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Nonlinear relations between observables in the scattering of spinor particles and the direct reconstruction of the scattering matrix

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Nonlinear relations between observables in the elastic scattering of two nonidentical particles are obtained, namely 17 relations expressing the components of higher-order polarization tensors in terms of lower-order ones (of degree $d \leq 2$) and 8 relations between the 19 simplest experiments. The relations are used to discuss the completeness of various optimal sets of experiments and to perform a "semidirect" reconstruction of the *np* scattering amplitudes at $E_{\text{lab}}=425$ MeV, $\theta_{\text{c.m.}}=65°$ (from data points supplemented by "data" generated from a phase-shift analysis).

I. INTRODUCTION

The purpose of this article is to discuss the direct reconstruction of the scattering matrix from experimental data for the elastic scattering of two nonidentical particles with spin $\frac{1}{2}$. Assuming Lorentz invariance, parity conservation, and time-reversal invariance, the scattering matrix M will involve 6 invariant amplitudes, i.e., 6 complex functions of the scattering angle and energy. There exist 36 linearly independent observables which provide us directly with the quantities $\text{Re}A_i^*A_k$, $\text{Im}A_i^*A_k$, and $|A_i|^2$ where A_i are the amplitudes, $1 \leq i \leq k \leq 6$. These quantities do not depend on the overall phase of the amplitudes, hence only 11 real functions can be obtained from the experiments (e.g., 6 amplitudes $|A_i|^2$ and 5 relative phases).

Since only 11 of the observables are functionally independent, nonlinear relations between them exist. These are obviously important for the planning of experiments; in particular it is useful to express more complicated experiments in terms of simpler ones. It is also of prime importance to know beforehand which sets of experiments provide the information needed to reconstruct the scattering matrix. The main results presented here are the following. (1) We obtain a set of 25 independent bilinear relations between the 36 linearly independent observables (Sec. II). (2) We express the 17 observables that are components of three- and four-component polarization tensors in terms of the 19 simplest observables. This involves no ambiguity and proves that the three- and four-component tensors are not needed for a complete reconstruction of the scattering matrix (Sec. III). (3) We obtain a convenient set of 8 nonlinear relations between the 19 simplest experiments (Sec. IV). (4) We use the nonlinear relations between observables to obtain different optimal sufficient and complete sets of experiments (Sec. V). We perform a reconstruction of the amplitudes from experimental data at $E_{\text{lab}}=425$ MeV, $\theta=65°$ (Sec. VI). Since the data are incomplete, we supplement it by "simulated data," calculated on the basis of phase-shift analysis.

The terminology we use was partly introduced in a recent publication.[1] Thus, in this paper, we call a set of experiments optimal if it consists of a measurement of the differential cross section, the polarization of the scattered particle $P_{n000}=A_{00n0}$, the polarization of the recoil particle $P_{0n00}=A_{000n}$, and the minimal needed number of components of two-component polarization tensors (all notations were reviewed in two recent publications[2,3] on the scattering formalism and are partly discussed in Sec. II below). A set is called *sufficient* if it makes it possible to reconstruct the scattering matrix up to certain discrete ambiguities (and the unavoidable continuous ambiguity of the overall phase). A set is called *complete* if it permits a reconstruction with no ambiguities (continuous or discrete). A set will be called *natural* if it involves the same number of "pure" experiments in the center-of-mass (c.m.s.) and laboratory (l.s.) systems. We recall that a pure experiment is one involving only spin projections on basis vectors; the c.m.s. and l.s. basis vectors are given in Sec. II. We are always talking about a set of experiments performed for one given energy and scattering angle.

The concept of a complete set of experiments was first introduced by Puzikov, Ryndin, and

Smorodinskiĭ[4] and a considerable amount of literature has been devoted to the direct reconstruction of scattering amplitudes.[1,4–13] The reason why we are going into considerable detail at the present stage is that the development of new experimental techniques, in particular the availability of beams of polarized nucleons and of targets polarized in arbitrary directions, has made complete experiments feasible. Indeed extensive projects for complete experiments exist at many laboratories and some have already been performed and analyzed.[14–17] Reference 1 was devoted to a study of the scattering of identical spinor particles (5 scattering amplitudes), in particular proton-proton scattering. The results were applicable to neutron-proton scattering only inasmuch as isospin invariance was assumed. This article applies to *np* scattering without isospin invariance and indeed a detailed test of isospin invariance would involve a reconstruction of all 6 scattering amplitudes, as treated in this article. For a discussion of isospin invariance and electromagnetic effects in *np* scattering, we refer to some recent publications[18–20] and references therein.

II. SCATTERING FORMALISM AND QUADRATIC RELATIONS BETWEEN OBSERVABLES

A. The formalism

The formalism we use was described in detail in Ref. 3. The scattering matrix is parametrized as

$$\begin{aligned} M(\vec{k}_f,\vec{k}_i)=\tfrac{1}{2}[&(a+b)+(a-b)(\vec{\sigma}_1\cdot\vec{n})(\vec{\sigma}_2\cdot\vec{n})\\ &+(c+d)(\vec{\sigma}_1\cdot\vec{m})(\vec{\sigma}_2\cdot\vec{m})\\ &+(c-d)(\vec{\sigma}_1\cdot\vec{l})(\vec{\sigma}_2\cdot\vec{l})\\ &+e(\vec{\sigma}_1+\vec{\sigma}_2,\vec{n})+f(\vec{\sigma}_1-\vec{\sigma}_2,\vec{n})]\ . \end{aligned} \tag{2.1}$$

Here $a,\ldots,f$ are the 6 scattering amplitudes, i.e., complex functions of the energy E and scattering angle θ. For identical particles (and *np* scattering under the assumption of isospin invariance) we have $f\equiv 0$. The Pauli matrices $\vec{\sigma}_1$ and $\vec{\sigma}_2$ act on the first and second particle spinors. The orthonormal vectors

$$\vec{l}=\frac{\vec{k}_f+\vec{k}_i}{|\vec{k}_f+\vec{k}_i|}\ ,\quad \vec{m}=\frac{\vec{k}_f-\vec{k}_i}{|\vec{k}_f-\vec{k}_i|}\ ,\quad \vec{n}=\frac{\vec{k}_i\times\vec{k}_f}{|\vec{k}_i\times\vec{k}_f|} \tag{2.2}$$

are the c.m.s. basis vectors; $\vec{k}_i$ and $\vec{k}_f$ are unit vectors in the direction of the initial- and scattered-particle c.m.s. momenta. We introduce three different triplets of basis vectors in the l.s., namely

$$(\vec{n},\vec{k},\vec{s}=\vec{n}\times\vec{k})\ ,\quad (\vec{n},\vec{k}\,',\vec{s}\,'=\vec{n}\times\vec{k}\,')\ ,\quad (\vec{n},\vec{k}\,'',\vec{s}\,''=\vec{n}\times\vec{k}\,'') \tag{2.3}$$

for the initial, scattered, and recoil particle, respectively (where $\vec{k}$, $\vec{k}\,'$, and $\vec{k}\,''$ are unit vectors along the incident-, scattered-, and recoil-particle momenta).

All experimental quantities are defined by the expression

$$\sigma X_{pqik}=\tfrac{1}{4}\,\mathrm{Tr}\sigma_{1p}\sigma_{2q}M\sigma_{1i}\sigma_{2k}M^+\ , \tag{2.4}$$

where the labels p, q, i, and k refer to the polarizations of the scattered, recoil, incident, and target particle, respectively. If an initial particle is unpolarized or a final polarization is undetected, the corresponding label is equal to 0 and we set $\sigma_0=I$. All c.m.s. and l.s. experiments are given in terms of the amplitudes in Tables 1 and 3 of Ref. 3.

For further use let us introduce a convenient notation for the 19 experiments involving at most two spin labels (and hence figuring in any optimal set of experiments):

$$\begin{aligned} &R_{12}\equiv \mathrm{Re}a^*b=\frac{\sigma}{2}(D_{0m0m}+D_{l0l0})\ ,\quad R_{13}\equiv \mathrm{Re}a^*c=\frac{\sigma}{2}(K_{0mm0}+K_{0ll0})\ ,\\ &R_{14}\equiv \mathrm{Re}a^*d=\frac{\sigma}{2}(C_{mm00}-C_{ll00})\ ,\quad R_{15}\equiv \mathrm{Re}a^*e=\frac{\sigma}{2}(A_{00n0}+A_{000n})\ ,\\ &R_{23}\equiv \mathrm{Re}b^*c=\frac{\sigma}{2}(C_{mm00}+C_{ll00})\ ,\quad R_{24}\equiv \mathrm{Re}b^*d=\frac{\sigma}{2}(K_{0mm0}-K_{0ll0})\ ,\\ &R_{26}\equiv \mathrm{Re}b^*f=\frac{\sigma}{2}(A_{00n0}-A_{000n})\ ,\quad R_{34}\equiv \mathrm{Re}c^*d=\frac{\sigma}{2}(D_{m0m0}-D_{l0l0})\ ,\\ &R_{56}\equiv \mathrm{Re}e^*f=\frac{\sigma}{2}(D_{0m0m}-D_{m0m0})\ , \end{aligned} \tag{2.5a}$$

$$I_{16}\equiv \mathrm{Im}a^*f=\frac{\sigma}{2}(D_{l0m0}-D_{0l0m})\ ,\quad I_{25}\equiv \mathrm{Im}b^*e=\frac{\sigma}{2}(D_{l0m0}+D_{0l0m})\ ,$$

$$I_{35}\equiv \mathrm{Im}c^*e=\frac{\sigma}{2}(K_{l00m}+K_{0lm0})\ ,\quad I_{36}\equiv \mathrm{Im}c^*f=\frac{\sigma}{2}(C_{lm00}-C_{ml00})\ ,$$

$$I_{45}\equiv \mathrm{Im}d^*e=\frac{\sigma}{2}(C_{lm00}+C_{ml00})\ ,\quad I_{46}\equiv \mathrm{Im}d^*f=\frac{\sigma}{2}(K_{l00m}-K_{0lm0})\ ,$$

$$|c|^2=\frac{\sigma}{2}(1-D_{n0n0}-C_{nn00}+K_{0nn0})\ ,\quad |d|^2=\frac{\sigma}{2}(1-D_{n0n0}+C_{nn00}-K_{0nn0})\ ,$$

$$|a|^2+|e|^2=\frac{\sigma}{2}(1+D_{n0n0}+C_{nn00}+K_{0nn0})\ ,\quad |b|^2+|f|^2=\frac{\sigma}{2}(1+D_{n0n0}-C_{nn00}-K_{0nn0})\ . \tag{2.5b}$$

The notations used in (2.5) are discussed in detail elsewhere.[2,3] We recall that σ is the unpolarized differential cross section, $A_{00n0}=P_1$ and $A_{000n}=P_2$ are asymmetries due to beam and target polarization, respectively. The quantities D_{p0i0}, D_{0q0k}, K_{0qi0}, K_{p00k}, and C_{pq00} are components of the depolarization, polarization transfer, and polarization correlation tensors.

