K^0-$\bar{K}^0$ mixing in the six-quark model

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Using the leading-logarithm approximation, strong-interaction corrections to K^0-$\bar{K}^0$ mixing in the six-quark model are computed in quantum chromodynamics. The full calculation involving the mixing of eight operators at some stages is done, as well as an approximate, much simpler calculation. Numerically, the exact and approximate results agree to high accuracy and both show that the corrections to the real and imaginary parts can be large. How to obtain the free-quark limit of these and other results is shown explicitly.

I. INTRODUCTION

The K^0-$\bar{K}^0$ mass matrix has played an important role in particle physics over the past decade. The small value of the real part of the off-diagonal elements found an explanation in the Glashow-Iliopoulos-Maiani mechanism[1] which invoked a fourth, charmed quark. Later calculations[2] of the magnitude of these mass-matrix elements led to a quantitative estimate of the charmed-quark mass. While these calculations were originally done without strong-interaction corrections, with the development of quantum choromodynamics (QCD) the short-distance effects due to strong interactions were soon computed[3,4] and found to change the answer rather little.

With the standard phase conventions an imaginary part of the off-diagonal mass-matrix elements is an expression of *CP* noninvariance and leads to the neutral-kaon eigenstates K_S^0 and K_L^0 not being *CP* eigenstates. With four quark flavors there is no imaginary part,[5] but in a six-quark model a phase in the heavy-quark couplings to the weak vector bosons leads to *CP* violation and an imaginary part in the mass matrix. The phenomenology of *CP* violation in the six-quark model has been discussed[6] without account of QCD corrections and found to be consistent with experiment and in particular with its observation in the K_S^0-K_L^0 system.

In this paper we calculate the QCD corrections to the K_S^0-K_L^0 mass matrix in the six-quark model. A brief account of this work was reported earlier.[7] Here we give a more complete treatment, including the mixing of eight operators at some stages of the calculation.

In the next section we give the details of how the effective $\Delta S=2$ weak Hamiltonian that contributes to the K^0-$\bar{K}^0$ mass matrix is calculated. Successively the *W* boson, *t* quark, *b* quark, and *c* quark are treated as heavy fields and removed from appearing in the theory. At each stage we get an effective theory with less fields and calculate coefficients of operators in the effective Hamiltonian. These are related to their values in the theory at the previous step by renormalization-group equations which we solve in the leading-logarithm approximation. After giving the full solution with all mixing included, we also give an analytic result based on dropping the mixing of six operators with two others.

In Sec. III we give numerical results. Careful attention is paid to how to match up the running coupling in effective theories with different numbers of quarks and how the free-quark results emerge as a limit. This limit is explicitly carried out in Appendix A. Numerical results for the strong-interaction corrections to the $\Delta S=2$ Hamiltonian are also given in several cases. The exact and approximate results are numerically close and indicate large strong-interaction corrections in some cases.

II. QCD CORRECTIONS TO THE K^0-$\bar{K}^0$ MASS MATRIX

We work within the standard model[8] where the gauge group of electroweak interactions is $SU(2)\times U(1)$ and six quarks, u,c,t with charge $\frac{2}{3}$ and d,s,b with charge $-\frac{1}{3}$, are assigned to left-handed doublets and right-handed singlets:

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L,$$

$$(u)_R,\ (d)_R,\ (c)_R,\ (s)_R,\ (t)_R,\ (b)_R\ .$$

The choice of quark fields is such that[9]

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix}_L = \begin{bmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3+c_2s_3e^{i\delta} & c_1s_2s_3-c_2c_3e^{i\delta} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}_L , \tag{1}$$

where $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $i \in (1,2,3)$. Equation (1) defines three real Cabibbo-type mixing angles θ_i and the CP-violating phase δ.

The portion (with $\Delta S = 2$) of the effective weak Hamiltonian density which contributes to the matrix element between a K^0 and $\bar{K}^0$ may be written uniquely as

$$\mathscr{H}_{\text{eff}}^{|\Delta S|=2} = s_1{}^2c_2{}^2(c_1c_2c_3 - s_2s_3e^{-i\delta})^2\mathscr{H}_1 + s_1{}^2s_2{}^2(c_1c_2c_3 + c_2s_3e^{-i\delta})^2\mathscr{H}_2$$
$$+2s_1{}^2s_2c_2(c_1c_2c_3 - s_2s_3e^{-i\delta})(c_1s_2c_3 + c_2s_3e^{-i\delta})\mathscr{H}_3 + \text{H.c.} \tag{2}$$

The components $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ of the complete Hamiltonian have relatively complicated expressions[10] in terms of time-ordered products of four weak charged currents contracted with W-boson fields, corresponding in the free-quark model to forming a "box diagram" with virtual quarks and W bosons in the loop.

In the free-quark model, successively treating the W boson, t quarks, and c quarks as heavy results in the following expressions:

$$\mathscr{H}_1 = -\frac{G_F{}^2m_c{}^2}{16\pi^2}[\bar{d}_\alpha\gamma_\mu(1-\gamma_5)s_\alpha][\bar{d}_\beta\gamma^\mu(1-\gamma_5)s_\beta] , \tag{3a}$$

$$\mathscr{H}_2 = -\frac{G_F{}^2m_t{}^2}{16\pi^2}[\bar{d}_\alpha\gamma_\mu(1-\gamma_5)s_\alpha][\bar{d}_\beta\gamma^\mu(1-\gamma_5)s_\beta] , \tag{3b}$$

$$\mathscr{H}_3 = -\frac{G_F{}^2m_c{}^2\ln(m_t{}^2/m_c{}^2)}{16\pi^2}[\bar{d}_\alpha\gamma_\mu(1-\gamma_5)s_\alpha][\bar{d}_\beta\gamma^\mu(1-\gamma_5)s_\beta] , \tag{3c}$$

where G_F is the Fermi constant, and m_c and m_t are the c- and t-quark masses. The color indices α and β are summed when repeated. Terms which are higher order in $m_t{}^2/m_W{}^2$, $m_c{}^2/m_t{}^2$, etc., have been dropped.

In the presence of strong interactions, as described by QCD, the results in Eqs. (3) will be modified. We shall derive in leading-logarithm approximation the form of the effective Hamiltonian when the W bosons and t, b, and c quarks are treated as heavy and their fields removed from explicitly appearing in the theory.

