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Space-times with constant vacuum energy density and a conformal Killing vector

R. N. Henriksen
Astronomy Group, Department of Physics, Queen's University at Kingston, Kingston, Ontario K7L 3N6 Canada

A. Gordon Emslie
Department of Physics, University of Alabama in Huntsville, Huntsville, Alabama 35899

P. S. Wesson
Department of Physics, University of Alberta, Edmonton, Alberta T6G 2J1 Canada
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We find a class of solutions to the Einstein field equations with constant vacuum energy density ("cosmological constant") that has a similarity symmetry of the second kind. We show this symmetry to be a global conformal symmetry. Nontrivial analytic solutions are given and one in particular (exhibiting intrinsic symmetry) is shown to evolve to a nonempty Robertson-Walker space-time with "steady-state" metric. This is found to be due to particle production associated with the negative matter pressure that is required by the assumed symmetry. These models can describe, classically, an origin of the Universe in terms of particle production from the vacuum, driving an exponential (de Sitter) expansion. This solution is inhomogeneous and anisotropic, but tends to homogeneity and isotropy at early times and large distances, and at late times and small distances. The solution therefore corresponds to the outward motion of a spherical disturbance which distorts the local homogeneity and isotropy in an asymptotically homogeneous and isotropic universe. The limiting homogeneous and isotropic forms are discussed.

I. INTRODUCTION

The homothetic[1] or similarity symmetries[2,3] of Einstein's field equations with zero cosmological constant Λ (i.e., vacuum energy density) are known to define an elegant and useful[4–6] class of solutions. The recent intense interest[7–9] in gauge theories of elementary particles with spontaneous symmetry breaking has, however, led to a better understanding of the cosmological term in the field equations, as being proportional to the ground-state energy density of the vacuum. Although this term is temperature and therefore time dependent, one view[9] is that the variation proceeds through a series of symmetry-breaking phase changes with the vacuum energy remaining constant in each phase. We therefore consider it to be of more than academic interest to extend the similarity symmetry to the case of nonzero cosmological constant.

This is particularly true because a large constant vacuum energy density can drive a de Sitter expansion in the early Universe, which removes many of the celebrated cosmological difficulties.[9] If such regions were in fact to be localized in a Robertson-Walker background, then solutions that are locally inhomogeneous but which tend to a homogeneous limit at large scales are of considerable interest. They might also be used to consider the effect of the vacuum energy on the horizon structure of black holes embedded in a homogeneous background.[4,5] We proceed then to find a class of inhomogeneous, spherically symmetric solutions which possess "similarity" symmetry. On purely dimensional grounds, we do not expect this symmetry to have the same form when $\Lambda \neq 0$ (however small) as when $\Lambda \equiv 0$, and in fact we shall find it to be[10] a "similarity of the second kind" and a conformal symmetry of the metric.

The symmetry is achieved at the cost of constraining the equation of state to a specific form. At large scale factors for the Universe this required form is rather conventional, but at small scale factors the pressure becomes negative. This behavior is not necessarily unphysical in the particle-production phases of some symmetry-breaking particle theories[11] and it is interesting that we find the simplest inhomogeneous symmetry in the presence of a vacuum term to require it (cf. also Ref. 12).

In Sec. II we present the basic symmetry and the form of the reduced field equations. In Sec. III we give analytic solutions in special cases and in Sec. IV we discuss the general integrations numerically. Section V is a summary discussion.

II. SELF-SIMILAR SYMMETRY, CANONICAL COORDINATES, AND THE REDUCED FIELD EQUATIONS

A. General formulation

We use the metric signature $(1,-1,-1,-1)$ and the sign convention for the Ricci tensor as in Robertson and Noonan.[13] Then the Einstein field equations with an ideal fluid model of matter are $(c=G=1)$

$$G_{ab}+\Lambda g_{ab}=-8\pi[(p+\rho)u_a u_b - p g_{ab}] , \qquad (1)$$

where Λ is the cosmological constant, p and ρ represent the total pressure and energy density of the matter (quantum fields), latin indices run from 0 to 3, and the other notations are standard. It is by now usual[7] to include the cosmological term on the right-hand side of Eq. (1) in the stress-energy tensor of the true vacuum (ground state) by writing

$$\begin{aligned} &p\equiv p_m+p_v ,\\ &\rho\equiv\rho_m+\rho_v ,\\ &p_v=-\rho_v,\;\; \rho_v\equiv\frac{\Lambda}{8\pi} , \end{aligned} \qquad (2)$$

so that (1) becomes simply the standard form

$$G_{ab}=-8\pi[(p+\rho)u_a u_b - p g_{ab}] , \qquad (3)$$

but with the vacuum and the normal excited states of matter given their separate equations of state as in (2). When ρ_v is held constant the formal correspondence is exact, but should ρ_v be varying with the temperature (and hence epoch) of the Universe, the distinction between it and any true cosmological constant should be retained.[9] Here we interpret Λ freely in terms of a vacuum state, but we treat it mathematically as a strict constant.

