advantage of the fact that $|n-1|$ is small. Thus our geometrical optics analysis will rest on the assumptions

$$\frac{\lambda}{a} \ll 1 , \tag{3.1a}$$

$$|n-1| \ll 1 . \tag{3.1b}$$

It is interesting to compare these conditions with Eq. (1.17). Because of the smallness of $|n-1|$ we see that there is a significant range of a's which are small enough for the Born approximation to be valid yet large enough for geometrical optics to be valid (for massless ν's 1 mm $\ll a \ll 10^7$ m; for $m_\nu = 30$ eV, 1 mm $\ll a \ll$ 300 m). We now proceed to analyze the various proposals for neutrino detection by coherent scattering using geometrical optics, and reproduce the Born approximation results in the regions of overlapping validity.

A. Total external reflection

The first proposal for a coherent-scattering experiment[19] involved the total external reflection of neutrinos from a plane surface with $n<1$. Rays incident on a plane surface at an angle $\theta \le \theta_c = \sqrt{2(1-n)}$ are assumed to have a reflection coefficient of unity, leading to a momentum transfer $\Delta p = 2p\theta$. The resulting pressure on the surface is therefore

$$P \sim (2\pi\theta_c)(p\theta_c)\theta_c j \sim (2)^{5/2} p j (n-1)^{3/2} , \tag{3.2}$$

where j is the neutrino flux. The factor $2\pi\theta_c$ represents the fraction of the total solid angle over which $\theta \le \theta_c$. It is strictly valid for an isotropic flux; however, a suppression by θ_c occurs for any flux which is not sharply peaked for $\theta < \theta_c$. There is also a factor of θ_c due to the fact that the area of the surface projected orthogonal to the ray is suppressed by θ_c. It was not included in the analyses of Refs. 19 and 20. Thus they found pressures of order $(n-1)$ rather than $(n-1)^{3/2}$, making their numerical estimates very over optimistic. Note there is no cancellation between ν's and $\bar{\nu}$'s since only one has $n<1$.

Equation (3.2) is valid for a single surface. Any real detector has two surfaces. To prevent a cancellation of forces it was proposed[19] that a stack of mirrors be built in conjunction with reflecting guides to enhance the flux on one surface.

We will now see that there is a more fundamental difficulty with a finite thickness detector.

Our theorem [Eq. (2.11)] states that $\Delta p/\Delta t = 0$. We must resolve this conflict in the range of overlapping validity. Consider plane waves incident on an infinite plane surface at an angle θ (Fig. 2). The

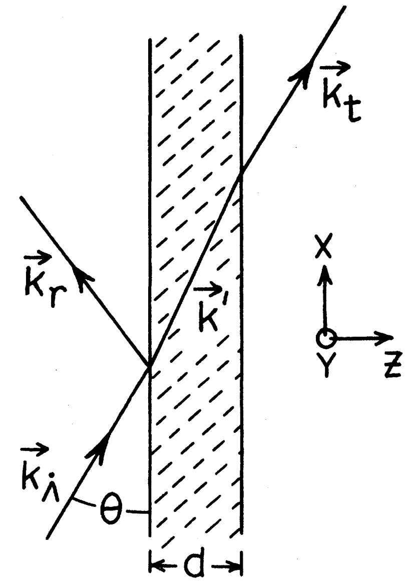

FIG. 2. The incident ($\vec{k}_i$), reflected ($\vec{k}_r$), transmitted ($\vec{k}_t$), and internal ($\vec{k}'$) wave vectors for a wave incident at angle θ on a reflecting surface of thickness d. The index of refraction is $n<1$. The z axis is normal to the surface of the slab.

boundary conditions are that $k' = nk$ and that $k_x = k_x'$, $k_y = k_y'$. One therefore has (for small $1-n$ and for θ^2 of order $1-n$) that

$$k_z' = k(\theta^2 - \theta_c^{\,2})^{1/2} , \tag{3.3}$$

where $\theta_c = \sqrt{2(1-n)}$. For $\theta < \theta_c$ this is imaginary so that the wave inside the scatterer decays exponentially in z with a decay length

$$z_0 = \frac{1}{2\pi} \frac{\lambda}{(\theta_c^{\,2} - \theta^2)^{1/2}} \ge \frac{1}{2\pi} \frac{\lambda}{\sqrt{2(1-n)}} . \tag{3.4}$$