The first step is to treat the W boson as heavy and remove it from explicitly appearing in the Hamiltonian. This is done in a manner similar to the analogous step in the derivation of the effective Hamiltonian for $\Delta S = 1$ weak nonleptonic decays.[11]

For this purpose it is convenient to separate the Hamiltonian into pieces $\mathscr{H}^{(\pm\pm)}$ that will not mix under renormalization by taking the four currents which were joined by W-boson propagators, and writing pairs of them as half the sum of color-symmetrized (superscript $+$) and -antisymmetrized (superscript $-$) pieces. Then

$$\mathscr{H} = \mathscr{H}^{(++)} + \mathscr{H}^{(+-)} + \mathscr{H}^{(-+)} + \mathscr{H}^{(--)} .$$

In the leading-logarithm approximation each of the Hamiltonians $\mathscr{H}_j$ with the W boson removed can be written as

$$\mathscr{H}_j = \left[\frac{\alpha_s(M_W{}^2)}{\alpha_s(\mu^2)}\right]^{2a^{(+)}}\mathscr{H}_j^{(++)} + \left[\frac{\alpha_s(M_W{}^2)}{\alpha_s(\mu^2)}\right]^{a^{(+)}+a^{(-)}}\mathscr{H}_j^{(+-)} + \left[\frac{\alpha_s(M_W{}^2)}{\alpha_s(\mu)}\right]^{a^{(-)}+a^{(+)}}\mathscr{H}_j^{(-+)}$$
$$+ \left[\frac{\alpha_s(M_W{}^2)}{\alpha_s(\mu^2)}\right]^{2a^{(-)}}\mathscr{H}_j^{(--)} , \tag{4}$$

where $a^{(+)} = \frac{6}{21}$ and $a^{(-)} = -\frac{12}{21}$, μ is the renormalization-point mass, $\alpha_s(M^2)$ the running fine-structure constant in a theory with six quarks, and the $\mathscr{H}_j^{(\pm\pm)}$ have the explicit form[10,12]

$$\begin{aligned}\mathscr{H}_1^{(\pm\pm)}(0)\equiv i\int d^4x[&T\{O_c^{(\pm)}(x)O_c^{(\pm)}(0)\}\\ &-2T\{\{[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)u_\alpha(x)][\bar{c}_\beta(x)\gamma^\mu(1-\gamma_5)d_\beta(x)]\\ &\pm[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)d_\alpha(x)][\bar{c}_\beta(x)\gamma^\mu(1-\gamma_5)u_\beta(x)]\}\\ &\times\{[\bar{s}_\lambda(0)\gamma^\nu(1-\gamma_5)c_\lambda(0)][\bar{u}_\delta(0)\gamma_\nu(1-\gamma_5)d_\delta(0)]\\ &\pm[\bar{s}_\lambda(0)\gamma^\nu(1-\gamma_5)d_\lambda(0)][\bar{u}_\delta(0)\gamma_\nu(1-\gamma_5)c_\delta(0)]\}\}]\ ,\end{aligned}\tag{5a}$$

$$\begin{aligned}\mathscr{H}_2^{(\pm\pm)}(0)\equiv i\int d^4x[&T\{O_t^{(\pm)}(x)O_t^{(\pm)}(0)\}\\ &-2T\{\{[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)u_\alpha(x)][\bar{t}_\beta(x)\gamma^\mu(1-\gamma_5)d_\beta(x)]\\ &\pm[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)d_\alpha(x)][\bar{t}_\beta(x)\gamma^\mu(1-\gamma_5)u_\beta(x)]\}\\ &\times\{[\bar{s}_\lambda(0)\gamma_\nu(1-\gamma_5)t_\lambda(0)][\bar{u}_\delta(0)\gamma^\nu(1-\gamma_5)d_\delta(0)]\\ &\pm[\bar{s}_\lambda(0)\gamma_\nu(1-\gamma_5)d_\lambda(0)][\bar{u}_\delta(0)\gamma^\nu(1-\gamma_5)t_\delta(0)]\}\}]\ ,\end{aligned}\tag{5b}$$

and

$$\begin{aligned}\mathscr{H}_3^{(\pm\pm)}(0)\equiv i\int d^4x[&T\{O_c^{(\pm)}(x)O_t^{(\pm)}(0)\}\\ &-T\{\{[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)u_\alpha(x)][\bar{c}_\beta(x)\gamma^\mu(1-\gamma_5)d_\beta(x)]\\ &\pm[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)d_\alpha(x)][\bar{c}_\beta(x)\gamma^\mu(1-\gamma_5)u_\beta(x)]\}\\ &\times\{[\bar{s}_\lambda(0)\gamma^\nu(1-\gamma_5)c_\lambda(0)][\bar{u}_\delta(0)\gamma_\nu(1-\gamma_5)d_\delta(0)]\\ &\pm[\bar{s}_\lambda(0)\gamma^\nu(1-\gamma_5)d_\lambda(0)][\bar{u}_\delta(0)\gamma_\nu(1-\gamma_5)c_\delta(0)]\}\}\\ &-T\{\{[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)u_\alpha(x)][\bar{t}_\beta(x)\gamma^\mu(1-\gamma_5)d_\beta(x)]\\ &\pm[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)d_\alpha(x)][\bar{t}_\beta(x)\gamma^\mu(1-\gamma_5)u_\beta(x)]\}\\ &\times\{[\bar{s}_\lambda(0)\gamma_\nu(1-\gamma_5)t_\lambda(0)][\bar{u}_\delta(0)\gamma^\nu(1-\gamma_5)d_\delta(0)]\\ &\pm[\bar{s}_\lambda(0)\gamma_\nu(1-\gamma_5)d_\lambda(0)][\bar{u}_\delta(0)\gamma^\nu(1-\gamma_5)t_\delta(0)]\}\}\\ &+T\{\{[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)c_\alpha(x)][\bar{t}_\beta(x)\gamma^\mu(1-\gamma_5)d_\beta(x)]\\ &\pm[\bar{s}_\alpha(x)\gamma^\nu(1-\gamma_5)d_\alpha(x)][\bar{t}_\beta(x)\gamma^\mu(1-\gamma_5)c_\beta(x)]\}\\ &\times\{[\bar{s}_\lambda(0)\gamma_\nu(1-\gamma_5)t_\lambda(0)][\bar{c}_\delta(0)\gamma^\nu(1-\gamma_5)d_\delta(0)]\\ &\pm[\bar{s}_\lambda(0)\gamma_\nu(1-\gamma_5)d_\lambda(0)][\bar{c}_\delta(0)\gamma^\nu(1-\gamma_5)t_\delta(0)]\}\}]\ .\end{aligned}\tag{5c}$$

Here

$$O_c^{(\pm)}=[(\bar{s}u)_{V-A}(\bar{u}d)_{V-A}\pm(\bar{s}d)_{V-A}(\bar{u}u)_{V-A}]-[(\bar{s}c)_{V-A}(\bar{c}d)_{V-A}\pm(\bar{s}d)_{V-A}(\bar{c}c)_{V-A}]$$

and similarly for $O_t^{(\pm)}$, in an abbreviated notation where

$$(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}=\bar{s}_\alpha\gamma_\mu(1+\gamma_5)d_\alpha\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta\ .$$

The matrix elements of the $\mathscr{H}_j$ are to be evaluated to all orders in the six-quark theory of strong interaction using the modified minimal-subtraction ($\overline{\text{MS}}$) scheme.