It is convenient when seeking spherically symmetric solutions to write Eqs. (3) in the form[14,15]

$$\Gamma^2-U^2=1-\frac{2m}{R} , \qquad (4a)$$

$$m_t=-4\pi R^2 p R_t , \qquad (4b)$$

$$m_r=4\pi R^2\rho R_r , \qquad (4c)$$

with the Bianchi identities

$$\begin{aligned} &\sigma_r=-2p_r/(p+\rho) ,\\ &\omega_t=-4R_t/R-2\rho_t/(p+\rho) . \end{aligned} \qquad (5)$$

The metric is taken in the spherically symmetric form

$$ds^2=e^{\sigma(r,t)}dt^2-e^{\omega(r,t)}dr^2-R^2(r,t)d\Omega^2 , \qquad (6)$$

where $d\Omega^2\equiv\sin^2\theta\, d\phi^2+d\theta^2$ and r is a Lagrangian coordinate comoving with the matter. The generalized Lorentz factor Γ, and the invariant radial four-velocity U, are given by

$$\Gamma=e^{-\omega/2}R_r,\quad U=e^{-\sigma/2}R_t . \qquad (7)$$

These equations are complete when $p_m(\rho_m)$ is given in Eq. (2).

These equations have been studied extensively in terms of their self-similarity of the first kind (invariance under a simple "stretching" or "scaling" group, e.g., Bluman and Cole[16]) by Cahill and Taub[2] and by Bicknell and Henriksen,[4,5] when the vacuum or cosmological term is absent. The presence of Λ or ρ_v destroys this simple symmetry by introducing a fundamental scale. However, we can hope in this case to find a self-similarity of the second kind[10] corresponding to a more complex invariance group. We do this by transforming to canonical[16] coordinates t',r' in which the symmetry group *is* just the stretching group so that the appropriate self-similar variable is again $\xi=t'/r'$. This proves to be always possible provided that the equation of state $p_m(\rho_m)$ is allowed to be found as part of the solution, rather than being given. This is the same constraint found by Demianski and Grishchuk[17] in their rotating, homogeneous, cosmologies when flat spacelike sections were imposed.

The transformation to the canonical coordinates is taken in the form

$$dt=e^{\Delta\sigma(t')/2}dt',\quad dr=e^{\Delta\omega(r')/2}dr' , \qquad (8)$$

under which $\sigma'=\sigma+\Delta\sigma(t')$ and $\omega'=\omega+\Delta\omega(r')$ in (6) and (7). In these coordinates we may take the similarity variable to be $\xi=t'/r'$ and make the usual ansatz[18,4] in Eqs. (4)–(7), namely,

$$\begin{aligned} &R=r'S(\xi) ,\;\; 8\pi\rho_m=\eta(\xi)/(r')^2 ,\\ &8\pi p_m=P(\xi)/(r')^2 ,\\ &m_m\equiv m-(4\pi/3)R^3\rho_v=(r'/2)M(\xi) ,\\ &\sigma=\sigma(\xi),\;\; \omega=\omega(\xi) . \end{aligned} \qquad (9)$$

This gives (4b), (4c), and Eqs. (5) as (a prime denotes $d/d\xi$)

$$M'=-PS^2S' , \qquad (10)$$

$$M-\xi M'=\eta S^2(S-\xi S') , \qquad (11)$$

$$\sigma'=-\frac{2}{\xi^2(P+\eta)}\frac{d}{d\xi}(\xi^2P) , \qquad (12)$$

$$\omega' = -4S'/S - 2\eta'/(P+\eta) \; ; \qquad (13)$$

while (4a) and (7) together give

$$1 - \frac{M}{S} - \frac{8\pi\rho_v}{3}(r')^2 S^2 = e^{-[\omega(\xi)+\Delta\omega(r')]}(S-\xi S')^2 - e^{-[\sigma(\xi)+\Delta\sigma(t')]}(S')^2 \; , \qquad (14)$$

and $\Delta\omega$, $\Delta\sigma$ must be chosen so that the assumed symmetry holds in these coordinates. We find the symmetry to have a quite different character depending on whether $\rho_v \gtrless 0$ and we proceed now to discuss these cases separately.