The set of observables (2.5) is invariant under certain "permutations" of the amplitudes $a,\ldots,f$, such as the simultaneous transposition $(a\leftrightarrow b, e\leftrightarrow f)$, as well as the individual transformations $(c\leftrightarrow d)$, $(a\leftrightarrow ie)$, and $(b\leftrightarrow if)$. Using permutation-group notations we can denote these transformations of the amplitude as

$$X_1=\begin{bmatrix} a & b & c & d & e & f\\ b & a & c & d & f & e\end{bmatrix},$$
$$X_2=\begin{bmatrix} a & b & c & d & e & f\\ a & b & d & c & e & f\end{bmatrix}, \tag{2.6}$$
$$X_3=\begin{bmatrix} a & b & c & d & e & f\\ ie & b & c & d & -ia & f\end{bmatrix},$$
$$X_4=\begin{bmatrix} a & b & c & d & e & f\\ a & if & c & d & e & -ib\end{bmatrix}.$$

The transformations $X_1,\ldots,X_4$ generate a finite group. Its order is 16 and it is isomorphic to a subgroup of the group of permutations of 6 elements. The 16 different transformations leaving the set (2.5) invariant are

$$\begin{aligned} &E,\ X_1,\ X_2,\ X_3,\ X_4,\ Y_1=X_2X_3\ ,\\ &Y_2=X_2X_4\ ,\ Y_3=X_3X_4\ ,\ Y_4=X_1X_2\ ,\\ &Y_5=X_1X_3\ ,\ Y_6=X_3X_1\ ,\ Z_1=X_2X_3X_4\ ,\\ &Z_2=X_3X_4X_1\ ,\ Z_3=X_4X_1X_2\ ,\\ &Z_4=X_2X_3X_1\ ,\ U=X_1X_2X_3X_4\ ,\end{aligned} \tag{2.7}$$

where E is the identity. The multiplication table is given in Table I.

The invariance group (2.7) of the set (2.5) allows us to classify optimal sets of experiments into conjugacy classes and to transfer information from one optimal set to all sets in the same class. We shall call this group the invariance group of the optimal set and denote it G_0.

B. Bilinear relations between observables

In order to obtain 25 bilinear relations among the observables we apply a method used by Bourrely and Soffer[12] for elastic proton-proton scattering. Define a Hermitian matrix of observables

$$\begin{aligned} &H=\varphi\otimes\varphi^\dagger\ ,\\ &\varphi^\dagger=(a^*+e^*,\sqrt{2}c^*,b^*+f^*,b^*-f^*\ ,\\ &\qquad -\sqrt{2}d^*,a^*-e^*)\end{aligned} \tag{2.8}$$

satisfying $\mathrm{Tr}H=4\sigma$. The specific choice of the vector φ is dictated by the desire to obtain formulas in which third- and fourth-order tensors appear separately on one side and diagonal second-order tensors and polarizations on the other [see formulas (2.11) below].

Fifteen independent quadratic relations among the observables are provided by the formulas

$$|H_{ij}|^2=H_{ii}H_{jj}\ ,\quad 1\le i<j\le 6\ , \tag{2.9}$$

expressing the moduli of all 15 off-diagonal matrix elements in terms of the diagonal ones. A set of 25 independent relations is obtained by adding 10 further formulas, namely

$$H_{ij}H_{jk}=H_{ik}H_{jj}\ ,\quad 2\le i+1=j<k\le 6\ , \tag{2.10}$$

making it possible to obtain the phases of 10 of the off-diagonal elements, e.g., those of H_{ik} $(3\le i+2\le k\le 6)$ in terms of the diagonal elements and the phases of the 5 remaining off-diagonal elements H_{ij} $(2\le i+1=j\le 6)$.

We shall not spell out the relations (2.10) in detail since they are quite complicated. Relations (2.9), on the other hand, can be written as

TABLE I. Multiplication table for the invariance group G_0 of the optimal set of experiments.

	E	X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Z_1	Z_2	Z_3	Z_4	U
E	E	X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Z_1	Z_2	Z_3	Z_4	U
X_1	X_1	E	Y_4	Y_5	Y_6	Z_3	Z_4	Z_2	X_2	X_3	X_4	U	Y_3	Y_1	Y_2	Z_1
X_2	X_2	Y_4	E	Y_1	Y_2	X_3	X_4	Z_1	X_1	Z_3	Z_4	Y_3	U	Y_5	Y_6	Z_2
X_3	X_3	Y_6	Y_1	E	Y_3	X_2	Z_1	X_4	Z_4	Z_2	X_1	Y_2	Y_5	U	Y_4	Z_3
X_4	X_4	Y_5	Y_2	Y_3	E	Z_1	X_2	X_3	Z_3	X_1	Z_2	Y_1	Y_6	Y_4	U	Z_4
Y_1	Y_1	Z_4	X_3	X_2	Z_1	E	Y_3	Y_2	Y_6	U	Y_4	X_4	Z_3	Z_2	X_1	Y_5
Y_2	Y_2	Z_3	X_4	Z_1	X_2	Y_3	E	Y_1	Y_5	Y_4	U	X_3	Z_4	X_1	Z_2	Y_6
Y_3	Y_3	Z_2	Z_1	X_4	X_3	Y_2	Y_1	E	U	Y_6	Y_5	X_2	X_1	Z_4	Z_3	Y_4
Y_4	Y_4	X_2	X_1	Z_3	Z_4	Y_5	Y_6	U	E	Y_1	Y_2	Z_2	Z_1	X_3	X_4	Y_3
Y_5	Y_5	X_4	Z_3	X_1	Z_2	Y_4	U	Y_6	Y_2	Y_3	E	Z_4	X_3	Z_1	X_2	Y_1
Y_6	Y_6	X_3	Z_4	Z_2	X_1	U	Y_4	Y_5	Y_1	E	Y_3	Z_3	X_4	X_2	Z_1	Y_2
Z_1	Z_1	U	Y_3	Y_2	Y_1	X_4	X_3	X_2	Z_2	Z_4	Z_3	E	Y_4	Y_6	Y_5	X_1
Z_2	Z_2	Y_3	U	Y_6	Y_5	Z_4	Z_3	X_1	Z_1	X_4	X_3	Y_4	E	Y_2	Y_1	X_2
Z_3	Z_3	Y_2	Y_5	Y_4	U	X_1	Z_2	Z_4	X_4	Z_1	X_2	Y_6	Y_1	Y_3	E	X_3
Z_4	Z_4	Y_1	Y_6	U	Y_4	Z_2	X_1	Z_3	X_3	X_2	Z_1	Y_5	Y_2	E	Y_3	X_4
U	U	Z_1	Z_2	Z_4	Z_3	Y_6	Y_5	Y_4	Y_3	Y_2	Y_1	X_1	X_2	X_4	X_3	E

$$[(K_{0mm0}\overset{+}{\underset{-}{\pm}}K_{0ll0})\overset{+}{\underset{+}{\pm}}(C_{nll0}\overset{+}{\underset{-}{\pm}}C_{ln0l})]^2+[\overset{+}{\underset{-}{\pm}}(K_{l00m}\overset{+}{\underset{-}{\pm}}K_{0lm0})\overset{+}{\underset{-}{\pm}}(C_{nlm0}\overset{-}{\underset{+}{\mp}}C_{nml0})]^2$$

$$=[(1-C_{nn00}{}^2)-(D_{n0n0}{}^2-K_{0nn0}{}^2)\overset{+}{\underset{-}{\pm}}2(K_{0nn0}-C_{nn00}D_{n0n0})]\overset{+}{\underset{-}{\mp}}2(P_1\overset{+}{\underset{-}{\pm}}P_2)[(1\overset{-}{\underset{+}{\mp}}C_{nn00})-(D_{n0n0}\overset{-}{\underset{+}{\mp}}K_{0nn0})]\ ,$$

$$[(C_{mm00}\overset{+}{\underset{-}{\pm}}C_{ll00})\overset{+}{\underset{+}{\pm}}(C_{lln0}\overset{-}{\underset{+}{\mp}}C_{ll0n})]^2+[\overset{-}{\underset{+}{\pm}}(C_{lm00}\overset{-}{\underset{+}{\mp}}C_{ml00})\overset{+}{\underset{-}{\pm}}(C_{mln0}\overset{-}{\underset{+}{\mp}}C_{lmn0})]^2$$

$$=[(1\overset{-}{\underset{+}{\mp}}C_{nn00})^2-(D_{n0n0}\overset{-}{\underset{+}{\mp}}K_{0nn0})^2]\overset{+}{\underset{-}{\mp}}2(P_1\overset{-}{\underset{+}{\mp}}P_2)[(1\overset{-}{\underset{+}{\mp}}C_{nn00})-(D_{n0n0}\overset{-}{\underset{+}{\mp}}K_{0nn0})]\ ,$$

$$[D_{0m0m}+\tfrac{1}{2}(D_{l0l0}-D_{m0m0})\pm C_{nl0l}]^2+[D_{0l0m}\pm\tfrac{1}{2}(C_{lnm0}-C_{nm0l})]^2$$

$$=\tfrac{1}{4}\{(1-C_{nn00}{}^2)+(D_{n0n0}{}^2-K_{0nn0}{}^2)+4(P_1{}^2-P_2{}^2)+2[D_{n0n0}\pm 2P_1(1+D_{n0n0})]$$

$$-2[C_{nn00}K_{0nn0}\pm 2P_2(C_{nn00}+K_{0nn0})]\}\ ,\qquad (2.11)$$

$$[\tfrac{1}{2}(D_{l0l0}+D_{m0m0})\pm C_{lnl0}]^2+[D_{l0m0}\pm\tfrac{1}{2}(C_{lnm0}+C_{nm0l})\mp C_{mnl0}]^2$$

$$=\tfrac{1}{4}\{(1-C_{nn00}{}^2)+(D_{n0n0}{}^2-K_{0nn0}{}^2)-4(P_1{}^2-P_2{}^2)+2[D_{n0n0}\pm 2P_2(1+D_{n0n0})]$$

$$-2[C_{nn00}K_{0nn0}\pm 2P_1(C_{nn00}+K_{0nn0})]\}\ ,$$

$$(C_{llll}\mp C_{llmm})[(C_{llll}\mp C_{llmm})-(1\pm C_{nn00})+(D_{n0n0}\pm K_{0nn0})]+(C_{llml}\pm C_{lllm})^2$$

$$=(1\pm C_{nn00})(D_{n0n0}\pm K_{0nn0})-(P_1\pm P_2)^2\ ,$$

$$(D_{m0m0}-D_{l0l0})^2+(C_{nm0l}+C_{lnm0})^2=(1-C_{nn00}{}^2)+(D_{n0n0}{}^2-K_{0nn0}{}^2)+2(K_{0nn0}C_{nn00}-D_{n0n0})\ .$$

We note that these relations, as well as those obtained from (2.10), mix together the simple and the complicated experiments. They cannot be directly used to express the three- and four-component quantities in terms of the simplest ones without introducing discrete ambiguities (due to solving quadratic equations).