The next step is to successively treat the t quark and b quark as heavy and remove their fields from explicitly appearing in the theory. For $\mathscr{H}_1$ this is particularly simple since the t- and b-quark fields do not appear explicitly in it. The effect of removing the t- and b-quark fields from the theory of strong interactions is to change the strong-coupling g and masses $m_u,\ldots,m_t$ in the six-quark theory to a coupling g', and the masses $m_{u'},\ldots,m_{b'}$ in an effective five-quark theory and then to a coupling g'' and masses $m_{u''},\ldots,m_{c''}$ in an effective four-quark theory of the strong interactions. Also the exponents $a^{(+)}$ $[a^{(-)}]$ change from $\frac{6}{21}$ $[-\frac{12}{21}]$ to $\frac{6}{23}$ $[-\frac{12}{23}]$ and then to $\frac{6}{25}$ $[-\frac{12}{25}]$ as one goes from the six-quark theory to the effective five-quark theory and then to the effective four-quark theory of strong interactions.

Thus the effective Hamiltonian density $\mathcal{H}_1$ becomes (dropping subscripts on strong coupling α)

$$\mathcal{H}_1=\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{12/21}\left[\frac{\alpha(m_t{}^2)}{\alpha'(m_b'{}^2)}\right]^{12/23}\left[\frac{\alpha'(m_b'{}^2)}{\alpha''(\mu^2)}\right]^{12/25}\mathcal{H}_1^{(++)}$$
$$+\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{-6/21}\left[\frac{\alpha(m_t{}^2)}{\alpha'(m_b'{}^2)}\right]^{-6/23}\left[\frac{\alpha'(m_b'{}^2)}{\alpha''(\mu^2)}\right]^{-6/25}\mathcal{H}_2^{(+-)}$$
$$+\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{-6/21}\left[\frac{\alpha(m_t{}^2)}{\alpha'(m_b'{}^2)}\right]^{-6/23}\left[\frac{\alpha'(m_b'{}^2)}{\alpha''(\mu^2)}\right]^{-6/25}\mathcal{H}_1^{(-+)}$$
$$+\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{-24/21}\left[\frac{\alpha(m_t{}^2)}{\alpha'(m_b'{}^2)}\right]^{-24/23}\left[\frac{\alpha'(m_b'{}^2)}{\alpha''(\mu^2)}\right]^{-24/25}\mathcal{H}_1^{(--)} . \quad (6)$$

The matrix elements of the effective Hamiltonian density $\mathcal{H}_1$ are at this stage to be evaluated in an effective four-quark theory of strong interactions. It only remains to treat the charm quark as heavy and remove it from explicitly appearing in $\mathcal{H}_1$. To leading order in the c-quark mass the matrix elements of $\mathcal{H}_1^{(\pm\pm)}$ can be expanded in the following fashion:

$$\langle\,|\mathcal{H}_1^{(\pm\pm)}|\,\rangle''=L^{(\pm\pm)}\left[\frac{m_c''}{\mu},g''\right]m_c''^2\langle\,|(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}|\,\rangle''' , \quad (7)$$

where

$$(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}=[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta] .$$

The double primed matrix elements are evaluated in an effective four-quark theory of strong interactions while the triple primed matrix elements are to be evaluated in an effective three-quark theory of strong interactions with coupling g''' and masses m_u''', m_d''', and m_s'''.

The operator $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$ is a color-symmetric four-fermion operator with the usual anomalous dimension

$$\gamma'''^{(+)}(g''')=\frac{g'''^2}{4\pi^2}+O(g'''^4) . \quad (8)$$

The mass parameter m_c'' depends on the renormalization point μ and its anomalous dimension is

$$\gamma_c''(g'')=\frac{g''^2}{2\pi^2}+O(g''^4) . \quad (9)$$

The components $\mathcal{H}_1^{(++)}$, $\mathcal{H}_1^{(+-)}$, $\mathcal{H}_1^{(-+)}$, and $\mathcal{H}_1^{(--)}$ are composed of a sum of time-ordered products of two local four-quark operators with color indices, respectively, symmetrized in both operators, symmetrized in the first operator and antisymmetrized in the second operator, antisymmetrized in the first operator and symmetrized in the second operator and finally antisymmetrized in both operators. They have the familiar anomalous dimensions,[11] $g''^2/2\pi^2+O(g''^4)$, $-g''^2/4\pi^2+O(g''^4)$, $-g''^2/4\pi^2+O(g''^4)$, and $-g''^2/\pi^2+O(g''^4)$, respectively. It follows that the Wilson coefficients $L^{(\pm\pm)}(m_c''/\mu,g'')$ obey the renormalization-group equations:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta''(g'')\frac{\partial}{\partial g''}+\gamma_c''(g'')m_c''\frac{\partial}{\partial m_c''}-\frac{g''^2}{2\pi^2}-\frac{g'''^2}{4\pi^2}\right]L^{(++)}\left[\frac{m_c''}{\mu},g''\right]=0 , \quad (10a)$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta''(g'')\frac{\partial}{\partial g''}+\gamma_c''(g'')m_c''\frac{\partial}{\partial m_c''}-\frac{5g''^2}{4\pi^2}-\frac{g'''^2}{4\pi^2}\right]L^{(+-)}\left[\frac{m_c''}{\mu},g''\right]=0 , \quad (10b)$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta''(g'')\frac{\partial}{\partial g''}+\gamma_c''(g'')m_c''\frac{\partial}{\partial m_c''}-\frac{5g''^2}{4\pi^2}-\frac{g'''^2}{4\pi^2}\right]L^{(-+)}\left[\frac{m_c''}{\mu},g''\right]=0 , \quad (10c)$$

$$\left[\mu\frac{\partial}{\partial\mu}+\beta''(g'')\frac{\partial}{\partial g''}+\gamma_c''(g'')m_c''\frac{\partial}{\partial m_c''}-\frac{2g''^2}{\pi^2}-\frac{g'''^2}{4\pi^2}\right]L^{(--)}\left[\frac{m_c''}{\mu},g''\right]=0\ . \tag{10d}$$

These may be solved in the standard fashion, introducing a running coupling constant $\bar{g}''(y,g'')$ defined by

$$\ln y=\int_{g''}^{\bar{g}''(y,g'')}dx\frac{1-\gamma_c''(x)}{\beta''(x)},\quad \bar{g}''(1,g'')=g''\ , \tag{11}$$

and noting that the coefficients $L^{(\pm\pm)}[1,\bar{g}''(m_c/\mu,g'')]$ may be replaced by their free-field values $L^{(\pm\pm)}(1,0)$ since the running fine-structure constant is taken as small at the scale of the charm-quark mass and because no large logarithms can be generated from higher-order QCD loop integrals when $m_c''/\mu=1$.