For $d \gg z_0$, total external reflection indeed occurs. For $d \ll z_0$, on the other hand, there is negligible decay of the wave from one surface to the other; i.e., the waves on the two surfaces are identical. The unique solution to this problem to leading order in d/z_0 is therefore a transmitted wave to the right of the slab with a transmission coefficient of unity, and *no* reflected wave, as would be expected from the Born approximation. Thus $\Delta p/\Delta t = 0$ when

$$d \ll \frac{\lambda}{\sqrt{1-n}} . \tag{3.5}$$

Let us compare (3.5) with Eq. (1.17). The relevant length a to be used here is the path length of the undeflected wave in the material; i.e.,

$$a = \frac{d}{\theta} .$$

The Born approximation is therefore valid when

$$\frac{d(1-n)}{\theta\lambda} \ll 1 . \tag{3.6}$$

But since $\theta < \theta_c$, (3.6) implies (3.5) so that when the Born approximation is valid $\Delta p/\Delta t = 0$ as required. The geometrical optics analysis fails because it does not consider reflections from the back surface of the slab. These reflections are negligible only when $d \gg z_0$. For massless neutrinos with $\lambda \simeq 2.4$ mm, however, the decay length in iron is of order 20 m so that $d \ll z_0$. For $m_\nu \simeq 30$ eV, z_0 is only of the order of 10 cm so that one could in principle build a reflector. However, the actual proposal[19] required large numbers of very thin reflectors with $d \sim \lambda \ll z_0$. Hence the detection of cosmological neutrinos by total reflection is impractical.

There is an even more stringent practical condition on the length L of the detector. The Born approximation gives zero momentum transfer for $L \ll \lambda/(n-1)$ (300 m for 30-eV ν's, 10^7 m for massless ν's). The geometrical optics analysis fails in this regime because of diffraction. We will analyze diffraction at a planar surface in the next section.

B. Refraction

We now consider the refraction of neutrinos by a scatterer of arbitrary shape to lowest order in $(1-n)$. It was argued in Ref. 20 that the refraction of an individual neutrino ray could impart a momentum [angular momentum] transfer of order $p(1-n)[ap(1-n)]$ to the target, so that

$$\frac{\Delta p}{\Delta t} \sim A(j_\nu - j_{\bar\nu}) p(1-n)\gamma , \tag{3.7}$$

where A is the target area and γ is a geometric factor due to the averaging of the neutrino flux over direction and impact points. Note that ν and $\bar\nu$ contributions cancel (they have opposite $n-1$) and that $jp(1-n)$ is essentially independent of neutrino mass and momentum (ν and $\bar\nu$ would add for the angular momentum transfer due to neutrino spin and the effect would scale as $1/p$). It was argued in Ref. 20 that γ could be nonzero even for an isotropic flux if the target configuration was properly chosen (e.g., a prism). We will now prove, however, that γ vanishes when one averages over the impact points on the surface, even for unidirectional flux.

Consider the refraction of a ray as it crosses the boundary between two substances with indices of refraction n and n' (Fig. 3). Let $\vec{k}$ and $\vec{k}\,'$ be the wave vectors of the incident and refracted waves. Snell's law states that

$$\vec{k}_\perp = \vec{k}\,'_\perp , \tag{3.8}$$

where $\vec{k}_\perp$ and $\vec{k}'_\perp$ are the components of the respective $\vec{k}$ vectors orthogonal to the local outward normal s. By the definition of the index of refraction