III. HIGHER-ORDER POLARIZATION TENSORS IN TERMS OF LOWER-ORDER ONES

The observables (2.5) are the only ones that will figure in "optimal sets of experiments" and they are in general easier to measure than the remaining observables, involving three or four polarizations.

The higher-order tensors can be expressed in terms of the lower-order ones without introducing any ambiguities at all (not even discrete ones) and we now proceed to obtain the corresponding formulas.

We shall first analyze the generic situation when none of the observables vanishes identically in the entire energy and angular region considered and later analyze exceptional cases.

Our analysis is based on an identity satisfied by any three complex numbers x, y, and z (Refs. 1, 6, 7, and 21):

$$x\,\mathrm{Im}y^*z + y\,\mathrm{Im}z^*x + z\,\mathrm{Im}x^*y = 0 \tag{3.1}$$

or equivalently, replacing z by iz:

$$x\,\mathrm{Re}y^*z - y\,\mathrm{Re}z^*x + iz\,\mathrm{Im}x^*y = 0 \tag{3.2}$$

(the asterisk denotes complex conjugation).

Multiplying (3.1) and (3.2) by another complex number u^* and taking the real and imaginary parts, we obtain the following 3 relations:

$$\mathrm{Re}u^*x\,\mathrm{Im}y^*z + \mathrm{Re}u^*y\,\mathrm{Im}z^*x + \mathrm{Re}u^*z\,\mathrm{Im}x^*y = 0\,, \tag{3.3}$$

$$\mathrm{Re}u^*x\,\mathrm{Re}y^*z - \mathrm{Re}u^*y\,\mathrm{Re}z^*x - \mathrm{Im}u^*z\,\mathrm{Im}x^*y = 0\,, \tag{3.4}$$

$$\mathrm{Im}u^*x\,\mathrm{Im}y^*z + \mathrm{Im}u^*y\,\mathrm{Im}z^*x + \mathrm{Im}u^*z\,\mathrm{Im}x^*y = 0\,. \tag{3.5}$$

Putting $u=x$ in (3.3) and (3.4) we obtain

$$|x|^2\mathrm{Re}y^*z = \mathrm{Re}x^*y\,\mathrm{Re}x^*z + \mathrm{Im}x^*y\,\mathrm{Im}x^*z\,, \tag{3.6}$$

$$|x|^2\mathrm{Im}y^*z = \mathrm{Re}x^*y\,\mathrm{Im}x^*z - \mathrm{Re}x^*z\,\mathrm{Im}x^*y\,. \tag{3.7}$$

Finally, putting $y=z$ in (3.6) we have

$$|x|^2|y|^2 = (\mathrm{Re}x^*y)^2 + (\mathrm{Im}x^*y)^2\,. \tag{3.8}$$

We can write (3.3), (3.4), and (3.5) symbolically as (u,xyz), $\{u,xyz\}$, and $[u,xyz]$, respectively. Then we can write 4 relations as

$$(u,xyz)\,,\ (x,uyz)\,,\ (y,uxz)\,,\ (z,uxy) \tag{3.9}$$

and any 3 of them are independent. The 3 relations

$$\{u,xyz\}\,,\ \{u,zxy\}\,,\ \{u,yzx\} \tag{3.10}$$

are independent and imply $[u,xyz]$. Any 3 of (3.9) together with any 2 of (3.10) are independent. Thus relations (3.3), (3.4), and (3.5) represent five functionally independent relations among any 4 different complex numbers u, x, y, and z. Identifying u, x, y, and z with any 4 of the 6 amplitudes $a,\ldots,f$ we obtain quadratic relations among the observables.

These relations can be used to express the higher-order tensors linearly in terms of the simpler ones. Multiple applications of the formulas (3.3)–(3.8) are required in some cases so that higher-order expressions are obtained. No "canonical" choice of the final expressions exists. In Table II, we present a set of formulas that we found particularly convenient. They express all components of the third- and fourth-order tensors in terms of the lower-order ones and we sometimes give several alternative expressions for the same quantity. Together with formulas (2.5) the formulas of Table II express all 36 bilinear combinations $R_{ik}, I_{ik}, |a|^2, \ldots, |f|^2$ $(1\le i\ne k\le 6)$ of the amplitudes in terms of the 19 simplest observables.

The expressions in Table II make sense as long as none of the denominators vanishes. Experiments measuring higher-order tensors are then not needed for a reconstruction, not even to resolve discrete ambiguities. The first 9 relations are bilinear in the observables, the first 8 of them transform among each other (as an octet) under the invariance group G_0, the ninth relation is invariant under the group. The remaining 8 relations are of order 3 or higher. Further higher-order relations can be obtained either by applying the group G_0 to the existing relations or directly, but we will not go into this here.

If the observables satisfy $R_{15}\ne0$, $R_{26}\ne0$, and either $R_{34}\ne0$, $R_{13}I_{45}-R_{14}I_{35}\ne0$, or $R_{23}I_{46}-R_{24}I_{36}\ne0$ then Table II can be used directly to obtain the higher-order tensors. The restriction on R_{34} or on quantities such as $R_{13}I_{45}-R_{14}I_{35}$ is not essential and a reconstruction of the higher-order tensors is still possible, using other formulas obtained from relations (3.3)–(3.8).

A case of particular interest is $R_{26}\approx0$, $R_{15}\ne0$. This occurs in the case of neutron-proton scattering where $R_{26}\ne0$ is a manifestation of isospin nonconservation (e.g., due to electromagnetic interactions). Anywhere except in the forward direction we can hence expect R_{26} to be small $(|f|^2\ll|a|^2)$.

Consider, e.g., the case $R_{26}=0$, $R_{15}\ne0$, $|c|^2\ne0$, $R_{13}I_{45}-R_{14}I_{35}=-R_{15}I_{34}\ne0$. Directly from Table II (for $R_{26}=0$) we obtain R_{36}, R_{46}, I_{23}, I_{24}, I_{34}, R_{16}, R_{25}, I_{12}, I_{56}, I_{26}, and $|b|^2$. The remaining quantities can be obtained, e.g., as

TABLE II. Three- and four-component tensors in terms of lower-order ones.

$$R_{35}=\frac{1}{R_{26}}(R_{23}R_{56}-I_{25}I_{36})=\mathrm{Re}c^*e=\frac{\sigma}{2}(C_{nll0}+C_{ln0l})$$

$$R_{45}=\frac{1}{R_{26}}(R_{24}R_{56}-I_{25}I_{46})=\mathrm{Re}d^*e=-\frac{\sigma}{2}(C_{lln0}+C_{ll0n})$$

$$R_{36}=\frac{1}{R_{15}}(R_{13}R_{56}-I_{16}I_{35})=\mathrm{Re}c^*f=\frac{\sigma}{2}(C_{lln0}-C_{ll0n})$$

$$R_{46}=\frac{1}{R_{15}}(R_{14}R_{56}-I_{16}I_{45})=\mathrm{Re}d^*f=\frac{\sigma}{2}(C_{ln0l}-C_{nll0})$$

$$I_{13}=\frac{1}{R_{26}}(R_{23}I_{16}-R_{12}I_{36})=\mathrm{Im}a^*c=\frac{\sigma}{2}(C_{nml0}-C_{nlm0})$$

$$I_{14}=\frac{1}{R_{26}}(R_{24}I_{16}-R_{12}I_{46})=\mathrm{Im}a^*d=-\frac{\sigma}{2}(C_{mln0}+C_{lmn0})$$

$$I_{23}=\frac{1}{R_{15}}(R_{13}I_{25}-R_{12}I_{35})=\mathrm{Im}b^*c=\frac{\sigma}{2}(C_{mln0}-C_{lmn0})$$

$$I_{24}=\frac{1}{R_{15}}(R_{14}I_{25}-R_{12}I_{45})=\mathrm{Im}b^*d=\frac{\sigma}{2}(C_{nml0}+C_{nlm0})$$

$$I_{34}=\frac{1}{R_{15}}(R_{14}I_{35}-R_{13}I_{45})=\frac{1}{R_{26}}(R_{24}I_{36}-R_{23}I_{46})=\mathrm{Im}c^*d=\frac{\sigma}{2}(C_{lnm0}+C_{nm0l})$$

$$R_{16}=R_{15}\frac{R_{13}I_{46}-R_{14}I_{36}}{R_{13}I_{45}-R_{14}I_{35}}=R_{26}\frac{R_{13}I_{46}-R_{14}I_{36}}{R_{23}I_{46}-R_{24}I_{36}}$$

$$=\frac{1}{R_{34}R_{15}R_{26}}[R_{14}R_{26}(R_{13}R_{56}-I_{16}I_{35})-R_{15}I_{46}(R_{23}I_{16}-R_{12}I_{36})]$$

$$=\mathrm{Re}a^*f=\frac{\sigma}{2}(C_{nl0l}-C_{lnl0})$$

$$R_{25}=R_{15}\frac{R_{23}I_{45}-R_{24}I_{35}}{R_{13}I_{45}-R_{14}I_{35}}=R_{26}\frac{R_{23}I_{45}-R_{24}I_{35}}{R_{23}I_{46}-R_{24}I_{36}}$$