B. Positive vacuum energy density

When $8\pi\rho_v = \Lambda > 0$, an inspection of the signs and coefficients of the various terms in (14) shows that the only way to maintain the assumed symmetry is to set

$$e^{-\Delta\sigma} = \frac{\Lambda(t')^2}{3}, \quad \Delta\omega = 0 \; , \qquad (15)$$

whence (14) is uniquely separated into the two equations

$$1 - \frac{M}{S} = e^{-\omega}(S - \xi S')^2 \equiv \Gamma^2(\xi) \qquad (16)$$

and

$$S^2 = e^{-\sigma}(\xi S')^2 \; , \qquad (17a)$$

which may also be written as

$$U^2 = \frac{\Delta R^2}{3} \; , \qquad (17b)$$

which is a Hubble law.

It is noteworthy that this separation of Eq. (4a) is closely analogous to that achieved by the spatially flat de Sitter solution. In that case, $\Gamma = 1$ and $U^2 = \Lambda R^2/3$, just as above except that the mass-energy of the matter is zero for the de Sitter case.

In these canonical coordinates, Eqs. (10)–(13) plus (16) and (17a) are complete for M, S, P, η, σ, ω as functions of $\xi = t'/r'$. We can now discover the similarity variable in arbitrary comoving coordinates t,r by transforming back to these coordinates using Eq. (8). This gives $r' = r$, and

$$t' = \text{const} \times e^{\sqrt{\Lambda/3}t} \; ,$$

so that the similarity variable is conveniently taken as

$$\xi = \sqrt{3/\Lambda}\, \frac{e^{\sqrt{\Lambda/3}t}}{r} \; , \qquad (18)$$

and we recall that $8\pi\rho_v = \Lambda$ when the vacuum energy density is constant. The generator of the corresponding Lie group symmetry may be taken as

$$\xi_+^a = e^{\sqrt{\Lambda/3}t}(\sqrt{3/\Lambda}, r, 0, 0) \; , \qquad (19)$$

for which the Lie derivative of the metric is

$$\mathcal{L}_{\xi_+} g_{ab} = 2e^{\sqrt{\Lambda/3}t} g_{ab} \qquad (20)$$

and the symmetry is thereby seen to be conformal. This constitutes an extention to inhomogeneous space-times of the de Sitter space conformal symmetry, namely,

$$\mathcal{L}_{\xi_{dS}} g_{ab} = 2e^{\sqrt{\Lambda/3}t} g_{ab} \; ,$$

$$\xi_{dS}^a = e^{\sqrt{\Lambda/3}t}(\sqrt{3/\Lambda}, 0, 0, 0) \; ,$$

when the equation of state of the "matter" is determined from the equations above.

Thus, in summary, we have found that the Einstein field equations with spherical symmetry, a fluid description of matter, and a nonzero cosmological constant (equivalently, a constant vacuum energy density) possess solutions having the conformal symmetry (19) and (20). This is so provided that Eqs. (10)–(13) and (16) and (17a) are satisfied and ξ is given by Eq. (18) in arbitrary comoving coordinates. The physical variables are found from Eqs. (7) and (9) with $r' = r$. We will explore these solutions further below, but we may note that in general from (16) that

$$M < S \; , \qquad (21a)$$

or dimensionally,

$$\frac{2Gm_m}{Rc^2} < 1 \qquad (21b)$$

in such solutions.

C. Negative vacuum energy density

When $8\pi\rho_v = \Lambda < 0$ inspection of Eq. (14) again reveals a unique separation if

$$\Delta\sigma = 0, \quad e^{-\Delta\omega} = \frac{|\Lambda|}{3}(r')^2 \; , \qquad (22)$$

which leads to (14) becoming separated as

$$e^{-\omega}(S - \xi S')^2 = S^2 \; , \qquad (23a)$$

or equivalently

$$\Gamma^2 = \frac{|\Lambda| R^2}{3} \qquad (23b)$$

and

$$e^{-\sigma} S'^2 = \frac{M}{S} - 1 \qquad (24a)$$

or its equivalent

$$U^2=M/S-1\ . \tag{24b}$$

This separation is quite different from that in Sec. II A, as here the "Hubble" variation is in space rather than in time and by Eq. (26b) it represents a complementary solution with

$$M/S>1 \tag{25a}$$

or

$$\frac{2Gm_m}{Rc^2}>1\ . \tag{25b}$$

The relation between this solution and that of Sec. II A is then much like that between solutions inside and outside an "apparent horizon."