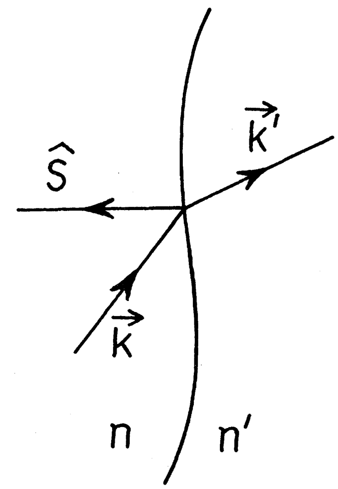

FIG. 3. Refraction at a boundary with normal vector $\hat{s}$ between two media with indices of refraction n and n'.

$$\frac{[(\hat{s}\cdot\vec{k})^2 + k_\perp{}^2]^{1/2}}{n} = \frac{[(\hat{s}\cdot\vec{k}\,')^2 + k_\perp{}^2]^{1/2}}{n'} . \tag{3.9}$$

To lowest order in $\delta n = n' - n$, Eq. (3.9) becomes

$$(\delta n)k = \frac{n\delta(\hat{s}\cdot\vec{k})}{k}(\hat{s}\cdot\vec{k}) . \tag{3.10}$$

Equations (3.8) and (3.10) are equivalent to the vector statement

$$\delta\vec{k} = +\frac{\delta n}{n} k \frac{\hat{s}}{\hat{s}\cdot\hat{k}} . \tag{3.11}$$

Now consider a refracting volume with a ray incident in the z direction (Fig. 4).

The momentum change at the front surface is

$$\delta\vec{k}_f = \frac{(n-1)k\hat{s}_f}{\hat{s}_f\cdot\hat{z}} = -(n-1)k\frac{\hat{s}_f}{|\hat{s}_f\cdot\hat{z}|} . \tag{3.12}$$

Similarly for the back surface

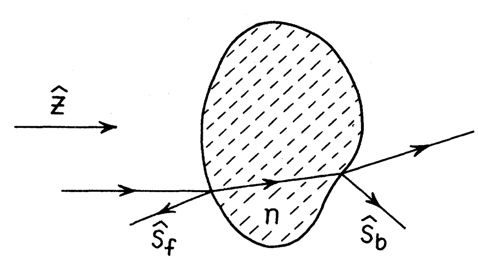

FIG. 4. Refraction of a ray incident along the z direction by a finite scatterer with index of refraction n. $\hat{s}_f$ and $\hat{s}_b$ are the outward normals to the surface at the entering and exiting positions, respectively.

$$\delta\vec{k}_b = +(1-n)k\frac{\hat{s}_b}{\hat{s}_b\cdot\hat{z}} = -(n-1)k\frac{\hat{s}_b}{|\hat{s}_b\cdot\hat{z}|} \tag{3.13}$$

($\hat{s}_f$ and $\hat{s}_b$ are defined in Fig. 4). In Eq. (3.13) we have replaced the magnitude and direction of the k vectors with their values before the refraction at the front surface. This is valid to leading order in $(n-1)$.

We now impose the homogeneity of the ν flux by integrating over the impact parameters (x,y) of rays traveling in the z direction. Adding the front and back surface deflections gives for the momentum transfer to the scatterer

$$\frac{\delta\vec{p}}{\delta t} = -(1-n)(j_\nu - j_{\bar{\nu}}) \times k\int dx\,dy\left[\frac{\hat{s}_f}{|\hat{s}_f\cdot\hat{z}|} + \frac{\hat{s}_b}{|\hat{s}_b\cdot\hat{z}|}\right] . \tag{3.14}$$

We have neglected the fact that the flux at the rear surface is not uniform due to the effects of focusing and defocusing from the front surface. This is again valid to leading order in $(n-1)$. Noting that

$$\hat{s}\frac{dx\,dy}{|\hat{s}\cdot\hat{z}|} = d\vec{a} , \tag{3.15}$$

we obtain

$$\frac{\delta\vec{p}}{\delta t} = (n-1)(j_\nu - j_{\bar{\nu}})k\int_s d\vec{a} , \tag{3.16}$$

where the integral is over the entire surface of the scatterer. However, by Gauss's theorem

$$\int_s da^i = \int_V d^3x\,\nabla_j(\delta_{ji}) = 0 . \tag{3.17}$$

Thus to order $(1-n)$ there is no momentum transfer to the refracting body.