A straightforward computation yields

$$L^{(++)}(1,0)=-\frac{1}{\pi^2}\left[\frac{3}{2}\right], \tag{12a}$$

$$L^{(+-)}(1,0)=L^{(-+)}(1,0)=-\frac{1}{\pi^2}\left[-\frac{1}{2}\right], \tag{12b}$$

and

$$L^{(--)}(1,0)=-\frac{1}{\pi^2}\left[\frac{1}{2}\right]. \tag{12c}$$

The factors in the square brackets stem from color summations. Solving the renormalization-group equations (10) using the leading-logarithm approximation then gives

$$L^{(++)}\left[\frac{m_c''}{\mu},g\right]=-\frac{1}{\pi^2}\left[\frac{\alpha''(\mu^2)}{\alpha''(m_c''^2)}\right]^{12/25}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{24/25}\left[\frac{3}{2}\right], \tag{13a}$$

$$L^{(+-)}\left[\frac{m_c''}{\mu},g\right]=L^{(-+)}\left[\frac{m_c''}{\mu},g\right]=-\frac{1}{\pi^2}\left[\frac{\alpha''(\mu^2)}{\alpha''(m_c''^2)}\right]^{-6/25}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{24/25}\left[-\frac{1}{2}\right], \tag{13b}$$

and

$$L^{(--)}\left[\frac{m_c''}{\mu},g\right]=-\frac{1}{\pi^2}\left[\frac{\alpha''(\mu^2)}{\alpha''(m_c''^2)}\right]^{-24/25}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{24/25}\left[\frac{1}{2}\right]. \tag{13c}$$

Using these results the effective Hamiltonian density $\mathscr{H}_1$ becomes

$$\begin{aligned}\mathscr{H}_1=-\frac{G_F^2}{16\pi^2}m_c^{*2}&[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma_\mu(1-\gamma_5)d_\beta]\\ \times\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}&\left[\frac{3}{2}\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{12/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{12/21}\right.\\ &-\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-6/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-6/23}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{-6/21}\\ &\left.+\frac{1}{2}\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-24/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-24/23}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{-24/21}\right],\end{aligned} \tag{14}$$

where m_c^* is the running charm-quark mass evaluated at $m_c''^2$, i.e.,

$$m_c^*=m_c''[\alpha_s''(m_c''^2)/\alpha_s''(\mu^2)]^{12/25}\ . \tag{15}$$

The Hamiltonian $\mathscr{H}_1$ already occurs in the four-quark model and our results agree with some of the previous results[3] for the QCD corrected $\mathscr{H}_1$, when the appropriate simplifications are made.

The derivation of the effective Hamiltonian density $\mathscr{H}_2$ proceeds along similar lines except that already at the step of removing the t-quark field from explicitly appearing each of the $\mathscr{H}_2^{(++)}$, $\mathscr{H}_2^{(+-)}$, $\mathscr{H}_2^{(-+)}$, and $\mathscr{H}_2^{(--)}$ collapses to a Wilson coefficient times

$$m_t{}^2[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta]$$

to leading order in the t-quark mass. From that point on the successive steps are marked by renormalization of this latter color index symmetric four-fermion operator. The final result is

$$\mathscr{H}_2=-\frac{G_F{}^2m_t^{*2}}{16\pi^2}[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta]\left[\frac{\alpha''(m_c{}''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha'(m_b{}'^2)}{\alpha''(m_c{}''^2)}\right]^{6/25}\left[\frac{\alpha(m_t{}^2)}{\alpha'(m_b{}'^2)}\right]^{6/23}$$
$$\times\left[\frac{3}{2}\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{12/21}-\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{-6/21}+\frac{1}{2}\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t{}^2)}\right]^{-24/21}\right], \tag{16}$$

where m_t^* is the running t-quark mass evaluated at $m_t{}^2$, i.e., $m_t^*=m_t[\alpha(m_t{}^2)/\alpha(\mu^2)]^{12/21}$.

The computation of the effective Hamiltonian density $\mathscr{H}_3$ in the presence of strong interactions is somewhat more complex. At the step of removing the t quark from $\mathscr{H}_3^{(\pm\pm)}$ eight operators are generated, even with the condition of keeping only those whose tree level matrix elements can yield a contribution of order $m_c{}''^2$ or mix under renormalization with operators whose matrix elements can. Expanding the matrix elements of $\mathscr{H}_3^{(\pm\pm)}$ in terms of matrix elements of these operators gives

$$\langle\,|\mathscr{H}_3^{(\pm\pm)}|\,\rangle=\sum_{j=1}^{7}L_j^{(\pm\pm)}\langle\,|O_j^{(\pm\pm)}|\,\rangle'+L_8^{(\pm\pm)}\langle\,|O_8|\,\rangle' \tag{17}$$

to leading order in the t-quark mass. The primed matrix elements are evaluated in an effective free-quark theory with strong coupling g'. Six of the operators[12]

$$O_1^{(\pm\pm)}=i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}\}\,, \tag{18a}$$

$$O_2^{(\pm\pm)}=i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A}\}\,, \tag{18b}$$

$$O_3^{(\pm\pm)}=i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A}+\cdots+(\bar{b}_\beta b_\beta)_{V-A}]\}\,, \tag{18c}$$

$$O_4^{(\pm\pm)}=i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A}+\cdots+(\bar{b}_\beta b_\alpha)_{V-A}]\}\,, \tag{18d}$$

$$O_5^{(\pm\pm)}=i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A}+\cdots+(\bar{b}_\beta b_\beta)_{V+A}]\}\,, \tag{18e}$$

$$O_6^{(\pm\pm)}=i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A}+\cdots+(\bar{b}_\beta b_\alpha)_{V+A}]\}\,, \tag{18f}$$