We again return to the original coordinates from the canonical coordinates using the transformation (8) which in this case yields $t'=t$ and

$$r'=\text{const}\times e^{\sqrt{|\Lambda|/3}r}\ .$$

We therefore find the similarity variable in the original coordinates to be

$$\xi=t\sqrt{|\Lambda|/3}e^{-\sqrt{|\Lambda|/3}r}\ . \tag{26}$$

The generator of the corresponding Lie group is say

$$\xi_-^a=(t,\sqrt{3/|\Lambda|},0,0)e^{\sqrt{|\Lambda|/3}r}\ ,$$

but this time the conformal symmetry $\mathscr{L}g_{ab}=2e^{\sqrt{\Lambda/3}r}g_{ab}$ holds only in the subspace with $d\theta=d\phi=0$. This is then a partially spatially conformal space-time in the sense of the partially self-similar space-times of Tomita.[19]

We note in summary that Eqs. (10)–(13), (23a), and (24a) now constitute a complete set for $P(\xi)$, $\eta(\xi)$, $M(\xi)$, $S(\xi)$, $\sigma(\xi)$, and $\omega(\xi)$ where ξ is given by Eq. (26), and $\Lambda=8\pi\rho_v<0$. Equations (7) and (9) continue to hold in the arbitrary coordinates r,t.

D. Reduced field equations

We proceed to find the equations to be solved in their simplest form in the cases where $\Lambda>0$ and $\Lambda<0$, respectively.

(i) $\Lambda>0$. Equations (10) and (11) are easily rearranged to give

$$\xi S'=\frac{\eta S^3-M}{S^2(P+\eta)} \tag{27}$$

and

$$\xi M'=-\frac{P}{P+\eta}(\eta S^3-M)\ . \tag{28}$$

Moreover, Eqs. (16) and (17a) may now be used with (27) and (28) to express the metric coefficients as

$$e^{\omega}=\frac{(M+PS^3)^2}{S^3(S-M)(P+\eta)^2} \tag{29}$$

and

$$e^{\sigma}=\left[\frac{\eta S^3-M}{S^3(P+\eta)}\right]^2\ . \tag{30}$$

Substituting these last equations into the Bianchi identities (12) and (13) gives, after a tedious calculation,

$$\frac{d\eta}{dS}=-\frac{3(P+\eta)}{S} \tag{31a}$$

or

$$P=-\eta-(S/3)d\eta/dS \tag{31b}$$

and

$$\frac{dP}{dS}=\frac{(P+\eta)}{2S(S-M)(\eta S^3-M)}\times[(M+PS^3)^2-4PS^3(S-M)]\ , \tag{32}$$

where we have used (27) to change the independent variable from ξ to S. Equation (31b) is the "equation of state" for this solution. The ratio of Eqs. (28) and (27) may be written as

$$P=-(1/S^2)dM/dS\ , \tag{33}$$

and Eqs. (31), (32), and (33) are complete for $P(S)$, $\eta(S)$, and $M(S)$. Equation (27) must be solved ultimately for $S(\xi)$ to complete the solution in terms of arbitrary coordinates.

In fact Eqs. (31b) and (33) are easily combined to give the integral

$$M=\eta\frac{S^3}{3}+\Delta\ , \tag{34}$$

where Δ is an arbitrary constant. This proves useful in finding our analytic solution below and serves as a check on the numerical integration of Eqs. (27), (31), (32), and (33) (Sec. IV). If we rewrite this integral using the ansatz (9) we find

$$m_m-\frac{4\pi}{3}\rho_m R^3=\frac{r\Delta}{2}\ , \tag{35}$$

so that with $\Delta=0$ we can expect the density of matter and spatial sections of the manifold to be homogeneous although the spatial sections may not be flat, as we are not dealing with dust. We shall see below that nevertheless the resulting space-time is not homogeneous.

(ii) $\Lambda<0$. For this case Eqs. (27) and (28) continue to apply and can be used with (23a) and (24a) to write

$$e^{\omega}=\left[\frac{M+PS^3}{(P+\eta)S^3}\right]^2 \tag{36}$$

and

$$e^{\sigma}=\frac{S}{(M-S)}\left[\frac{\eta S^3-M}{\xi S^2(P+\eta)}\right]^2 . \tag{37}$$

These in turn combine with the Bianchi identities to yield

$$\frac{d\eta}{dS}=\frac{P+\eta}{2S(S-M)}(\eta S^3+3M-4S) \tag{38}$$

and

$$\frac{dP}{dS}=\frac{P+\eta}{S}\left[\frac{M-PS^3}{\eta S^3-M}\right] . \tag{39}$$

These last two equations in combination with (27) and (33) form a complete set for numerical analysis. We have, however, been unable to discover an integral analogous to (34) for this case. Although we include the $\Lambda<0$ solutions formally here for completeness, we will not study it much in this paper as it seems the unphysical case (see, e.g., Sec. II E).