$$=\frac{1}{R_{34}R_{15}R_{26}}[R_{15}R_{24}(R_{23}R_{56}-I_{25}I_{36})-R_{26}I_{45}(R_{13}I_{25}-R_{12}I_{35})]$$

$$=\mathrm{Re}b^*e=\frac{\sigma}{2}(C_{nl0l}+C_{lnl0})$$

$$I_{12}=R_{15}\frac{R_{14}R_{23}-R_{13}R_{24}}{R_{13}I_{45}-R_{14}I_{35}}=R_{26}\frac{R_{14}R_{23}-R_{13}R_{24}}{R_{23}I_{46}-R_{24}I_{36}}$$

$$=\frac{1}{R_{34}R_{15}R_{26}}[R_{15}R_{23}(R_{24}I_{16}-R_{12}I_{46})-R_{13}R_{26}(R_{14}I_{25}-R_{12}I_{45})]$$

$$=\mathrm{Im}a^*b=\frac{\sigma}{2}(C_{mnl0}-C_{lnm0})$$

$$I_{56}=R_{15}\frac{I_{36}I_{45}-I_{35}I_{46}}{R_{13}I_{45}-R_{14}I_{35}}=R_{26}\frac{I_{36}I_{45}-I_{35}I_{46}}{R_{23}I_{46}-R_{24}I_{36}}$$

$$=\frac{1}{R_{34}R_{15}R_{26}}[R_{15}I_{46}(R_{23}R_{56}-I_{25}I_{36})-R_{26}I_{45}(R_{13}R_{56}-I_{16}I_{35})]$$

$$=\mathrm{Im}e^*f=\frac{\sigma}{2}(C_{nm0l}-C_{mnl0})$$

$$I_{15}=\frac{R_{13}(R_{24}R_{56}-I_{25}I_{46})-R_{14}(R_{23}R_{56}-I_{25}I_{36})}{R_{24}I_{36}-R_{23}I_{46}}$$

$$=\frac{1}{R_{34}(R_{26})^2}[R_{26}{}^2R_{13}I_{45}+(R_{23}R_{56}-I_{25}I_{36})(R_{24}I_{16}-R_{12}I_{46})]=\mathrm{Im}a^*e=\frac{\sigma}{2}(C_{lllm}+C_{llml})$$

TABLE II. (*Continued.*)

$$I_{26}=\frac{R_{23}(R_{14}R_{56}-I_{16}I_{45})-R_{24}(R_{13}R_{56}-I_{16}I_{35})}{R_{14}I_{35}-R_{13}I_{45}}$$
$$=\frac{1}{R_{34}(R_{15})^2}[R_{15}{}^2R_{23}I_{46}+(R_{13}R_{56}-I_{16}I_{35})(R_{14}I_{25}-R_{12}I_{45})]=\mathrm{Im}b^*f=\frac{\sigma}{2}(C_{llml}-C_{lllm})$$
$$|a|^2=\frac{1}{R_{34}(R_{26})^2}[R_{13}R_{14}R_{26}{}^2+(R_{23}I_{16}-R_{12}I_{36})(R_{24}I_{16}-R_{12}I_{46})]$$
$$=\frac{R_{12}(R_{13}I_{46}-R_{14}I_{36})+I_{16}(R_{14}R_{23}-R_{13}R_{24})}{R_{23}I_{46}-R_{24}I_{36}}=\frac{\sigma}{2}(D_{n0n0}+K_{0nn0}+C_{llll}-C_{llmm})$$
$$|b|^2=\frac{1}{R_{34}(R_{15})^2}[R_{23}R_{24}R_{15}{}^2+(R_{13}I_{25}-R_{12}I_{35})(R_{14}I_{25}-R_{12}I_{45})]$$
$$=\frac{R_{12}(R_{23}I_{45}-R_{24}I_{35})+I_{25}(R_{13}R_{24}-R_{14}R_{23})}{R_{13}I_{45}-R_{14}I_{35}}=\frac{\sigma}{2}(D_{n0n0}-K_{0nn0}+C_{llll}+C_{llmm})$$
$$|e|^2=\frac{1}{R_{34}(R_{26})^2}[I_{35}I_{45}R_{26}{}^2+(R_{23}R_{56}-I_{25}I_{36})(R_{24}R_{56}-I_{25}I_{46})]$$
$$=\frac{R_{56}(R_{23}I_{45}-R_{24}I_{35})+I_{25}(I_{35}I_{46}-I_{36}I_{45})}{R_{23}I_{46}-R_{24}I_{36}}=\frac{\sigma}{2}(1+C_{nn00}-C_{llll}+C_{llmm})$$
$$|f|^2=\frac{1}{R_{34}(R_{15})^2}[I_{36}I_{46}R_{15}{}^2+(R_{13}R_{56}-I_{16}I_{35})(R_{14}R_{56}-I_{16}I_{45})]$$
$$=\frac{R_{56}(R_{13}I_{46}-R_{14}I_{36})-I_{16}(I_{35}I_{46}-I_{36}I_{45})}{R_{13}I_{45}-R_{14}I_{35}}=\frac{\sigma}{2}(1-C_{nn00}-C_{llll}-C_{llmm})$$

$$I_{13}=\frac{|c|^2R_{12}-R_{13}R_{23}}{I_{23}},\quad R_{35}=\frac{|c|^2I_{25}-R_{23}I_{35}}{I_{23}},\quad I_{14}=\frac{R_{13}I_{34}+R_{34}I_{13}}{|c|^2},$$
$$R_{45}=\frac{R_{34}R_{35}+I_{34}I_{35}}{|c|^2},\quad I_{15}=\frac{R_{13}I_{35}+R_{35}I_{13}}{|c|^2},\quad |a|^2=\frac{R_{13}{}^2+I_{13}{}^2}{|c|^2}. \tag{3.11}$$

A complete unambiguous reconstruction of all observables is again seen to be possible. If $I_{23}=0$ or $|c|^2=0$ a slight modification provides a reconstruction in terms of other denominators (this can be seen using the group G_0).

Finally, if in some region, e.g., that of high energies, *both* polarizations vanish, i.e., $R_{15}=R_{26}=0$, or for some reason these two polarizations are not measured, then an unambiguous reconstruction of the scattering matrix is impossible without measuring higher-order tensors. To see this it is sufficient to notice that the transformation

$$(a,b,c,d,e,f)\rightarrow(a^*,b^*,c^*,d^*,-e^*,-f^*) \tag{3.12}$$

leaves all observables in the set (2.5) invariant, except for A_{00n0} and A_{000n}, which change sign. For $R_{15}=R_{26}=0$ the two vectors in (3.12) must be distinguished by a measurement of, e.g., R_{i5} or R_{k6} $(i=2,3,4;k=1,3,4)$. Note that (3.12) is formally equivalent to the reflection $\vec{n}\rightarrow-\vec{n}$ in the scattering matrix.

IV. RELATIONS BETWEEN THE 19 SIMPLEST EXPERIMENTS

The 19 observables in the set (2.5) are expressed in terms of 6 complex amplitudes $a,\ldots,f$, the overall phase of these amplitudes does not enter, hence 19 observables are expressed in terms of 11 real functions. It follows that 8 independent nonlinear relations exist between them. Such relations are particularly important since they immediately provide a criterion for the incompleteness of a set of experiments.

For example, write expression (3.4) for the amplitudes $\{a,ebf\}$. We obtain

$$R_{15}R_{26}-R_{12}R_{56}+I_{16}I_{25}=0 \tag{4.1}$$

or in terms of c.m.s. experimental quantities

$$A_{00n0}{}^2 - A_{000n}{}^2 = (D_{0l0m}{}^2 - D_{l0m0}{}^2) + (D_{0m0m} - D_{m0m0})(D_{0m0m} + D_{l0l0}) \; . \tag{4.2}$$

It follows that if the two asymmetries A_{00n0} and A_{000n} are known no additional information is provided by measuring the five linearly independent components of the two depolarization tensors, D_{p0i0} and D_{0q0k}, rather than just four of them. The relation (4.2) makes it possible to calculate (up to a possible twofold ambiguity) the fifth component in terms of four measured ones. Alternatively, (4.2) can serve as a test of the experimental errors involved in a measurement of all five linearly independent components of these two tensors.

Relation (4.1) is the only second-order relation among the simplest observables. All other relations involve multiple applications of (3.3)–(3.8) and again no "canonical" choice of 8 relations exists. On the other hand, no end of possible relations can be written. We give a useful set of them in Table III. Relation (T III.1) is the same as (4.1) and is unique in being of second order. Relations (T III.2), (T III.3), and (T III.4) are, up to linear combinations, the only three third-order relations within the set (2.5) involving nondiagonal experiments only.

Relations (T III.1) and (T III.4) are invariant under the group G_0, relations (T III.2) and (T III.3) transform amongst each other.

The relations (T III.5), . . . , (T III.8) involve diagonal observables ($|c|^2$, $|d|^2$, $|a|^2+|e|^2$, or $|b|^2+|f|^2$) on the left-hand side, nondiagonal on the right. Relations (T III.6) and (T III.8) are obtained from (T III.5) and (T III.7) respectively, by applying the operator X_2 (X_1) of G_0. New relations within each of the (T III.5), . . . , (T III.8) sets can be obtained by applying the group G_0; we only give one representative of each G_0 class of relations within each set of formulas. Sets of 8 independent relations are obtained by taking, e.g., (T III.1), . . . , (T III.4) and one of each set of formulas (T III.5), . . . , (T III.8).

Relations (T III.5), . . . , (T III.8) are quite cumbersome when written explicitly in terms of c.m.s. or l.s. observables, though this is easily done using formulas (2.5). Relations (T III.1), . . . , (T III.4) are rewritten in terms of these observables in Table IV.

A way of seeing that the relations (T III.1), . . . , (T III.8) are independent is to use them for a complete reconstruction of all 19 "optimal" observables from 11 given ones. Let us assume that 11 nondiagonal experiments are performed, e.g., providing the set

$$\{R_{12}, R_{13}, R_{14}, R_{15}, R_{23}, R_{24}, R_{26}, I_{16}, I_{25}, I_{35}, I_{36}\} \; . \tag{4.3}$$

Relations (T III.1), . . . , (T III.4) now represent a system of linear inhomogeneous equations for R_{56}, R_{34}, I_{45}, and I_{46}. The determinant of the system is nonzero, hence it has a unique solution. Relations (T III.5), . . . , (T III.8) (one of each) then provide all diagonal optimal observables in terms of the set (4.3). The set (4.3) is hence complete: from (4.3) we obtain (2.5) linearly (and thus with no ambiguities); from (2.5) we obtain all observables using the formulas of Table II.