originate from the portion of $\mathscr{H}_3^{(\pm\pm)}$,

$$i\int d^4x\, T\{O_c^{(\pm)}(x)O_t^{(\pm)}\}$$

which is an integral of a time-ordered product of two pieces of the effective $\Delta S=1$ weak nonleptonic Hamiltonian, one containing a t quark and the other a c quark. Note that $O_j^{(\pm\pm)}=O_j^{(\pm\mp)}$ for $j\in(1,\ldots,6)$. The two additional operators needed are

$$O_7^{(\pm\pm)}=i\int d^4x\, T\{\{[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)u_\alpha(x)][\bar{c}_\beta(x)\gamma^\mu(1-\gamma_5)d_\beta(x)]$$
$$\pm[\bar{s}_\alpha(x)\gamma_\mu(1-\gamma_5)d_\alpha(x)][\bar{c}_\beta(x)\gamma^\mu(1-\gamma_5)u_\beta(x)]\}$$
$$\times\{[\bar{s}_\lambda\gamma_\nu(1-\gamma_5)c_\lambda][\bar{u}_\delta\gamma^\nu(1-\gamma_5)d_\delta]\pm[\bar{s}_\lambda\gamma_\nu(1-\gamma_5)d_\lambda][\bar{u}_\delta\gamma^\nu(1-\gamma_5)c_\delta]\}\} \tag{18g}$$

and

$$O_8=\frac{m_c{}'^2}{g'^2}[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma_\mu(1-\gamma_5)d_\beta]\,. \tag{18h}$$

The factor of $1/g'^2$ is inserted into the definition of O_8 so that to lowest order the anomalous dimension matrix $\gamma_{ij}'^{(\pm\pm)}(g')$ has all its entries proportional to g'^2. If O_8 did not contain the factor of $1/g'^2$ then the elements $\gamma_{i8}'^{(\pm\pm)}(g')$ would be (to lowest order) constants independent of g' for $i\in(1,\ldots,7)$. Then in solving the renormalization-group equations $L_8^{(\pm\pm)}$ would have to be treated in a different fashion from the $L_j^{(\pm\pm)}$, $j\in,\ldots,7)$. On the other hand, with our definition[13] of O_8 it can be treated on the same footing as all the other operators. Of course in calculating its renormalization we must now be careful to include the coupling constant renormalization.

The matrix elements of the operators $O_1^{(\pm\pm)}$ and $O_2^{(\pm\pm)}$ cannot produce a factor of $m_c''^2$ at three level. However, they must in principle be included since under renormalization they mix with the operators $O_3^{(\pm\pm)}$, $O_4^{(\pm\pm)}$, etc., whose matrix elements can produce a factor of $m_c''^2$. The anomalous-dimension matrices $\gamma_{ij}'^{(\pm\pm)}(g')$ for these eight operators are[10]

$$\gamma_{ij}'^{(++)}(g')=\frac{g'^2}{8\pi^2}\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\ 3 & 1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0\\ 0 & 0 & \frac{7}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & 32\\ 0 & 0 & \frac{22}{9} & \frac{8}{3} & -\frac{5}{9} & \frac{5}{3} & 0 & 16\\ 0 & 0 & 0 & 0 & 3 & -3 & 0 & -32\\ 0 & 0 & -\frac{5}{9} & \frac{5}{3} & -\frac{5}{9} & -\frac{13}{3} & 0 & -16\\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -24\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} \end{bmatrix}+O(g'^4)\,, \tag{19a}$$

$$\gamma_{ij}'^{(+-)}(g')=\frac{g'^2}{8\pi^2}\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\ 3 & 1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0\\ 0 & 0 & \frac{7}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & 32\\ 0 & 0 & \frac{22}{9} & \frac{8}{3} & -\frac{5}{9} & \frac{5}{3} & 0 & 16\\ 0 & 0 & 0 & 0 & 3 & -3 & 0 & -32\\ 0 & 0 & -\frac{5}{9} & \frac{5}{3} & -\frac{5}{9} & -\frac{13}{3} & 0 & -16\\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} \end{bmatrix}+O(g'^4)\,, \tag{19b}$$

$$\gamma_{ij}'^{(-+)}(g')=\frac{g'^2}{8\pi^2}\begin{bmatrix} -5 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\ 3 & -5 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0\\ 0 & 0 & -\frac{47}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & -16\\ 0 & 0 & \frac{22}{9} & -\frac{10}{3} & -\frac{5}{9} & \frac{5}{3} & 0 & 0\\ 0 & 0 & 0 & 0 & -3 & -3 & 0 & 16\\ 0 & 0 & -\frac{5}{9} & \frac{5}{3} & -\frac{5}{9} & -\frac{31}{3} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} \end{bmatrix}+O(g'^4)\,, \tag{19c}$$

$$\gamma'^{(--)}_{ij}(g')=\frac{g'^2}{8\pi^2}\begin{bmatrix} -5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -5 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{47}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & -16 \\ 0 & 0 & \frac{22}{9} & -\frac{10}{3} & -\frac{5}{9} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & 0 & 16 \\ 0 & 0 & -\frac{5}{9} & \frac{5}{3} & -\frac{5}{9} & -\frac{31}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} \end{bmatrix}+O(g'^4)\ . \tag{19d}$$

The coefficients $L_j^{(\pm\pm)}(m_t/\mu,g)$ satisfy renormalization-group equations which can be solved in the standard way. In this solution values are needed for the coefficients $L_j^{(\pm\pm)}(1,\bar{g}(m_t/\mu,g))$, where g is the running coupling in the six-quark theory. These are found by noting that in the leading-logarithm approximation the $L_j^{(\pm\pm)}(1,\bar{g}(m_t/\mu,g))$ can be replaced by their free-field values $L_j^{(\pm)}(1,0)$ for $j\in(1,\ldots,7)$:

$$L_1^{(\pm\pm)}(1,0)=1\ , \tag{20a}$$

$$L_2^{(\pm\pm)}(1,0)=1\ , \tag{20b}$$

$$L_3^{(\pm\pm)}(1,0)=L_4^{(\pm\pm)}(1,0)=L_5^{(\pm\pm)}(1,0)=L_6^{(\pm\pm)}(1,0)=0\ , \tag{20c}$$

and

$$L_7^{(\pm\pm)}(1,0)=-1\ . \tag{20d}$$

For the coefficient $L_8^{(\pm\pm)}(1,\bar{g}(m_t/\mu,g))$ the situation is somewhat more subtle since the operator O_8 contains a factor of $1/g'^2$. Explicit calculation gives that in the $\overline{\text{MS}}$ regularization scheme

$$L_7^{(\pm\pm)}(m_t/\mu=1,\bar{g})\propto\bar{g}^2\ln(m_t^2/\mu^2)\,|_{\mu=m_t}=0\ . \tag{21}$$

The last step follows, not only because the factor of $\bar{g}^2$ is small, but also because the logarithm vanishes at $\mu=m_t$.