E. Singular solution with $M=S$

We have seen above that the $\Lambda>0$ solution requires $M<S$ and conversely for the $\Lambda<0$ solution. In this section we ask whether a special solution exists with $M=S$ everywhere. When $\Lambda>0$, Eq. (10) requires $P=-1/S^2$, (16) requires $S=\xi$ (an arbitrary multiplicative constant can be absorbed in r), (17) gives $e^{\sigma}=1$, (11) is an identity, (12) is an identity, and only (13) need be solved as

$$\omega'=-(4/\xi)-2\eta'/(P+\eta) , \tag{40}$$

for which an equation of state for the matter must be given $[P(\eta)]$. Such an exercise is somewhat futile in the absence of a definite field-theory model,[11] but we note that a homogeneous model is obtained if we use $\eta=P/a_s{}^2$ ($a_s{}^2$ may be negative). With this equation of state, the solution of (40) is $e^{\omega}=\xi^{\gamma}$, where $\gamma=-4a_s{}^2/(1+a_s{}^2)$ (and where we have absorbed another constant into t). The metric is thus

$$ds^2=dt^2-\xi^{\gamma}dr^2-r^2\xi^2d\Omega^2 , \tag{41}$$

which, with a redefinition of the radial coordinate $(dr_*{}^2=r^{-\gamma}dr^2)$, reduces to a Kantowski-Sachs metric of closed form.[20] In the special case $a_s{}^2=-\frac{1}{3}$, the metric assumes the isotropic form

$$ds^2=dt^2-\xi^2(dr^2+r^2d\Omega^2) . \tag{42}$$

The intrinsic geometry of the (flat) spacelike hypersurfaces is interesting in that the radial proper separation of particles is small at large r (small ξ) and increases as r decreases, while proper distances on the two-spheres are independent of r.

When $\Lambda<0$ and $M=S$, (24) shows immediately that $S=\text{const}$, while (23) shows $e^{\omega}=1$. Equation (11) gives $\eta=1/S^2=\text{const}$, Eq. (10) is an identity, and Eq. (13) is an identity. We have then only to solve (12) when an expression for P is given. With P also constant, we find $e^{\sigma}=\text{const}\times\xi^{-4\eta/(P+\eta)}$ where ξ is given by Eq. (26). Thus we have (making an appropriate choice of the t variable) a static space-time wherein there is a blue-shift in all directions from $r=0$. Such solutions do not seem to have much physical significance, and so subsequently we will concentrate on the $\Lambda>0$ solutions.

III. ANALYTIC SOLUTIONS FOR $\Lambda=8\pi\rho_v>0$

A. $\Delta=0$

We find first a solution that we expect to be homogeneous in the energy density. We define

$$y\equiv\eta S^2, \quad \alpha\equiv P/\eta , \tag{43}$$

so that Eqs. (31b) and (33) may be written as [using (34)]

$$\alpha=-\frac{1}{3}\left[1+\frac{d\ln y}{d\ln S}\right] , \tag{44}$$

$$P=y\alpha/S^2 . \tag{45}$$

A direct substitution into Eq. (32) and changing the independent variable from S to y gives

$$\frac{d\alpha}{dy}=-\frac{1}{4}\frac{(\alpha+1)(3\alpha+1)}{(3-y)} ,$$

which integrates to

$$\alpha=\frac{C(1-y/3)^{1/2}-\frac{1}{3}}{1-C(1-y/3)^{1/2}} \tag{46}$$

on recalling that $M<S$ requires $y<3$ in the present case, and writing C as an arbitrary constant of integration. From Eq. (27)

$$\frac{d\ln\xi}{d\ln S}=\frac{3}{2}(1+\alpha) ,$$

which can be combined with (44) to give

$$\frac{d\ln y}{d\ln\xi}=-2C(1-y/3)^{1/2}\ ,$$

or on quadrature

$$\frac{y}{3}=\mathrm{sech}^2(C\ln\xi+D)\ , \tag{47}$$

where D is another constant of integration. By (18) this corresponds to an arbitrary scaling $r \rightarrow ar$ and thus may be set equal to zero with no loss of generality. With (47) there follows from (46)

$$\alpha=\frac{C\tanh(C\ln\xi)-\frac{1}{3}}{1-C\tanh(C\ln\xi)}\ , \tag{48}$$

and hence

$$\frac{d\ln S}{d\ln\xi}=1-C\tanh(C\ln\xi)\ ,$$

so that

$$S=K\xi\,\mathrm{sech}(C\ln\xi)\ , \tag{49}$$

which completes the solution (K is an arbitrary positive constant).

The metric coefficients are given by (30) and (29) as

$$e^{\sigma}=[1-C\tanh(C\ln\xi)]^2 \tag{50}$$

and

$$e^{\omega}=C^2S^2\ . \tag{51}$$

We see from (50) that there is no apparent horizon in the solution if $|C|<1$. The sign of C may be taken positive as the solution is invariant under $C\rightarrow -C$.