We may similarly analyze momentum transfer. For the moment, let us consider an isotropic scattering medium, i.e., $\vec{j}=0$ (in this case the Born approximation gave a null result).

In this case the ray moves along the direction of its momentum as it traverses the medium, giving no change in angular momentum; such changes occur only at the surface, $\delta\vec{J} = \vec{r}\times\delta\vec{k}$. Averaging over impact parameters gives

$$\frac{\delta\vec{J}}{\delta t} = (n-1)(j_\nu - j_{\bar{\nu}})k\int_s \vec{r}\times d\vec{a} . \tag{3.18}$$

But

$$\int_s (r\times da)^i = \int_s \epsilon_{ijk} r_j da^k = \int_V d^3x\,\partial_k r_j \epsilon_{ijk} = 0 , \tag{3.19}$$

so once again we obtain a null result to leading order in $(1-n)$.

This derivation includes only the effects of changes in the *orbital* angular momentum of the neutrinos. Indeed the spin angular momentum transfer upon refraction is

$$\delta\vec{s} = \mp\delta\hat{k}\tfrac{1}{2} \tag{3.20}$$

for ν or $\bar{\nu}$, respectively. Comparing this to

$$\delta\vec{L} = -\vec{r}\times\delta\vec{k} ,$$

we see

$$\frac{\delta S}{\delta L} \sim \frac{1}{rk} \sim \frac{\lambda}{a} .$$

Thus δS is of the same order as the *corrections* to geometrical optics due to the fact that rays cannot simultaneously be momentum eigenstates and be narrow compared to a. It is nevertheless amusing to note that after averaging over impact parameters, $\delta\vec{s}=0$:

$$\delta\vec{s} = \mp\frac{1}{2}\left[\frac{\delta\vec{k}}{k} - \hat{k}\frac{\delta k}{k}\right] . \tag{3.21}$$

But noting that δk has opposite signs at the front and back surfaces, we find

$$\frac{\delta\vec{s}}{\delta t} = \pm\frac{1}{2k}\frac{\delta\vec{p}}{\delta t} = 0 . \tag{3.22}$$

In Ref. 19 it was proposed to measure the Δp and ΔJ of a prism in the neutrino background. The above proof demonstrates that Δp and for an isotropic medium ΔJ are zero to leading order in $(n-1)$ for *any* shape of the refracting body. The Born approximation of Sec. II extends this statement down to arbitrarily small refractors.

We must now examine a subtlety of the above proof. The arguments we have used fail for rays incident on the refracting surface at an angle $\theta\sim\sqrt{|n-1|}$ for two reasons.

(1) The linearized form of Snell's law [Eq. (3.10)] breaks down because $\delta(\hat{s}\cdot\vec{k})\sim\hat{s}\cdot\vec{k}$.

(2) The geometrical optics reflection coefficient becomes of order unity and $(1-T)=O(1)$, where T is the transmission coefficient. The transmitted wave may also be evanescent so that it carries no momentum. We have been assuming that all of the ν's are transmitted.

The problem in its worst form occurs when the refractor contains a plane surface of area A. The momentum transfer which we associated with this surface is, from Eq. (3.14),

$$\frac{\delta\vec{p}}{\delta t}=(n-1)j_\nu(\hat{s}\cdot\hat{k}A)\frac{\hat{s}}{\hat{s}\cdot\hat{k}}k \ , \tag{3.23}$$

where $\hat{s}\cdot\hat{k}Aj_\nu$ is the total flux into the plane. This is of order $(n-1)$.

What actually happens for $\hat{s}\cdot\hat{k}\sim\sqrt{|n-1|}$ is that ν's are reflected or refracted through angles of order $\hat{s}\cdot\hat{k}$. The resulting momentum transfer is

$$\frac{\delta\vec{p}}{\delta t}\sim(k\hat{s}\cdot\hat{k})j_\nu(\hat{s}\cdot\hat{k}A) \ . \tag{3.24}$$

Note that this is also of order $(n-1)$; but it is not necessarily the same as Eq. (3.23). Thus we find that for the case of a homogeneous unidirectional beam of neutrinos incident on a scatterer containing a plane surface at angles of $O(\sqrt{1-n})$ with respect to the plane; we may have $\Delta\vec{p}\neq 0$ and $\Delta\vec{J}\neq 0$ to $O(1-n)$.