The final aim is to derive an effective Hamiltonian independent of the heavy W-boson, t-quark, b-quark, and c-quark fields. To do this the b quark and c quark must still be considered as heavy and removed from explicitly appearing in the theory. Removal of the b quark is similar to the previous step. There are still eight operators whose renormalization is characterized by the anomalous-dimension matrices $\gamma''^{(\pm\pm)}(g'')$:

$$\gamma''^{(++)}_{ij}(g'')=\frac{g''^2}{8\pi^2}\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{7}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & 32 \\ 0 & 0 & \frac{23}{9} & \frac{7}{3} & -\frac{4}{9} & \frac{4}{3} & 0 & 16 \\ 0 & 0 & 0 & 0 & 3 & -3 & 0 & -32 \\ 0 & 0 & -\frac{4}{9} & \frac{4}{3} & -\frac{4}{9} & -\frac{14}{3} & 0 & -16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \end{bmatrix}+O(g''^4)\ , \tag{22a}$$

$$\gamma''^{(+-)}_{ij}(g'')=\frac{g''^2}{8\pi^2}\begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{7}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & 32 \\ 0 & 0 & \frac{23}{9} & \frac{7}{3} & -\frac{4}{9} & \frac{4}{3} & 0 & 16 \\ 0 & 0 & 0 & 0 & 3 & -3 & 0 & -32 \\ 0 & 0 & -\frac{4}{9} & \frac{4}{3} & -\frac{4}{9} & -\frac{14}{3} & 0 & -16 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \end{bmatrix}+O(g''^4)\ , \tag{22b}$$

$$\gamma''^{(-+)}_{ij}(g'')=\frac{g''^2}{8\pi^2}\begin{bmatrix} -5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -5 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{47}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & -16 \\ 0 & 0 & \frac{23}{9} & -\frac{11}{3} & -\frac{4}{9} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & 0 & 16 \\ 0 & 0 & -\frac{4}{9} & \frac{4}{3} & -\frac{4}{9} & -\frac{32}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \end{bmatrix}+O(g''^4)\,, \tag{22c}$$

$$\gamma''^{(--)}_{ij}(g'')=\frac{g''^2}{8\pi^2}\begin{bmatrix} -5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -5 & -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{47}{9} & \frac{11}{3} & -\frac{2}{9} & \frac{2}{3} & 0 & -16 \\ 0 & 0 & \frac{23}{9} & -\frac{11}{3} & -\frac{4}{9} & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 & 0 & 16 \\ 0 & 0 & -\frac{4}{9} & \frac{4}{3} & -\frac{4}{9} & -\frac{32}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} \end{bmatrix}+O(g''^4)\,. \tag{22d}$$

Finally at the step of removing the charm quark, only one operator

$$m''^2_c[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta]$$

appears and its anomalous dimension follows from mass renormalization and the renormalization of the color-symmetric local four-fermion operator $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$. This program for deriving the effective Hamiltonian $\mathscr{H}_3$ in the presence of strong interactions is a straightforward generalization of that used to derive the effective Hamiltonian for weak nonleptonic decays. Its complexity is such that, unlike the case of $\mathscr{H}_1$ and $\mathscr{H}_2$, we cannot write a simple analytic expression for $\mathscr{H}_3$.

However there are some further approximations, beyond the leading-logarithm approximation, which make the derivation of a simple analytic expression for $\mathscr{H}_3$ possible. As can be seen from Eqs. (20), the operators $O_3^{(\pm)},\ldots,O_6^{(\pm)}$ are induced through strong interactions and thus their contribution is less important than $O_7^{(\pm\pm)}$ which has a nonzero coefficient even in the absence of the strong interactions. It follows since O_1 and O_2 do not mix directly with O_7 and O_8, that to a good approximation, at the stage of removing the t quark, the set of eight operators can be truncated to the two operators $O_7^{(\pm\pm)}$ and O_8. Again, on removing the b quark there are two operators. Finally, on removing the charm quark only an operator proportional to O_8 occurs. This is the approximate solution we presented previously.[7] It yields an analytic expression[14] for $\mathscr{H}_3$:

$$\mathscr{H}_3=\frac{G_F{}^2m_c^{*2}}{64\pi\alpha''(m_c''^2)}[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma_\mu(1-\gamma_5)d_\beta]\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}$$
$$\times\left\{\frac{72}{35}\left[5\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{12/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}+2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}\right.\right.$$
$$\left.-7\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{12/2}$$

$$+\frac{48}{143}\left[13\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-6/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-6/23}-2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-6/23}\right.$$
$$\left.-11\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{-6/21}$$
$$+\frac{24}{899}\left[-31\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-24/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-24/23}+2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-24/23}\right.$$
$$\left.\left.+29\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{-24/21}\right\}. \qquad (23)$$

The matrix elements of the three parts of the effective Hamiltonian for K^0-$\bar{K}^0$ mixing in Eqs. (14), (16), and (23) are to be evaluated using the mass-independent $\overline{\text{MS}}$ scheme in an effective theory of strong interactions with three light quark flavors u, d, and s. The effects of QCD can be ascertained by comparing $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ given by Eqs. (14), (16), and (23) with their free-quark values in Eqs. (3).

III. NUMERICAL RESULTS

We are now in a position to evaluate numerically the coefficients of $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$ in the pieces $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ of the effective $\Delta S=2$ Hamiltonian. For given values of the parameters, these coefficients can be compared with their free-quark values.

There is only one operator $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$, so that any renormalization point (μ) dependence of its coefficient and of the matrix element of the operator must cancel between them, at least if everything is computed exactly. In ratios where the matrix element cancels, such as in the ratio of imaginary to real parts of $\langle K^0|\mathscr{H}_{\text{eff}}|\bar{K}^0\rangle$, the μ dependence of the coefficients must cancel out to obtain a renormalization-point-independent answer for these physical quantities. Our results for $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ in leading-logarithm approximation in Eqs. (14), (16), and (23) all have the same μ dependence through the factor $[\alpha''(m_c''^2)/\alpha(\mu^2)]^{6/27}$, and satisfy this last criterion, leading to predictions for such ratios which are renormalization point independent.

To evaluate the effective Hamiltonian we need values of the masses, including μ, and expressions for the running fine-structure constants $\alpha(Q^2)$, $\alpha'(Q^2)$, $\alpha''(Q^2)$, and $\alpha'''(Q^2)$ in effective field theories with 6, 5, 4, and 3 quarks, respectively. We use the standard

$$\alpha(Q^2)=\frac{12\pi}{33-2N_f}\frac{1}{\ln(Q^2/\Lambda^2)}, \qquad (24)$$

where N_f, the number of quark flavors, is six.