The set (4.3) is by no means unique; many such complete sets of 11 nondiagonal experiments exist (they can be classified into orbits under G_0). We shall not go into an analysis here since such sets, excluding the cross section σ, represent little experimental interest.

Relations (T III.1), . . . , (T III.8) also make it possible to identify incomplete sets. As an example consider a different set of 11 nondiagonal optimal observables:

$$\{R_{12}, R_{13}, R_{14}, R_{15}, R_{23}, R_{24}, R_{26}, R_{34}, I_{16}, I_{36}, I_{46}\} . \tag{4.4}$$

Relations (T III.1), . . . , (T III.3) provide R_{56}, I_{35}, and I_{45} linearly in terms of I_{25}. Relation (T III.4) is then satisfied identically and I_{25} remains unknown. Substituting for R_{56}, I_{35}, and I_{45} in terms of I_{25} into, e.g., (T III.5a) and (T III.5b) [or any other pair of relations in one of the sets (T III.5), . . . , (T III.8)], we find that I_{25} cancels out and we obtain an identity involving the set (4.4). The experiments in this set are hence not independent: the same information can be obtained from 10 experiments only and a reconstruction of the scattering matrix is hence impossible.

TABLE III. Relations between the 19 simplest observables (the "optimal" observables).

$$R_{15}R_{26}-R_{12}R_{56}+I_{16}I_{25}=0 \quad \text{(T III.1)}$$

$$R_{56}(R_{12}R_{34}-R_{14}R_{23})+I_{16}(R_{23}I_{45}-R_{34}I_{25})+I_{36}(R_{14}I_{25}-R_{12}I_{45})=0 \quad \text{(T III2)}$$

$$R_{56}(R_{12}R_{34}-R_{13}R_{24})+I_{16}(R_{24}I_{35}-R_{34}I_{25})+I_{46}(R_{13}I_{25}-R_{12}I_{35})=0 \quad \text{(T III.3)}$$

$$R_{26}(R_{13}I_{45}-R_{14}I_{35})+R_{15}(R_{24}I_{36}-R_{23}I_{46})=0 \quad \text{(T III.4)}$$

$$|c|^2=\frac{R_{13}(R_{23}R_{56}-I_{25}I_{36})-I_{35}(R_{23}I_{16}-R_{12}I_{36})}{R_{15}R_{26}} \quad \text{(T III.5a)}$$

$$=\frac{R_{23}(R_{14}I_{35}-R_{13}I_{45})+R_{34}(R_{13}I_{25}-R_{12}I_{35})}{R_{14}I_{25}-R_{12}I_{45}} \quad \text{(T III.5b)}$$

$$=\frac{R_{13}R_{23}R_{15}R_{26}+(R_{23}I_{16}-R_{12}I_{36})(R_{13}I_{25}-R_{12}I_{35})}{R_{12}R_{15}R_{26}} \quad \text{(T III.5c)}$$

$$=\frac{R_{13}R_{34}R_{15}R_{26}+(R_{23}I_{16}-R_{12}I_{36})(R_{13}I_{45}-R_{14}I_{35})}{R_{14}R_{15}R_{26}} \quad \text{(T III.5d)}$$

$$=R_{34}\frac{R_{13}R_{26}(R_{12}I_{35}-R_{13}I_{25})-R_{23}R_{15}(R_{12}I_{36}-R_{23}I_{16})}{R_{14}R_{26}(R_{12}I_{35}-R_{13}I_{25})-R_{24}R_{15}(R_{12}I_{36}-R_{23}I_{16})} \quad \text{(T III.5e)}$$

$$=(R_{24}I_{36}-R_{23}I_{46})\frac{R_{23}R_{15}{}^2I_{36}+(R_{13}R_{56}-I_{16}I_{35})(R_{13}I_{25}-R_{12}I_{35})}{R_{15}R_{26}[R_{23}(R_{14}R_{56}-I_{16}I_{45})-R_{24}(R_{13}R_{56}-I_{16}I_{35})]} \quad \text{(T III.5f)}$$

$$|d|^2=\frac{R_{14}(R_{24}R_{56}-I_{25}I_{46})-I_{45}(R_{24}I_{16}-R_{12}I_{46})}{R_{15}R_{26}} \quad \text{(T III.6a)}$$

$$=\frac{R_{24}(R_{13}I_{45}-R_{14}I_{35})+R_{34}(R_{14}I_{25}-R_{12}I_{45})}{R_{13}I_{25}-R_{12}I_{35}} \quad \text{(T III.6b)}$$

$$=\frac{R_{14}R_{24}R_{15}R_{26}+(R_{24}I_{16}-R_{12}I_{46})(R_{14}I_{25}-R_{12}I_{45})}{R_{12}R_{15}R_{26}} \quad \text{(T III.6c)}$$

$$=\frac{R_{14}R_{34}R_{15}R_{26}+(R_{24}I_{16}-R_{12}I_{46})(R_{14}I_{35}-R_{13}I_{45})}{R_{13}R_{15}R_{26}} \quad \text{(T III.6d)}$$

$$=R_{34}\frac{R_{14}R_{26}(R_{12}I_{45}-R_{14}I_{25})-R_{24}R_{15}(R_{12}I_{46}-R_{24}I_{16})}{R_{13}R_{26}(R_{12}I_{45}-R_{14}I_{25})-R_{23}R_{15}(R_{12}I_{46}-R_{24}I_{16})} \quad \text{(T III.6e)}$$

$$=(R_{23}I_{46}-R_{24}I_{36})\frac{R_{24}R_{15}{}^2I_{46}+(R_{14}R_{56}-I_{16}I_{45})(R_{14}I_{25}-R_{12}I_{45})}{R_{15}R_{26}[R_{24}(R_{13}R_{56}-I_{16}I_{35})-R_{23}(R_{14}R_{56}-I_{16}I_{45})]} \quad \text{(T III.6f)}$$

TABLE III. (*Continued.*)

$$|a|^2+|e|^2=\frac{R_{12}(R_{14}I_{36}-R_{13}I_{46})+I_{16}(R_{13}R_{24}-R_{14}R_{23})+R_{56}(R_{24}I_{35}-R_{23}I_{45})-I_{25}(I_{35}I_{46}-I_{36}I_{45})}{R_{24}I_{36}-R_{23}I_{46}} \quad \text{(TIII.7a)}$$

$$=\frac{R_{26}{}^2(R_{13}R_{14}+I_{35}I_{45})+(R_{23}R_{56}-I_{25}I_{36})(R_{24}R_{56}-I_{25}I_{46})+(R_{23}I_{16}-R_{12}I_{36})(R_{24}I_{16}-R_{12}I_{46})}{R_{34}R_{26}{}^2} \quad \text{(TIII.7b)}$$

$$=\frac{R_{15}R_{26}[R_{13}(R_{14}I_{36}-R_{13}I_{46})-I_{35}(I_{35}I_{46}-I_{36}I_{45})]+(R_{14}I_{35}-R_{13}I_{45})[R_{56}(R_{23}R_{56}-I_{25}I_{36})+I_{16}(R_{23}I_{16}-R_{12}I_{36})]}{(R_{13}R_{56}-I_{16}I_{35})(R_{24}I_{36}-R_{23}I_{46})} \quad \text{(TIII.7c)}$$

$$=\frac{(R_{14}I_{35}-R_{13}I_{45})^2(R_{12}I_{16}+R_{56}I_{25})+R_{15}{}^2[(R_{13}I_{46}-R_{14}I_{36})(R_{13}R_{24}-R_{14}R_{23})+(R_{24}I_{35}-R_{23}I_{45})(I_{35}I_{46}-I_{36}I_{45})]}{(R_{14}I_{35}-R_{13}I_{45})[I_{16}(R_{24}I_{35}-R_{23}I_{45})-R_{56}(R_{13}R_{24}-R_{14}R_{23})]} \quad \text{(TIII.7d)}$$

$$|b|^2+|f|^2=\frac{R_{12}(R_{24}I_{35}-R_{23}I_{45})+I_{16}(I_{35}I_{46}-I_{36}I_{45})+R_{56}(R_{14}I_{36}-R_{13}I_{46})-I_{25}(R_{13}R_{24}-R_{14}R_{23})}{R_{14}I_{35}-R_{13}I_{45}} \quad \text{(TIII.8a)}$$

$$=\frac{R_{15}{}^2(R_{23}R_{24}+I_{36}I_{46})+(R_{13}R_{56}-I_{16}I_{35})(R_{14}R_{56}-I_{16}I_{45})+(R_{13}I_{25}-R_{12}I_{35})(R_{14}I_{25}-R_{12}I_{45})}{R_{34}R_{15}{}^2} \quad \text{(TIII.8b)}$$

$$=\frac{R_{15}R_{26}[R_{23}(R_{24}I_{35}-R_{23}I_{45})-I_{36}(I_{36}I_{45}-I_{35}I_{46})]+(R_{24}I_{36}-R_{23}I_{46})[R_{56}(R_{13}R_{56}-I_{16}I_{35})+I_{25}(R_{13}I_{25}-R_{12}I_{35})]}{(R_{23}R_{56}-I_{25}I_{36})(R_{14}I_{35}-R_{13}I_{45})} \quad \text{(TIII.8c)}$$

$$=\frac{(R_{24}I_{36}-R_{23}I_{46})^2(R_{12}I_{25}+R_{56}I_{16})+R_{26}{}^2[(R_{23}I_{45}-R_{24}I_{35})(R_{14}R_{23}-R_{13}R_{24})+(R_{14}I_{36}-R_{13}I_{46})(I_{36}I_{45}-I_{35}I_{46})]}{(R_{24}I_{36}-R_{23}I_{46})[I_{25}(R_{14}I_{36}-R_{13}I_{46})-R_{56}(R_{14}R_{23}-R_{13}R_{24})]} \quad \text{(TIII.8d)}$$