Note that if the ν flux is a smooth function of angles (as is the case for cosmic neutrinos), the fraction of phase space for which this occurs is of order $\sqrt{1-n}$; so the $\delta\vec{p}/\delta t$ and $\delta\vec{J}/\delta t$ become order $(n-1)^{3/2}$. [Of course the results of Sec. II imply that the coefficient must vanish when (2.14) is satisfied.] Thus this remains an impractical mode of detection.

The nonzero result we have obtained for a unidirectional flux must be reconciled with the null result from perturbation theory in the region of overlapping validity. The conflict is obvious when one considers that nontrivial reflection coefficient implies that the wave differs from the zeroth-order wave by order unity (i.e., the "region of disagreement" of Fig. 5). (There is also a problem for the reflected waves, but its resolution is the same as below.)

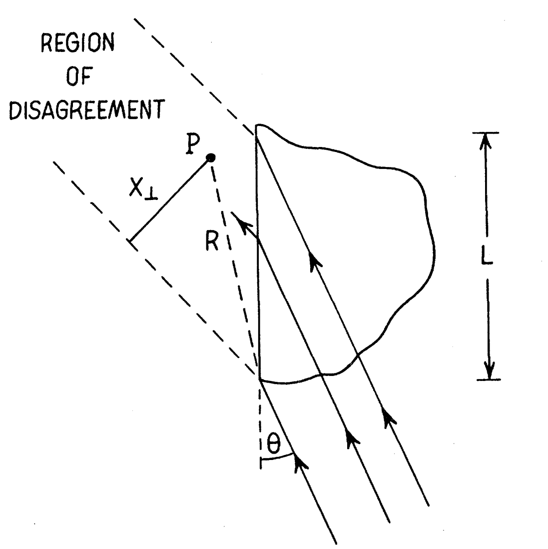

FIG. 5. Rays incident on a plane surface at angle $\theta\sim\theta_c$. The rays in the "region of disagreement" have lost a substantial fraction of their intensity due to reflection at the surface, in disagreement with the Born approximation.

Consider a point P in this region. The lower corner of the surface diffracts the wave since geometrical optics attenuates waves passing to the right of the corner due to the reflection at the planar surface. Standard Fresnel diffraction lore implies that the effect of its diffraction is of order 1 at P when

$$\frac{x_\perp^2}{\lambda R}\lesssim 1 \ . \tag{3.25}$$

When this is true, geometrical optics is not self-consistent. By geometry, for any P in the region of disagreement

$$\frac{x_\perp^2}{\lambda R}<\frac{L\theta^2}{\lambda} \ . \tag{3.26}$$

Thus a sufficient condition for the failure of geometrical optics is

$$\frac{L\theta^2}{\lambda}\lesssim 1 \ . \tag{3.27}$$

But since $\theta\lesssim\sqrt{|n-1|}$, the condition

$$\frac{L(n-1)}{\lambda}\lesssim 1 \tag{3.28}$$

is sufficient for the failure of geometrical optics. The condition for the validity of the Born approximation is just

$$\frac{L(n-1)}{\lambda}\ll 1 \ , \tag{3.29}$$

so that geometrical optics must fail whenever the Born approximation is valid.