Equations (34) and (47)–(51) give the complete general solution for the case $\Lambda>0$, $\Delta=0$ (these results are summarized in Sec. V). The spatial sections of the space-time have a line element given by

$$dl^2=\frac{3K^2}{\Lambda}\exp(2\sqrt{\Lambda/3}t)(d\chi^2+\cos^2\chi\,d\Omega^2)\ , \tag{52}$$

where $\cos\chi=\mathrm{sech}(C\ln\xi)$, and are thus homogeneous closed spaces of constant curvature. This constancy of three-space curvature follows naturally from the assumed symmetry, i.e., it is due to the absence of any further dimensional scales in the problem other than $\Lambda^{-1/2}$ and c.

Asymptotically, as $\xi\rightarrow 1$, one finds easily that

$$S\simeq K\xi,\ \ \eta\simeq 3/S^2\ ,$$
$$P\simeq -1/S^2,\ \ M\simeq S \tag{53}$$

and

$$ds^2\simeq dt^2-\frac{3K^2}{\Lambda}e^{2\sqrt{\Lambda/3}t}\left[C^2\frac{dr^2}{r^2}+d\Omega^2\right]\ . \tag{54}$$

Here $\simeq$ means "asymptotically equal." We recall that our solutions are restricted to the region $M<S$ [Eq. (21a)], so that $\xi=1$ is a natural starting hypersurface for the solution. The metric (54) is again a Kantowski-Sachs[20] space-time (cf Sec. II E). Indeed, setting

$$\frac{dr_*}{r_*}=C\frac{dr}{r}$$

renders it in the "isotropic" form (42), and a scaling in time removes K. Our solution can then be matched onto the $M=S$ solution of Sec. II E at $\xi=1$ (with $a_s{}^2=-\frac{1}{3}$) and continued to $S=0$ ($\xi=0$) to obatin a global solution in $0<\xi<\infty$. We show below that the solution tends to de Sitter space as $\xi\rightarrow\infty$. Thus, the patched global solution described a thin region of inhomogeneity and anisotropy (a "bubble") near $\xi=1$, which propagates into the vacuum-dominated $M=S$ region and leaves behind de Sitter space-time. In this sense it may have application to a "phase changing"[9] instability of the vacuum. We also observe that the numerical calculations of Sec. IV indicate that there may be a family of solutions with this behavior in the vicinity of the analytic solution.

The analytic solution in fact only touches the point $M=S$, as may be seen by observing M/S [see, e.g., (60)] to have a maximum at $\xi=1$, equal to unity. It actually passes this critical or bifurcation point with $dP/dS=\lim(0/0)$ and $e^{\omega}=\lim(0/0)$ [see (29) and (32)]. Hence it may be continued to $\xi=0$ to give another global solution. In that limit the solution is given also by (55), (56), and (57) provided the formal substitution $C\rightarrow -C$ (new $C>0$) is made everywhere. This solution therefore describes a (nonempty) spherical wave of inhomogeneity near $\xi=1$, which propagates into a (nonempty) de Sitter universe ($\xi<1$) and leaves behind a (nonempty) de Sitter universe ($\xi>1$).

As $\xi\rightarrow\infty$ we have

$$S\rightarrow 2K\xi^{1-C},\ \ M\rightarrow 8K\xi^{1-3C}\ ,$$
$$\eta\rightarrow 3/K^2\xi^2,\ \ P\rightarrow\frac{3C-1}{3(1-C)}\eta\ . \tag{55}$$

The metric becomes

$$ds^2=(1-C)^2dt^2-4K^2\xi^{2(1-C)}(C^2dr^2+r^2d\Omega^2)\ . \tag{56}$$

For $C<1$ the scale factor S approaches ∞ as $\xi\rightarrow\infty$ corresponding to an open universe. With $\bar{t}=(1-C)t$ and $\bar{r}=2K(3/\Lambda)^{(1-C)/2}r^C$, the metric may be written

$$ds^2=d\bar{t}^2-e^{2\sqrt{\Lambda/3}t}(d\bar{r}^2+\bar{r}^2d\Omega^2)\ , \tag{57}$$

which corresponds to the Robertson-Walker de Sitter metric. It is remarkable that this is achieved here for a nonempty space-time, which is therefore a self-consistent model of a steady-state cosmology.

If $C>1$, $S\to 0$ as $\xi\to\infty$, corresponding to a closed universe. The maximum value of S is achieved when

$$\xi=\exp\left[\frac{1}{C}\tanh^{-1}\left[\frac{1}{C}\right]\right]$$

and is

$$S=K\left[1-\frac{1}{C^2}\right]^{1/2}\exp\left[\frac{1}{C}\tanh^{-1}\left[\frac{1}{C}\right]\right].$$

If $C=1$, S approaches the limiting value $2K$ as $\xi\to\infty$.