TABLE IV. Relations (T III.1) to (T III.4) in terms of pure experiments in the center-of-mass and laboratory systems. Equations (T IV.2b), (T IV.3b), (T IV.4b) are examples of such relations in the l.s. For instance, a choice is possible between the two given quantities in the curved brackets on the left-hand side of (T IV.2b) and (T IV.3b). Here $\alpha=\theta/2-\theta_1$, $\beta=\theta/2+\theta_2$ where θ, θ_1, and θ_2 are the c.m.s. scattering, l.s. scattering, and recoil angles, respectively.

c.m.s.

$$A_{00n0}^2-A_{000n}^2=(D_{0l0m}^2-D_{l0m0}^2)+(D_{0m0m}-D_{m0m0})(D_{0m0m}+D_{l0l0}) \qquad \text{(T IV.1a)}$$

$$(A_{00n0}^2-A_{000n}^2)(D_{m0m0}-D_{l0l0})$$
$$=(D_{0m0m}-D_{m0m0})(C_{mm00}^2-C_{ll00}^2)+(D_{0m0m}+D_{l0l0})(C_{lm00}^2-C_{ml00}^2)+2[D_{0l0m}(C_{mm00}C_{ml00}+C_{ll00}C_{lm00})-D_{l0m0}(C_{mm00}C_{lm00}+C_{ll00}C_{ml00})] \qquad \text{(T IV.2a)}$$
$$=(D_{0m0m}-D_{m0m0})(K_{0mm0}^2-K_{0ll0}^2)+(D_{0m0m}+D_{l0l0})(K_{l00m}^2-K_{0lm0}^2)+2[D_{0l0m}(K_{0mm0}K_{0lm0}-K_{0ll0}K_{l00m})-D_{l0m0}(K_{0mm0}K_{l00m}-K_{0ll0}K_{0lm0})] \qquad \text{(T IV.3a)}$$

$$A_{00n0}(K_{0mm0}C_{lm00}+K_{0ll0}C_{ml00}+C_{ll00}K_{0lm0}-C_{mm00}K_{l00m})=A_{000n}(K_{0mm0}C_{ml00}+K_{0ll0}C_{lm00}+C_{ll00}K_{l00m}-C_{mm00}K_{0lm0}) \qquad \text{(T IV.4a)}$$

l.s.

$$4(A_{00n0}^2-A_{000n}^2)=(D_{0s''0s}+D_{0k''0k})^2+(D_{0s''0k}-D_{0k''0s})^2-(D_{s'0s0}+D_{k'0k0})^2-(D_{s'0k0}-D_{k'0s0})^2 \qquad \text{(T IV.1b)}$$

$$2(A_{000n}^2-A_{00n0}^2)\begin{Bmatrix}[(D_{s'0s0}-D_{k'0k0})\cos(\alpha-\theta/2)+(D_{s'0k0}+D_{k'0s0})\sin(\alpha-\theta/2)]\\ [(D_{0k''0k}-D_{0s''0s})\cos(\beta-\theta/2)-(D_{0s''0k}+D_{0k''0s})\sin(\beta-\theta/2)]\end{Bmatrix}$$
$$=(A_{00ss}^2+A_{00sk}^2-A_{00ks}^2-A_{00kk}^2)[(D_{s'0s0}+D_{k'0k0})\cos(\alpha-\theta/2)-(D_{s'0k0}-D_{k'0s0})\sin(\alpha-\theta/2)]$$
$$+(A_{00ss}^2-A_{00sk}^2+A_{00ks}^2-A_{00kk}^2)[(D_{0s''0s}+D_{0k''0k})\cos(\beta-\theta/2)-(D_{0s''0k}-D_{0k''0s})\sin(\beta-\theta/2)]$$
$$+2(A_{00ss}A_{00ks}+A_{00kk}A_{00sk})[(D_{s'0s0}+D_{k'0k0})\sin(\alpha-\theta/2)+(D_{s'0k0}-D_{k'0s0})\cos(\alpha-\theta/2)]$$
$$+2(A_{00ss}A_{00sk}+A_{00kk}A_{00ks})[(D_{0s''0s}+D_{0k''0k})\sin(\beta-\theta/2)+(D_{0s''0k}-D_{0k''0s})\cos(\beta-\theta/2)] \qquad \text{(T IV.2b)}$$
$$=(K_{s'00s}^2+K_{s'00k}^2-K_{k'00s}^2-K_{k'00k}^2)[(D_{s'0s0}+D_{k'0k0})\cos(\alpha-\theta/2)+(D_{s'0k0}-D_{k'0s0})\sin(\alpha-\theta/2)]$$
$$+(K_{s'00s}^2-K_{s'00k}^2+K_{k'00s}^2-K_{k'00k}^2)[(D_{0s''0s}+D_{0k''0k})\cos(\beta-\theta/2)-(D_{0s''0k}-D_{0k''0s})\sin(\beta-\theta/2)]$$
$$+2(K_{s'00s}K_{k'00s}+K_{k'00k}K_{s'00k})[(D_{s'0s0}+D_{k'0k0})\sin(\alpha-\theta/2)-(D_{s'0k0}-D_{k'0s0})\cos(\alpha-\theta/2)]$$
$$+2(K_{s'00s}K_{s'00k}+K_{k'00k}K_{k'00s})[(D_{0s''0s}+D_{0k''0k})\sin(\beta-\theta/2)+(D_{0s''0k}-D_{0k''0s})\cos(\beta-\theta/2)] \qquad \text{(T IV.3b)}$$

$$A_{00n0}[\sin(\beta-\theta/2)(A_{00ss}K_{0k''k0}-A_{00kk}K_{0s''s0}+A_{00sk}K_{0s''k0}-A_{00ks}K_{0k''s0})+\cos(\beta-\theta/2)(A_{00ss}K_{0s''k0}+A_{00kk}K_{0k''s0}-A_{00sk}K_{0k''k0}-A_{00ks}K_{0s''s0})]$$
$$=A_{000n}[\sin(\beta-\theta/2)(A_{00ss}K_{0s''s0}-A_{00kk}K_{0k''k0}-A_{00sk}K_{0k''s0}+A_{00ks}K_{0s''k0})$$
$$-\cos(\beta-\theta/2)(A_{00ss}K_{0k''s0}+A_{00kk}K_{0s''k0}+A_{00sk}K_{0s''s0}+A_{00ks}K_{0k''k0})] \qquad \text{(T IV.4b)}$$

V. COMMENTS ON THE DIRECT RECONSTRUCTION OF THE SCATTERING MATRIX

Let us first show that optimal complete natural sets of 12 experiments exist, i.e., that 12 well chosen experiments (in the c.m.s. or l.s.) are sufficient to reconstruct the scattering matrix.

A. Set involving A_{00cd} and K_{0bc0}

Consider a set consisting of the unpolarized cross section σ, the two asymmetries A_{000n} and A_{00n0}, any 7 of the 8 components of the polarization correlation tensor C, and the polarization transfer tensor K with both labels in the scattering plane and any two of the three diagonal quantities D_{n0n0}, K_{0nn0}, and C_{nn00}. Relation (T III.4), or equivalently (T IV.4a) or (T IV.4b) will then unambiguously provide us with the missing component of K or C.

Thus, the set

$$\{R_{13},R_{14},R_{15},R_{23},R_{24},R_{26},I_{35},I_{36},I_{45},I_{46}\}$$

is obtained by performing 9 measurements. From Table II we also obtain I_{34}, I_{12}, R_{25}, R_{16}, and I_{56} in terms of the above set. We make use of the arbitrariness of the overall phase and postulate that the amplitude c is real and positive (we assume $c\neq0$). Denoting the real and imaginary parts of an amplitude x as x_1 and x_2, respectively, we express the real and imaginary parts of $a,\ldots,f$ in terms of c, a_2, and the measured or calculated observables:

$$\begin{aligned}
&a_1=\frac{R_{13}}{c}\ ,\quad b_1=\frac{R_{23}}{c}\ ,\\
&c_1=c\ ,\quad d_1=\frac{R_{14}c-I_{34}a_2}{R_{13}}\ ,\\
&e_1=\frac{R_{15}c-I_{35}a_2}{R_{13}}\ ,\quad f_1=\frac{R_{16}c-I_{36}a_2}{R_{13}}\ ,\\
&a_2\ ,\quad b_2=\frac{R_{23}a_2+I_{12}c}{R_{13}}\ ,\\
&c_2=0\ ,\quad d_2=\frac{I_{34}}{c}\ ,\\
&e_2=\frac{I_{35}}{c}\ ,\quad f_2=\frac{I_{36}}{c}\ .
\end{aligned} \tag{5.1}$$

In order to obtain a_2 and c we make use of the cross section σ and two additional diagonal observables, e.g., K_{0nn0} and C_{nn00}. We find

$$a_2=X_1c+\frac{Y_1}{c}=X_2c+\frac{Y_2}{c}\ , \tag{5.2a}$$

$$c=\left[\frac{Y_2-Y_1}{X_1-X_2}\right]^{1/2}\ , \tag{5.2b}$$

where

$$\begin{aligned}
&X_1=-\frac{A_0E_K-A_KE_0}{2\Delta_K}\ ,\\
&Y_1=-\frac{\sigma R_{13}{}^2(A_K-K_{0nn0}A_0)}{\Delta_K}\ ,\\
&X_2=-\frac{A_0E_C-A_CE_0}{2\Delta_C}\ ,\\
&Y_2=-\frac{\sigma R_{13}{}^2(A_C-C_{nn00}A_0)}{\Delta_C}\ ,
\end{aligned} \tag{5.3a}$$

$$\begin{aligned}
&A_0=R_{13}{}^2+R_{23}{}^2+I_{34}{}^2+I_{35}{}^2+I_{36}{}^2\ ,\\
&E_0=R_{13}{}^2+R_{14}{}^2+R_{15}{}^2+R_{16}{}^2+I_{12}{}^2\ ,\\
&A_K=R_{13}{}^2-R_{23}{}^2-I_{34}{}^2+I_{35}{}^2-I_{36}{}^2\ ,\\
&E_K=R_{13}{}^2-R_{14}{}^2+R_{15}{}^2-R_{16}{}^2-I_{12}{}^2\ ,\\
&A_C=R_{13}{}^2-R_{23}{}^2+I_{34}{}^2+I_{35}{}^2-I_{36}{}^2\ ,\\
&E_C=-R_{13}{}^2+R_{14}{}^2+R_{15}{}^2-R_{16}{}^2-I_{12}{}^2\ ,
\end{aligned} \tag{5.3b}$$

$$\Delta_K=A_0B_K-A_KB_0\ ,\quad \Delta_C=A_0B_C-A_CB_0\ , \tag{5.3c}$$

$$\begin{aligned}
&B_0=R_{23}I_{12}-R_{14}I_{34}-R_{15}I_{35}-R_{16}I_{36}\ ,\\
&B_K=-R_{23}I_{12}+R_{14}I_{34}-R_{15}I_{35}+R_{16}I_{36}\ ,\\
&B_C=-R_{23}I_{12}-R_{14}I_{34}-R_{15}I_{35}+R_{16}I_{36}\ .
\end{aligned} \tag{5.3d}$$

This reconstruction is possible as long as $c\neq0$, $\Delta_K\neq0$, $\Delta_C\neq0$. These quantities may vanish for some energies and angles but they do not vanish identically. The above reconstruction made use of 12 measured observables. Note that if only one of the diagonal measurements was performed (e.g., K_{0nn0}) in addition to σ, then (5.1) and (5.2a) would provide the real and imaginary parts of all amplitudes in terms of c and we would have a quadratic equation for $|c|^2$. This would provide $|c|^2$ (and hence $c=|c|>0$) with at most a twofold ambiguity.