C. Angular momentum transfer to anisotropic medium ($\vec{j}\neq 0$)

This is the only case in which the Born approximation gave a nonzero result [Eq. (2.13)]. We now derive Eq. (2.13) using geometrical optics. In our previous optical analysis we showed that the angular momentum transfer due to refraction at the surface vanished after averaging over impact positions. However, for anisotropic media, there is also a change in the angular momentum of a ray as it traverses the refracting body. This is simply because the direction of propagation of the ray is *not* in the direction of its wave vector (i.e., its momentum). This phenomenon is well known in the case of birefringence in crystals. It is very easy to analyze in the case of massless neutrinos. The equation of motion in the medium is

$$(i\partial\!\!\!/ - j\!\!\!/)\nu = 0 \;. \tag{3.30}$$

Defining

$$\nu = e^{-ij\cdot x}\nu' \;, \tag{3.31}$$

and assuming a homogeneous medium (i.e., spatially constant), we find

$$i\partial\!\!\!/ \nu' = 0 \;. \tag{3.32}$$

Hence ν' propagates freely. Thus the rays of ν' are straight lines along the direction of the wave vector $\vec{k}'$ of ν'. But by (3.31) the rays of ν and ν' coincide. Thus defining $\hat{v}$ to be the direction of the ray of ν, we have

$$\hat{v} = \hat{k}' \;. \tag{3.33}$$

However, from Eq. (3.31)

$$\vec{k} = \vec{k}' + \vec{j} \tag{3.34}$$

with $\vec{k}$ the wave vector of ν. Thus

$$\hat{v} = \frac{(\vec{k} - \vec{j})}{|\vec{k} - \vec{j}|} \;. \tag{3.35}$$

To first order in $\vec{j}$ [i.e., to first order in $(n-1)$],

$$\hat{v} = \hat{k} - \frac{\vec{j}}{k} + \hat{k}\frac{\hat{k}\cdot\vec{j}}{k} \;. \tag{3.36}$$

In transversing the material the ν's are displaced by

$$\Delta\vec{r} = \hat{v}L \;, \tag{3.37}$$

where L is the length of the ray in the material. To order $(n-1)$ for rays incident in the z direction,

$$\Delta\vec{r} = \hat{v}t(x,y) \;, \tag{3.38}$$

where $t(x,y)$ is the thickness in the z direction of the refractor. Thus the angular momentum transferred to a given neutrino is

$$-\Delta\vec{J} = +\Delta\vec{r}\times\vec{k} = -t(x,y)\,\vec{j}\times\hat{z} \;. \tag{3.39}$$

Averaging over impact parameters gives

$$\begin{aligned} \frac{\Delta\vec{J}}{\Delta t} &= -(\text{flux})\cdot\hat{z}\times\vec{j}\int dx\,dy\,t(x,y) \\ &= -\vec{F}\times\vec{M} \;, \end{aligned} \tag{3.40}$$

where $\vec{F}$ is the directional flux. This is exactly the result of the Born approximation [Eq. (2.13)].

D. Superconducting refraction detector

Another recently proposed detector[21] for low-energy neutrinos is based on refraction in a thin superconducting strip as shown in Fig. 6. The author concluded that refraction at the surface induces a momentum transfer Δp_s to the superconducting electrons in the direction of the current I, i.e., in the direction of the drift velocity $\vec{v}_s$ of the electrons in the superconductor. This momentum transfer would in turn give rise to a magnetic field which could in principle be measured using a SQUID.

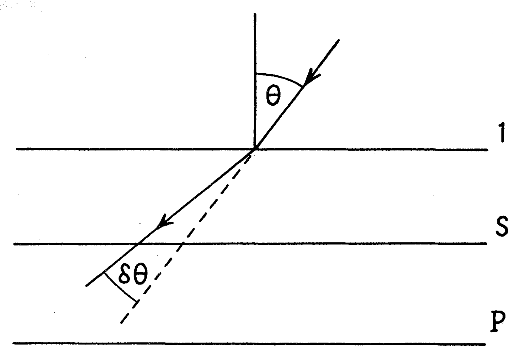

FIG. 6. Refraction at a surface between a medium with $n=1$ and a superconducting medium, as considered in Ref. 21.