Corresponding formulas hold for $\alpha'(Q^2)$, $\alpha''(Q^2)$, and $\alpha'''(Q^2)$ with $N_f=5$, 4, and 3, respectively, but also with Λ replaced by Λ', Λ'', and Λ'''. That the Λ parameters are different follows if one demands matching of the value of the appropriate running couplings at boundaries between region. Explicitly setting $\alpha(m_t^2)=\alpha'(m_t^2)$ yields[15]

$$\Lambda=\Lambda'\left[\frac{\Lambda'}{m_t}\right]^{2/21}, \qquad (25)$$

while the corresponding matchings at m_b and m_c give

$$\Lambda'=\Lambda''\left[\frac{\Lambda''}{m_b}\right]^{2/23} \qquad (26)$$

and

$$\Lambda''=\Lambda'''\left[\frac{\Lambda'''}{m_c}\right]^{2/25}. \qquad (27)$$

Such a use of different values for Λ, Λ', ... does of course effect the numerical results and makes those reported here somewhat different from those we reported earlier in our short paper[7] on this subject where the differences in Λ', Λ', ... were not taken into account.[16] Also this allows us to reproduce the free-quark (no QCD) results[17] as Λ, Λ', ... approach zero, consistent with Eqs. (25), (26), and (27). This is carried through explicitly in Appendix A.

The effective Hamiltonian for K^0-$\bar{K}^0$ mixing is often used in conjunction with that for $\Delta S=1$ nonleptonic decays. For completeness we record in Appendix B the values of the coefficients of the operators in the $\Delta S=1$ weak nonleptonic Hamiltonian.

In the numerical work we use $m_c^*=1.5$ GeV, from charmonium spectroscopy; $m_b'=4.5$ GeV, from Γ spectroscopy; $m_t^*=30$ GeV, just to choose one possible value which is experimentally acceptable at the present; $M_W=80$ GeV; and $\Lambda''^2=0.01$ and 0.1 GeV.[2] It is Λ'' in the effective four-quark theory which is presumably the quantity being extracted from QCD analysis of deep-inelastic scattering experiments.[18] We set $\alpha'''(\mu^2)=1$, although it is easy to change this by again recalling that all the pieces of the Hamiltonian have the same μ dependence and multiplying all the answers by an appropriate factor.

Values for η_1, η_2, η_3, which are defined, respectively, as the ratios of coefficients of $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$ in $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ with strong interactions included, to those in the free-quark model, are presented in Table I. The results for η_3 are those calculated from the full mixing with eight operators. However, the approximate analytic results in Eq. (23) yield the same result to two place accuracy. It is evidently an excellent approximation to the full answer using all eight operators.

All the coefficients of $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ are lowered by QCD from their free-quark model values. The corrections to $\mathscr{H}_2$ and $\mathscr{H}_3$ are rather appreciable, but stable to varying Λ'' (or m_t for that matter). $\mathscr{H}_1$ changes by a factor of 1.4 between Λ''^2 of 0.01 and 0.1 GeV^2. A complete analysis of the effects of all this on the K^0-$\bar{K}^0$ system with the attendant phenomenology can be found elsewhere.[19]

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APPENDIX A: REPRODUCING THE FREE-QUARK MODEL

As we take the limit of Λ approaching zero, the running strong coupling approaches zero and the theory goes over to that of a free-quark model. In our expressions for the leading-logarithm QCD corrections to various quantities, one finds typical factors like $\alpha_s''(m_c^2)/\alpha_s'(m_b^2)$ raised in fractional powers. To evaluate such a factor as we approach the free-quark model, recall that the running fine-structure constant in an effective theory with four quarks,

TABLE I. QCD correction factors η_1, η_2, and η_3 to the pieces $\mathscr{H}_1$, and $\mathscr{H}_2$, and $\mathscr{H}_3$ of the effective Hamiltonian for K^0-$\bar{K}^0$ mixing.

Parameters	η_1	η_2	η_3
$\Lambda''^2=0.01$ GeV^2	0.69	0.59	0.41
$\Lambda''^2=0.1$ GeV^2	0.99	0.60	0.40

$$\alpha_s''(Q^2)=\frac{12\pi}{25\ln(Q^2/\Lambda''^2)}\ , \tag{A1}$$

while that in an effective theory with five quarks,

$$\alpha_s'(Q^2)=\frac{12\pi}{23\ln(Q^2/\Lambda'^2)}\ . \tag{A2}$$

Matching the running couplings at the boundary between four and five quarks (i.e., at m_b^2) gives as in Eq. (26)

$$\Lambda'=\Lambda''\left[\frac{\Lambda''}{m_b}\right]^{2/23}\ . \tag{A3}$$

Therefore,

$$\begin{aligned}\frac{\alpha_s''(m_c^2)}{\alpha_s'(m_b^2)}&=\frac{23}{25}\frac{\ln(m_b^2/\Lambda'^2)}{\ln(m_c^2/\Lambda''^2)}\\&=\frac{\ln(m_b^2/\Lambda''^2)}{\ln(m_c^2/\Lambda''^2)}\\&=1-\frac{\ln(m_b^2/m_c^2)}{\ln(\Lambda''^2/m_c^2)}\\&=1+\frac{25}{12\pi}\alpha_s\ln\left[\frac{m_b^2}{m_c^2}\right]+O(\alpha_s^2)\ .\end{aligned} \tag{A4}$$

The question of what is the argument of α_s in Eq. (A4) is a higher-order effect (in α_s).

In a similar way one finds

$$\frac{\alpha_s'(m_b^2)}{\alpha_s(m_t^2)}=1+\frac{23}{12\pi}\alpha_s\ln\left[\frac{m_t^2}{m_b^2}\right]+O(\alpha_s^2)\ , \tag{A5}$$

$$\frac{\alpha_s(m_t^2)}{\alpha_s(m_W^2)}=1+\frac{21}{12\pi}\alpha_s\ln\left[\frac{m_W^2}{m_t^2}\right]+O(\alpha_s^2)\ , \tag{A6}$$

and

$$\frac{\alpha_s'''(\mu^2)}{\alpha_s''(m_c^2)}=1+\frac{27}{12\pi}\alpha_s\ln\left[\frac{m_c^2}{\mu^2}\right]+O(\alpha_s^2)\ . \tag{A7}$$

Now take, for example, the QCD-correlated expressions for $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$ in the effective weak $\Delta S=2$ Hamiltonian density that contributes to the K^0-$\bar{K}^0$ mass matrix. For $\mathscr{H}_1$, given in Eq. (14), to get the free-quark limit we need only keep the first term in Eqs. (A4), (A5), (A6), and (A7). The correct result, Eq. (3a) emerges immediately. Similarly for $\mathscr{H}_2$, the QCD-corrected result in Eq. (16) goes over to the free-quark equation (3b).