We have thus shown that, e.g., the set of 12 observables

$$\{\sigma,A_{00n0},A_{000n},A_{00ss},A_{00kk},A_{00sk},A_{00ks}\ ,\ K_{0s''s0},K_{0s''k0},K_{0k''s0},A_{00nn},K_{0nn0}\} \tag{5.4}$$

constitutes an optimal complete natural set and, e.g., the set of 11 observables

$$\{\sigma, A_{00n0}, A_{000n}, A_{00ss}, A_{00kk}, A_{00sk}, A_{00ks}, K_{0s''s0}, K_{0s''k0}, K_{0k''s0}, A_{00nn}\} \tag{5.5}$$

constitutes an optimal sufficient natural set. Since any 7 of the 8 linearly independent quantities $A_{00ss}, \ldots, K_{0s''s0}, \ldots$, could have been chosen and any 2 of $A_{00nn}, K_{0nn0}, D_{n0n0}$ in (5.4) or any one of them in (5.5), there exist 24 complete sets of type (5.4) and 24 sufficient sets of type (5.5).

B. Set involving components of D_{a0c0}, D_{0b0d}, and C_{ab00}

A different natural set of observables consists of all 5 linearly independent scattering-plane components of the 2 depolarization tensors D, 4 scattering-plane components of the polarization correlation tensor C, the 2 polarizations and an appropriate number of diagonal experiments. In view of relations (T III.1) and (T III.2) [i.e., (T IV.1a) and (T IV.2a), or (T IV.1b) and (T IV.2b)], the above 11 nondiagonal quantities actually represent 9 experiments. Once A_{00n0}, A_{000n}, C_{mm00}, C_{ll00}, C_{lm00}, and C_{ml00} are measured (T III.1) and (T III.2) can be solved to obtain the missing components of D.

Thus, performing 9 experiments we obtain the 11 quantities

$$\{R_{12}, R_{14}, R_{15}, R_{23}, R_{26}, R_{34}, R_{56}, I_{16}, I_{25}, I_{36}, I_{45}\} .$$

Directly from Table II we can obtain R_{35}, R_{46}, I_{13}, and I_{24}. Making an appropriate choice for the overall phase we arrange for a to be real and positive. All amplitudes can then be expressed in terms of the above nondiagonal observables and, e.g., $a_1 = a$ and b_2:

$$\begin{aligned}
&a_1 = a , \quad b_1 = \frac{R_{12}}{a} , \quad c_1 = \frac{1}{R_{12}}(R_{23}a - I_{13}b_2) ,\\
&d_1 = \frac{R_{14}}{a} , \quad e_1 = \frac{R_{15}}{a} , \quad f_1 = \frac{1}{R_{12}}(R_{26}a - I_{16}b_2) ,\\
&a_2 = 0 , \quad b_2 ,\\
&c_2 = \frac{I_{13}}{a} , \quad d_2 = \frac{1}{R_{12}}(I_{24}a + R_{14}b_2) ,\\
&e_2 = \frac{1}{R_{12}}(I_{25}a + R_{15}b_2) , \quad f_2 = \frac{I_{16}}{a} .
\end{aligned} \tag{5.6}$$

Now let us make use of the cross section σ and two diagonal observables, e.g., D_{n0n0} and C_{nn00}. We find

$$b_2 = W_1 a + \frac{Z_1}{a} = W_2 a + \frac{Z_2}{a} , \tag{5.7a}$$

$$a = \left[\frac{Z_2 - Z_1}{W_1 - W_2} \right]^{1/2} , \tag{5.7b}$$

where

$$\begin{aligned}
W_1 &= -\frac{S_D T_0 - S_0 T_D}{2\delta_D} ,\\
Z_1 &= -\frac{\sigma R_{12}^2 (T_D - T_0 D_{n0n0})}{\delta_D} ,\\
W_2 &= -\frac{S_C T_0 - S_0 T_C}{2\delta_C} ,\\
Z_2 &= -\frac{\sigma R_{12}^2 (T_C - T_0 C_{nn00})}{\delta_C} ,
\end{aligned} \tag{5.8a}$$

$$\begin{aligned}
T_0 &= R_{12}^2 + R_{14}^2 + R_{15}^2 + I_{13}^2 + I_{16}^2 ,\\
S_0 &= R_{12}^2 + R_{23}^2 + R_{26}^2 + I_{24}^2 + I_{25}^2 ,\\
T_D &= R_{12}^2 - R_{14}^2 + R_{15}^2 - I_{13}^2 + I_{16}^2 ,\\
S_D &= R_{12}^2 - R_{23}^2 + R_{26}^2 - I_{24}^2 + I_{25}^2 ,\\
T_C &= -R_{12}^2 + R_{14}^2 + R_{15}^2 - I_{13}^2 - I_{16}^2 ,\\
S_C &= R_{12}^2 - R_{23}^2 - R_{26}^2 + I_{24}^2 + I_{25}^2 ,
\end{aligned} \tag{5.8b}$$

$$\delta_D = T_0 Q_D - T_D Q_0 , \quad \delta_C = T_0 Q_C - T_C Q_0 , \tag{5.8c}$$

$$\begin{aligned}
Q_0 &= -R_{23}I_{13} - R_{26}I_{16} + R_{14}I_{24} + R_{15}I_{25} ,\\
Q_D &= R_{23}I_{13} - R_{26}I_{16} - R_{14}I_{24} + R_{15}I_{25} ,\\
Q_C &= R_{23}I_{13} + R_{26}I_{16} + R_{14}I_{24} + R_{15}I_{25} .
\end{aligned} \tag{5.8d}$$

This reconstruction is possible if $\delta_D \neq 0$, $\delta_C \neq 0$, and $a \neq 0$; these quantities certainly do not vanish identically. The same comments apply as after formulas (5.3). Thus, e.g., the set of 12 "observables"

$$\{\sigma, R_{14}, R_{15}, R_{23}, R_{26}, R_{56}, I_{16}, I_{25}, I_{36}, I_{45}, D_{n0n0}, C_{nn00}\} \tag{5.9}$$

is an optimal complete natural set and, e.g., the set of 11 "observables"

$$\{\sigma, R_{14}, R_{15}, R_{23}, R_{26}, R_{56}, I_{16}, I_{25}, I_{36}, I_{45}, D_{n0n0}\} \tag{5.10}$$

is an optimal sufficient natural one. Choosing different diagonal experiments or different components of the tensor D we obtain many more such sets. The discussion is similar when the tensor C

is replaced by the tensor K.

To conclude this section let us call attention to the importance of evaluating the completeness of a set of experiments. Thus it is possible to measure as many as 25 observables without obtaining any information about one of the amplitudes, 26 that still leave a continuous ambiguity, and 27 that leave a discrete ambiguity. An example of such a "disastrous" choice of 25 experiments is, e.g., the set of all R_{ik} and I_{ik} for $i,k\neq5$ and $|a|^2$, $|b|^2$, $|c|^2$, $|d|^2$, $|f|^2$.

VI. NUMERICAL APPLICATIONS OF THE NONLINEAR RELATIONS AND A "SEMIDIRECT" RECONSTRUCTION AT $E_{\rm lab}=425$ MeV, $\theta=65°$

A. Formulation of the problem and the scattering data

It is at this stage somewhat too early to attempt a serious direct reconstruction of the np scattering amplitudes since a complete experiment has yet to be performed. To illustrate the use of the nonlinear relations between observables and to perform a preliminary study of the propagation of errors and the numerical problems involved, we adopt a hybrid approach. We choose the point $(E_{\rm lab},\theta_{\rm c.m.})=(425\text{ MeV},65°)$ at which, or close to which, 11 different np scattering experiments have actually been performed. Taking the data from a recent compilation[22] and interpolating linearly in energy and/or angle whenever necessary, we obtain the following values of optimal observables:

$$A_{00n0}=0.087\pm0.012\ ,\quad A_{000n}=0.107\pm0.017\ , \tag{6.1a}$$

$$K_{0s''s0}=0.022\pm0.096\ ,\quad K_{0s''k0}=-0.079\pm0.110\ ,\quad K_{0k''s0}=-0.080\pm0.160\ , \tag{6.1b}$$

$$D_{s'0s0}=0.449\pm0.029\ ,\quad D_{s'0k0}=0.671\pm0.031\ , \tag{6.1c}$$

$$K_{0nn0}=-0.364\pm0.184\ ,\quad A_{00nn}=0.174\pm0.126\ , \tag{6.1d}$$

$$\sigma=1.725\pm0.086\text{ mb/sr}\ , \tag{6.1e}$$

$$D_{n0n0}=0.886\pm0.034\ . \tag{6.1f}$$

Conspicuously missing in the set (6.1) are components of the asymmetry tensor A_{00cd}. These will presumably soon be measured, since they actually involve the simplest of the two-component experiments, once a polarized beam and target are simultaneously available. We make use of the fact that an np phase-shift analysis is available[23] at 425 MeV. This analysis provides the following values:

$$A_{00ss}=-0.050\pm0.013\ ,\quad A_{00kk}=-0.201\pm0.025\ ,\quad A_{00sk}=A_{00ks}=0.062\pm0.006\ , \tag{6.2a}$$

and also

$$D_{k'0s0}=-0.821\pm0.009 \tag{6.2b}$$

We shall, below, treat the values (6.2) on the same footing as (6.1), i.e., as if they represented measured data. A reconstruction of amplitudes that makes use of observables from (6.1) and (6.2) will be called a "semidirect reconstruction."