The (dimensional) matter density is given exactly by (9), (43), (47), and (49) as

$$\rho_m=\frac{\Lambda}{8\pi K^2}e^{-2\sqrt{\Lambda/3}t} \tag{58}$$

so that the solution is matter homogeneous. By (9), (43), and (49), p_m is not in general homogeneous; however, in the limit $\xi\to\infty$, α [Eq. (48)] tends to the constant value $(3C-1)/3(1-C)$ so that the material pressure is also homogeneous in this limit. As remarked above if the solution is continued to $\xi=0$ with or without patching to the $M=S$ solution, then it is also homogeneous in this limit.

B. $\Delta\neq 0$

The only analytic solution of the type $P=a_s{}^2\eta$ which is independent of the previous discussions is with $a_s{}^2=-\frac{1}{6}$. Equation (31a) gives $\eta=\eta_1 S^{-5/2}$ (η_1 a constant), (34) gives $M=(\eta_1/3)S^{1/2}+\Delta$, and (33) is satisfied with $P=-(\eta_1/6)S^{-5/2}$. Substitution into (32) yields a compatibility condition which determines $\Delta=-\eta_1{}^2/36$. Finally we may solve Eq. (27) to obtain

$$S^{1/2}=-\eta_1/24+\text{const}\times\xi^{2/5}\ . \tag{59}$$

The metric coefficients are easily found from (29) and (30). This solution is not homogeneous and has (for $\eta_1>0$) a singularity at a finite value of ξ, so we shall not discuss it further here.

IV. NUMERICAL INTEGRATIONS

We have integrated Eqs. (27), (31a), (32), and (33) numerically to investigate whether or not the analytic solution of Sec. III A is typical. We calculated the analytic curve numerically by starting at $S_0=1$, $C\ln\xi_0=1$, $\eta_0=3\,\text{sech}^2(1)$, $M_0=\text{sech}^2(1)$, and $P_0=[\tanh(1)-\frac{1}{3}]/[1-\tanh(1)]$, and integrating forwards and backwards in S. The choice for P_0 requires $C\simeq 0.9522$, and is serendipitously in the interesting range $(\frac{1}{3}\leq C\leq 1)$. This curve is labeled M_0 in Fig. 1, which is the P-S plane of the solution space. The curve continues smoothly to touch at $M=S$ at small S ($\ln\xi=0$) as expected. As S increases, P rises to a positive maximum, declines rapidly, and then asymptotically as ξ^{-2}.

The other curves in Fig. 1 were generated by varying M about its analytic solution M_0 as indicated, while holding other quantities at $S=1$ fixed. The curves generated by taking $M<M_0$ led to negative M at large S and so we discarded them for our purposes. For $M_0\leq M\leq M_0+\sim 0.08$, a dense stable family of solutions was found, of which the analytic curve is clearly a typical member (they all terminate at $M=S$ at small S and display similar behavior at large S). When M was increased to $M_0+0.08$ and beyond, the singularity formed in the $P-S$ and $\xi-S$ planes when $\eta S^3-M=0$ was encountered.

V. SUMMARY AND DISCUSSION

We have succeeded in this paper in finding a class of nonempty, spherically symmetric (although the hypersurfaces are anisotropic), inhomogeneous solutions to the Einstein field equations with a finite "cosmological constant" (constant vacuum energy density). Those with a positive constant possess a conformal symmetry also found in the de Sitter universe, but the group generators are not purely timelike for these space-times as they are in the de Sitter case.

A family of analytic solutions has been found for $\Lambda>0$ and a homogeneous matter energy density. The asymptotically open members of this family describe a region of inhomogeneity near $\xi=1$ propagating into the nonempty spatially flat, Robertson-Walker region at $\xi<1$ and leaving behind another at $\xi>1$. An open solution in $\xi>1$ may also be patched to the $M=S$ solution for $\xi<1$. This may describe the transformation of a vacuum-dominated universe into a (nonempty) de Sitter universe. This exact classical solution, which has a conformal Killing vector and allows negative pressure, imitates remarkably the quantum, conformal field-theory description of the origin of the Universe given by Brout, Englert, and Gunzig[11] and by Brout, Englert, and Spindel.[21] One can hope also that this solution may eventually shed light on the degree of homogeneity attainable after a phase change in the vacuum.