For $\mathscr{H}_3$ the expression is much more complicated, even dropping the mixing of six operators to obtain the approximate result in Eq. (23):

$$\mathscr{H}_3=\frac{G_F{}^2 m_c^{*2}}{64\pi\alpha''(m_c''^2)}[\bar{s}_\alpha\gamma^\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma_\mu(1-\gamma_5)d_\beta]\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}$$
$$\times\left\{\frac{72}{35}\left[5\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{12/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}+2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}\right.\right.$$
$$\left.-7\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t^2)}\right]^{12/21}$$
$$+\frac{48}{143}\left[13\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-6/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-6/23}-2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-6/23}\right.$$
$$\left.-11\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t^2)}\right]^{-6/21}$$
$$+\frac{24}{899}\left[-31\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-24/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-24/23}+2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-24/23}\right.$$
$$\left.\left.+29\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]\left[\frac{\alpha(M_W{}^2)}{\alpha(m_t^2)}\right]^{-24/21}\right\}. \quad \text{(A8)}$$

In the limit process, the leading term cancels and we must keep the order-α_s terms in (A4) and (A6). We consider each quantity in large square brackets separately:

$$\left[5\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{12/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}+2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{12/23}-7\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{7/23}\right]$$
$$\underset{\Lambda\to 0}{\longrightarrow}\left[\frac{-60-10+35}{12\pi}\alpha_s\ln\left[\frac{m_b{}^2}{m_c{}^2}\right]+\frac{-60-24+49}{12\pi}\alpha_s\ln\left[\frac{m_t{}^2}{m_b{}^2}\right]\right]=-\frac{35}{12\pi}\alpha_s\ln\left[\frac{m_t{}^2}{m_c{}^2}\right], \quad \text{(A9)}$$

$$\left[13\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{6/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{6/23}-2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{6/23}\right.$$
$$\left.-11\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-7/23}\right]$$
$$\underset{\Lambda\to 0}{\longrightarrow}\left[\frac{78+10+55}{12\pi}\alpha_s\ln\left[\frac{m_b{}^2}{m_c{}^2}\right]+\frac{78-12+77}{12\pi}\alpha_s\ln\left[\frac{m_t{}^2}{m_b{}^2}\right]\right]=\frac{143}{12\pi}\alpha_s\ln\left[\frac{m_t{}^2}{m_c{}^2}\right], \quad \text{(A10)}$$

and

$$\left[-31\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{24/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{24/23}+2\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-5/21}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{24/23}\right.$$
$$\left.+29\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{-5/25}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{-7/23}\right]$$
$$\underset{\Lambda\to 0}{\longrightarrow}\left[\frac{-744-10-145}{12\pi}\alpha_s\ln\left[\frac{m_b{}^2}{m_c{}^2}\right]+\frac{-744+48-203}{12\pi}\alpha_s\ln\left[\frac{m_t{}^2}{m_b{}^2}\right]\right]=-\frac{899}{12\pi}\alpha_s\ln\left[\frac{m_t{}^2}{m_c{}^2}\right]. \quad \text{(A11)}$$

TABLE II. Coefficients of operators in the weak effective Hamiltonian for $\Delta S=1$ decays.

Λ''^2	0.01 GeV^2	0.1 GeV^2
C_1	$-1.0 \quad +0.034\ \tau$	$-0.93 \quad +0.049\ \tau$
C_2	$1.60 \quad -0.034\ \tau$	$1.55 \quad -0.049\ \tau$
C_3	$-0.033-0.006\ \tau$	$-0.022-0.014\ \tau$
C_5	$0.018+0.004\ \tau$	$0.011+0.009\ \tau$
C_6	$-0.10 \quad -0.10\ \tau$	$-0.048-0.11\ \tau$

Parameters

$M_W=80$ GeV, $m_t=30$ GeV, $m_b=4.5$ GeV,
$m_c=1.5$ GeV, $\alpha'''(\mu^2)=1$,
$\tau=s_2{}^2+(s_2c_2s_3e^{-i\delta}/c_1c_3)$

Therefore, going back to $\mathscr{H}_3$

$$\mathscr{H}_3 \underset{\Lambda\to 0}{\longrightarrow} \frac{G_F{}^2m_c{}^2}{64\pi\alpha_s}[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta]$$
$$\times\left[\frac{72}{35}\left[-\frac{35}{12\pi}\right]+\frac{48}{143}\left[\frac{143}{12\pi}\right]+\frac{24}{899}\left[-\frac{899}{12\pi}\right]\right]\alpha_s\ln\left[\frac{m_t{}^2}{m_c{}^2}\right]$$
$$=-\frac{G_F{}^2m_c{}^2}{16\pi^2}\ln\left[\frac{m_t{}^2}{m_c{}^2}\right](\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha)(\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta)\,, \qquad \text{(A12)}$$

which is exactly the free-quark (no QCD) result for $\mathscr{H}_3$ in Eq. (3c). Similar computations give the free-quark limit in other situations, such as the $\Delta S=1$ effective weak Hamiltonian.

APPENDIX B: COEFFICIENTS OF OPERATORS IN $\Delta S=1$ HAMILTONIAN

The effective Hamiltonian for $\Delta S=1$ nonleptonic decays is frequently used with the for $\Delta S=2$ K^0-$\bar{K}^0$ mixing. With the use of different values for Λ, Λ',... in the expressions for $\alpha(Q^2)$, $\alpha'(Q^2)$,... corresponding to effective field theories with 6, 5,... quark flavors, the numerical results for the $\Delta S=1$ nonleptonic weak Hamiltonian are changes from those in Ref. 11, where the differences in Λ, Λ',... were not taken into account. We have recomputed the coefficients in $\mathscr{H}^{(\Delta S=1)}=\sum_i C_iQ_i$ using the same choice of parameters as in this paper: $M_W=80$ GeV, $m_t=30$ GeV, $m_b=4.5$ GeV, $m_c=1.5$ GeV, and μ chosen so that $\alpha'''(\mu^2)=1$. The results are given in Table II and replace those in Tables I and III in Ref. 11.

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$$O_q^{(\pm)}=[(\bar{s}u(_{V-A}(\bar{u}d)_{V-A}(\bar{u}u)_{V-A}]-[u\to q]\ .$$

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