Unfortunately, there cannot be a momentum transfer parallel to the surface, although one can easily be misled by a simple application of Snell's law. Indeed using Snell's law and the notation of Fig. 6, one easily finds in leading order in $\delta\theta$

$$\delta\theta\cos\theta = -(n-1)\sin\theta \;. \tag{3.41}$$

If we also write $k_x = k\sin\theta$, where the x axis is chosen along the current direction, we find using Snell's Law

$$\begin{aligned} \Delta k_x &= \Delta(k\sin\theta) \\ &= k\Delta\sin\theta + \sin\theta\Delta k \\ &= \delta\theta k\cos\theta + \sin\theta\Delta k \\ &= -k(n-1)\sin\theta + \sin\theta\Delta k \;. \end{aligned} \tag{3.42}$$

Only the first term in this last expression is kept in Ref. 21, i.e., the Δk term is neglected. However, from the boundary condition of the wave equation one also has

$$\Delta\left[\frac{k}{n}\right] = 0 \;, \tag{3.43}$$

i.e.,

$$\frac{k+\Delta k}{n} = k, \quad \Delta k = k(n-1) \;.$$

Hence Δk contribution cancels the first contribution leading to $\Delta k_x = 0$. Actually, it is easy to show directly that the momentum parallel to the surface is continuous. To do this we write the wave equation in both mediums,

$$(\nabla^2+k^2)A_1 e^{i\vec{k}_1\cdot\vec{x}}=0 , \tag{3.44}$$

$$(\nabla^2+n^2k^2)A_2 e^{i\vec{k}_2\cdot\vec{x}}=0 ; \tag{3.45}$$

let us take the surface to be the xy plane, i.e., $z=0$. Then continuity at the surface requires

$$A_1 e^{i\vec{k}_1\cdot\vec{x}}\,|_{z=0}=A_2 e^{i\vec{k}_2\cdot\vec{x}}\,|_{z=0} , \tag{3.46}$$

which immediately leads to

$$(\vec{k}_1)_{||}=(\vec{k}_2)_{||} , \tag{3.47}$$

where $||$ indicates the component of $\vec{k}_a$ in the xy plane. Hence there is no transfer of momentum in the direction of the current. There is, however, a component of momentum transfer perpendicular to the surface. If one was only considering one strip (s in Fig. 6), there would be an equal and opposite contribution when the ray exits the lower surface. The detector proposed in Ref. 21 included two strips s and p, which have the same index of refraction. Hence there is no momentum transfer when the ray exits layer s. One would therefore surmise that a drift velocity would arise perpendicular to the current direction. However the strip s is assumed to be superconducting, and cannot sustain static fields. The superconductor will rearrange the electrons such as to cancel this field, the net result being no effect.

IV. SUMMARY AND CONCLUSIONS

The standard hot big-bang cosmological model predicts the existence of a sea of relic neutrinos characterized by a momentum distribution of relativistic form (even if $T_\nu \ll m_\nu$) with an effective temperature $T_\nu \simeq 1.9\,°\mathrm{K}$. If the neutrino chemical potentials are small (i.e., $|\xi_i| \equiv |\mu_i|/T_\nu \ll 1$), then $\langle p\rangle \simeq 3.2T_\nu \simeq 5.2\times10^{-4}$ eV and $N_{\nu_i} \simeq N_{\bar{\nu}_i} \simeq 50/\mathrm{cm}^3$. Limits on ξ_i from nucleosynthesis yield $|\xi_i| \lesssim 1$, unless the nucleosynthesis scenario is altered in a complicated way, while there is a strong theoretical prejudice that $|\xi_i| \ll 1$.

There have recently been several proposals that these neutrinos could be detected by their coherent interaction with matter. Unfortunately, all but one of these proposals is incorrect. We have shown in general that the energy, momentum and (for a microscopically isotropic target) angular momentum transfer from a homogeneous (when time-averaged) incident neutrino beam to a microscopic or macroscopic static target vanishes to first order in the weak interaction. As this general result contradicts the claims of all but one of the proposals, we have also reexamined each of these proposals using the language of geometrical optics; in each case we have pointed out the flaw in the original suggestion.

The one exception to these negative conclusions is that torque can be exerted by the neutrino sea on a polarized target (or in principle on a target with a nonzero current density) if there is a neutrino-antineutrino asymmetry and an anisotropy in the neutrino flux. This effect was first pointed out by Stodolsky[18] some years ago, but we would like to give the relevant formulas including the effects of neutral currents and possible neutrino masses.