Because of its potential importance, we summa-

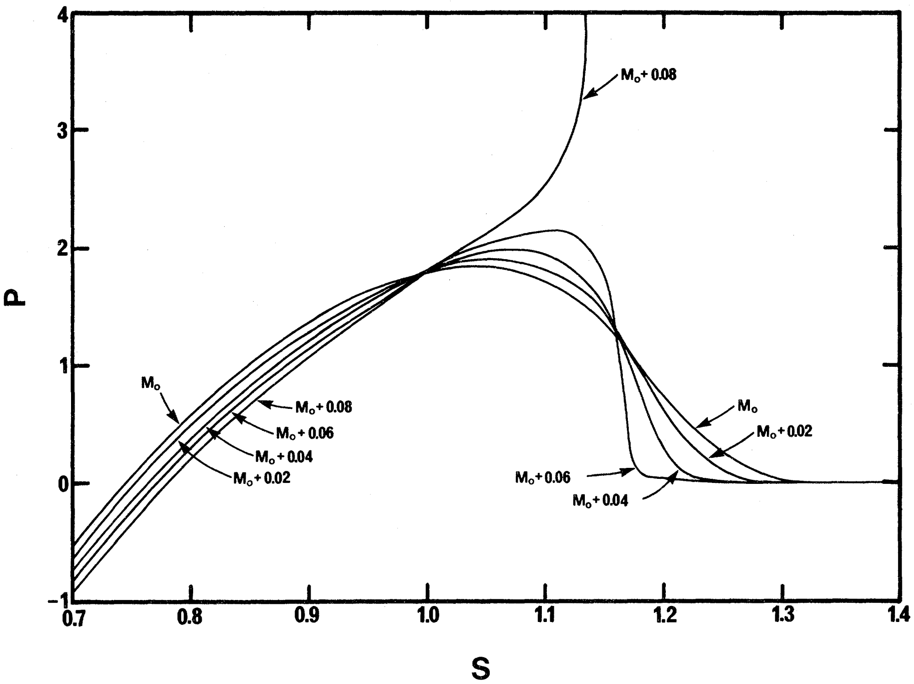

FIG. 1. Numerically computed solutions of the reduced field equations in the P-S plane. The curve labeled M_0 is the analytic solution of Sec. III [$\Delta=0$ in Eq. (34)]. Other curves are labeled by the value of their mass excess over the analytic solution. The curves for mass excesses $\gtrsim 0.08$ exhibit a singularity at some value of S, at which $dP/dS \to \infty$ as $\eta S^3 - M \to 0$. Solutions with mass defects relative to the analytic solution encounter negative M at large S.

rize the results of Sec. III A here. The solution is explicitly (in terms of the conformally invariant variables)

$$\begin{aligned}
&S = K\xi\,\mathrm{sech}(C\ln\xi)\ ,\\
&\eta = 3/(K^2\xi^2)\ ,\\
&P = \frac{3C\tanh(C\ln\xi)-1}{K^2\xi^2[1-C\tanh(C\ln\xi)]}\ ,\\
&M = K\xi\,\mathrm{sech}^3(C\ln\xi)\ ,\\
&e^{\sigma/2} = [1-C\tanh(C\ln\xi)]\ ,
\end{aligned} \tag{60}$$

and

$$e^{\omega/2} = CK\xi\,\mathrm{sech}(C\ln\xi)\ .$$

The arbitrary constant K may, if desired, be absorbed into ξ, and the significance of C has been discussed in Sec. III.

It may seem strange that the above behavior is obtained with no apparent coupling between ρ_m and ρ_v (ρ_v is held constant throughout). However, there is a kind of coupling through the matter equation of state that is required as part of the assumed symmetry [Eq. (31b)]. Thus, for a solution with a homogeneous matter density, (4c) and (4b) may be combined to give the first law as

$$\frac{d}{dt}\left[\frac{4\pi}{3}R^3\rho_m\right] = -4\pi R^2 R_t p_m\ , \tag{61}$$

from which we see that $p_m < 0$ is synonymous with a nonadiabatic increase in the matter energy in the Universe. Moreover, Fig. 1 shows that this epoch terminates ($p_m \geq 0$) in a relatively rapid transition, just as in Ref. 21.

Similar behavior has been found by Henriksen[12] for a vacuum energy $\rho_v \propto t^{-2}$ in a Robertson-Walker universe, the solution of which is also based on a homothetic symmetry. Of course in the present solution, a detailed numerical correspondence would require an enormous Λ (by the standards of the present epoch, say $\rho_v \sim 10^{89}$ erg cm^{-3}) which certainly cannot be extrapolated to late epochs. Thus, as we suggested in the Introduction to this paper, the constant ρ_v phase must be very transitory, lasting perhaps only until the epoch of positive p_m, after which it must undergo one or more phase transitions. Our patched solutions might be useful in describing "bubbles" where the new vacuum is forming in the midst of the false vacuum.[9] Such fluctuations associated with phase transitions can be relevant to theories of galaxy formation.

We have also given for completeness the class of solutions found when $\rho_v < 0$, although what, if any, physical significance they have is not clear. Formally, they have only a partial conformal symmetry which may suggest an interesting mathematical class of solutions to investigate (see, e.g., Tomita[19]).

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