Specializing (2.1) to the interaction of neutrinos with a target fermion a, we have

$$\bar{\psi}\gamma^\mu\psi \rightarrow \sum_i \bar{\nu}_i\gamma^\mu(1+\gamma_5)\nu_i , \qquad j_\mu \rightarrow \frac{G_F}{\sqrt{2}}\bar{\psi}_a\gamma_\mu(g_V^{ai}+\gamma_5 g_A^{ai})\psi_a , \tag{4.1}$$

where the fields are normalized so that $\langle \psi_a^\dagger\psi_a\rangle = N_a$ is the number density of target particles. We will assume that the neutrino anisotropy is due to a velocity $\vec{v}_a$ of the target through the rest frame of the neutrino sea (i.e., the flux in the target rest frame is in the $-\vec{v}_a$ direction). We expect $v_a \sim 10^{-3}c$, which is typical of the motion of the earth through the galaxy (and which is also the speed needed to account for the dipole anisotropy observed in the microwave radiation).

From (2.13) we therefore have

$$\vec{\tau}=\frac{\Delta E}{2}(\hat{v}_e\times\langle\vec{\sigma}\rangle) , \tag{4.2}$$

where $\vec{\tau}$ is the torque on the target, $\langle\vec{\sigma}\rangle$ is the average value of $\vec{\sigma}$ in the target,

$$\Delta E=\frac{4}{\sqrt{2}}G_F v_e N_a^{\mathrm{tot}}\sum_i (N_{\nu_i}-N_{\bar{\nu}_i})g_A^{ai}K(p,m_i) \tag{4.3}$$

is the energy difference between the configuration with $\langle\vec{\sigma}\rangle$ parallel to $\vec{v}_e$ and antiparallel, $N_a^{\mathrm{tot}}=N_a V$ is the number of target particles, and $K(p,m_i)$ defined in (1.13) is $1(\frac{1}{2})$ for $p \gg m_i$ ($p \ll m_i$). For example, for a single electron one has (in $SU_2\times U_1$) $g_A^{ei}=\frac{1}{2}$ for $i=\nu_e$ and $-\frac{1}{2}$ for $i=\nu_\mu,\nu_\tau$ so that (with $m_i=0$)

$$\Delta E=\frac{4}{\sqrt{2}}G_F\frac{v_e}{2}[(N_{\nu_e}-N_{\bar{\nu}_e})-(N_{\nu_\mu}-N_{\bar{\nu}_\mu})-(N_{\nu_\tau}-N_{\bar{\nu}_\tau})] . \tag{4.4}$$

Of course, if the electron spin were polarized at right angles to $\vec{v}_e$ it would precess with an angular frequency $\Delta\phi/\Delta t = \Delta E$.

However, ΔE given in (4.4) is incredibly tiny for any reasonable asymmetries. For $N_{\nu_e} - N_{\bar{\nu}_e} \sim 50/\text{cm}^3$, $N_{\nu_i} - N_{\bar{\nu}_i} = 0$ $(i = \mu, \tau)$, for example, which corresponds to $\xi_e \simeq O(1)$, (4.4) yields

$$\Delta E = \frac{\Delta\phi}{\Delta t} \simeq 10^{-38}\ \text{eV} \sim 10^{-23}\frac{\text{rad}}{\text{sec}} . \tag{4.5}$$

The effect could of course be enormously enhanced by considering a large ferromagnet.[18] But even for $N_e^{\text{tot}} \simeq 10^{27}$, the energy difference is still only $\Delta E \sim 10^{-11}$ eV, which is many orders of magnitude smaller than the interaction energy of the magnet with any feasible magnetic field.

Our final conclusion is therefore rather negative: Barring an enormous deviation from the standard cosmological model, the detection of the relic neutrinos by this coherent interaction with matter is very unlikely.

Note added. After this manuscript was completed a letter by Cabibbo and Maiani appeared.[29] These authors studied the case of coherent scattering of massive neutrinos using a nonrelativistic formulation and arrive at conclusions similar to ours.

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