The laboratory-system scattering and recoil angles at the considered point are

$$\theta_1=29.9°\ ,\quad \theta_2=54.8°\ , \tag{6.3}$$

respectively.

In order to reconstruct both amplitudes and other experimental quantities we use a Monte Carlo method, similar to the one used by Johnson *et al.*[14] for pp scattering. Each of the variables from the (6.1) or (6.2) set used in a reconstruction will be randomly and independently generated about its measured (or phase-shift) value within the given error. Since large experimental errors are involved we found it preferable to use a uniform random distribution, rather than, say, a Gaussian one. This obviously increases the spread of the calculated quantities. On the other hand, an artificial confinement of the data causing a peak at the given imprecise experimental value, can, and in fact does, make a semidirect reconstruction of the amplitudes impossible.

B. Use of nonlinear relations to generate further optimal data points

The relation (T IV.4b) can be used to obtain the experimentally unknown component $K_{0k''k0}$ from the quantities (6.1a), (6.1b), and (6.2a). An input of $N=20\,000$ simulations turns out to provide a good stability and the result is

$$K_{0k''k0}=0.130\pm0.091\quad[-0.139,0.396]\ . \tag{6.4}$$

To the right of the mean value we give the standard deviation (statistical error) and in brackets the interval that gives the observed lower and upper bounds of the output distribution, respectively.

Note that the phase-shift-analysis value of (6.4) is $K_{0k''k0}=0.158\pm0.019$ and the two values are in good agreement. This is encouraging from the point of view of applying the nonlinear relations in energy regions where phase-shift analyses are not available. It also indicates that at this energy and angle a measurement of $K_{0k''k0}$ would not be an efficient way of decreasing the influence of errors on an amplitude reconstruction since the statistical error in (6.4) is less than the experimental errors in (6.1b).

In order to use relation (T IV.1b) we must select 5 linearly independent components of the two depolarization tensors. Let us choose these to be

$$\{D_{s'0s0},D_{s'0k0},D_{k'0s0},D_{0s''0s},D_{0s''0k}\}\ .$$

The first two have been measured (6.1c), for the third we take the phase-shift-analysis value (6.2b).

Relation (T IV.1b) now reduces to the equation of a circle in the two unknown experimental quantities

$$x=D_{0s''0s},\quad y=D_{0s''0k}\ ,$$

namely

$$\begin{aligned}&(x-x_0)^2+(y-y_0)^2=r^2\ ,\\&x_0=\frac{\cos\theta_2}{2\sin\theta_1}(D_{s'0k0}+D_{k'0s0})\ ,\\&y_0=x_0\tan\theta_2\ ,\\&r^2=(A_{00n0}{}^2-A_{000n}{}^2)\\&\quad+\tfrac{1}{4}[2D_{s'0s0}+\cot\theta_1(D_{s'0k0}+D_{k'0s0})]^2\\&\quad+\tfrac{1}{4}(D_{s'0k0}-D_{k'0s0})^2\ .\end{aligned}\tag{6.5}$$

A first realistic application consists in attaching systematically an error of ±0.100 (this is larger than the usual experimental errors for this case) to each value of $D_{0s''0k}$ ($D_{0s''0s}$) ranging from -1 to $+1$ and then calculating the corresponding $D_{0s''0s}$ ($D_{0s''0k}$). Since a square root is present in each such process, some simulations may generate a negative radicand, preventing thereby the determination of x or y. Such events are rejected and a number of approximately 20000 simulations in each calculation again provides a very good stability of the results.

Up to 100000 simulations were tried and rejected in some cases. The results are shown in Tables V and VI, where the numbers to the right of the means have the same meaning as those in Eq. (6.4). A rejection of all simulations is indicated by a blank for the unattained output value. The asterisk in Table V means that only one event in 100000 was accepted in that particular case.

An idealized version of the preceding application can be made if one assumes that $D_{0s''0s}$ ($D_{0s''0k}$) is derived from *precise* experimental values of $D_{0s''0k}$ ($D_{0s''0s}$). In this refined case, the outline of the ring in the (x,y) plane becomes smooth as shown in Fig. 1. We see that only a rather restricted crescent in the $(D_{0s''0s},D_{0s''0k})$ plane is allowed by the previously known data (6.1a), (6.1c), and (6.2b), indicating that a measurement of this optimal variable would not decrease errors in an amplitude reconstruction significantly.

C. Determination of three-component polarization tensors

The determination of components of higher-order tensors from the lower-order ones can be useful in the direct reconstruction. The sets of experiments (6.1a), (6.1b), the phase-shift values (6.2a), and the calculated observable (6.4) now provide the set

$$S=\{R_{13},R_{14},R_{15},R_{23},R_{24},R_{26},I_{35},I_{36},I_{45},I_{46}\}$$

discussed in Sec. V. Table II allows us to determine $\{R_{16},R_{25},I_{12},I_{34},I_{56}\}$, or equivalently $C_{nl0l},C_{lnl0},C_{lnm0},C_{mnl0},C_{nm0l}$ which are finally reconverted into the eight linearly dependent observables of C_{anc0} and C_{nb0d}. As indicated previously the constant distribution of the input statistic can give unacceptable output values so that it is necessary to impose a physical constraint, namely,

TABLE V. Determination of $D_{0s''0k}$ from assumed imprecise values of $D_{0s''0s}$.

$D_{0s''0s}$	$D_{0s''0k}$		$D_{0s''0s}$	$D_{0s''0k}$	
-0.100 ± 0.100	0.215 ± 0.040	[0.103,0.305]	0.100 ± 0.100	0.116 ± 0.064	[−0.082,0.258]
-0.300 ± 0.100	0.235 ± 0.032	[0.166,0.307]	0.300 ± 0.100	-0.087 ± 0.104	[−0.388,0.133]
-0.500 ± 0.100	0.185 ± 0.039	[0.082,0.283]	0.500 ± 0.100	-0.272	(*)
-0.700 ± 0.100	0.036 ± 0.071	[−0.160,0.189]	0.700 ± 0.100		
-0.900 ± 0.100	-0.187 ± 0.066	[−0.383,−0.044]	0.900 ± 0.100		

TABLE VI. Determination of $D_{0s''0s}$ from assumed imprecise values of $D_{0s''0k}$.

$D_{0s''0k}$	$D_{0s''0s}$		$D_{0s''0k}$	$D_{0s''0s}$	
-0.100 ± 0.100	0.281 ± 0.048	[0.134,0.386]	0.100 ± 0.100	0.115 ± 0.099	[−0.270,0.309]
-0.300 ± 0.100	0.340 ± 0.030	[0.259,0.405]	0.300 ± 0.100	-0.065 ± 0.079	[−0.274,0.119]
-0.500 ± 0.100	0.328 ± 0.027	[0.258,0.399]	0.500 ± 0.100		
-0.700 ± 0.100	0.239 ± 0.044	[0.125,0.342]	0.700 ± 0.100		
-0.900 ± 0.100	0.030 ± 0.097	[−0.270,0.197]	0.900 ± 0.100		

that the resulting observables be included in the interval [-1,1]. This selection rejected between 3% and 10% of the 300 000 events submitted, from which we obtain a relatively good stability of the results, presented in Table VII. As expected these are rather small and accompanied by important statistical errors. A measurement of one of the components of C_{nb0d} or C_{anc0} would hence be helpful in eliminating ambiguities due to the presence of experimental errors (as opposed to the idealized case of no errors where the three- and four-component tensors are not needed.)

D. Semidirect reconstruction of the scattering amplitudes

Let us now consider the measured observables (6.1a), (6.1b), (6.1d), (6.1e) together with the phase-shift-analysis quantities (6.2a) and the observable (6.4) calculated using (T IV.4b). They form the optimal complete natural set of 12 observables (5.4). We can hence use formulas (5.1) to (5.3) to perform a numerical "semidirect" reconstruction of the scattering amplitudes at $E_{\rm lab}=425$ MeV, $\theta_{\rm c.m.}=65°$. Once again simulations leading to a negative radicand in (5.2b) were rejected. They represent approximately 59.8% of the events. A self-consistency constraint was imposed, namely that the outcoming amplitudes reproduce the value of σ within its experimental error. This selection rejected almost 39.7% of the events. Only 5000 simulations were successfully treated, which is far from sufficient to guarantee the stability of the results. Statistical errors are large and the output distribution is quite broad, so that the means here should be taken only as qualitative indications.

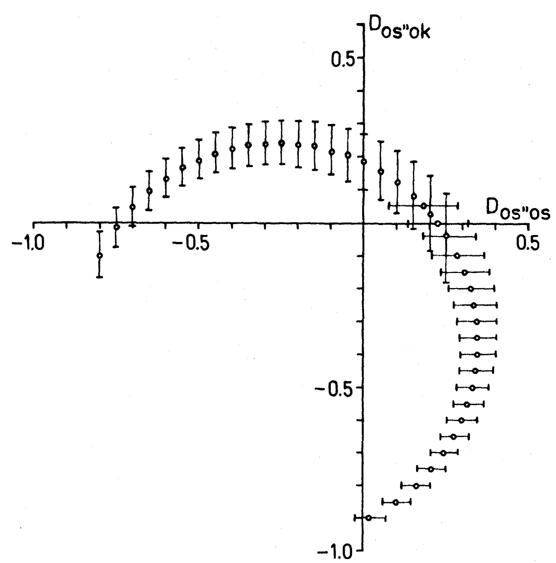

FIG. 1. Predicted values of $D_{0s''0s}$ and $D_{0s''0k}$ when no correlation is assumed between their errors. The missing part of the circle corresponds to values never attained for the number of simulations tried.

The results of the reconstruction are given in Table VIII. Two reconstructions are represented in this table. The semidirect reconstruction (from the data supplemented by phase-shift-analysis values for A_{00cd}) and from the observables (5.4) completely calculated on the basis of the phase-shift analysis. The second set of values for the scattering amplitudes does not coincide with the values given by Bystricky, Lechanoine, and Lehar,[23] since our phase convention is Im$c=0$, Re$c>0$, whereas their overall phase is determined by the one-pion-exchange amplitudes used for higher partial waves. The numbers in Table VIII are ordered in the same way as in Eq. (6.4): After the mean value we give the standard deviation and the interval representing the output distribution of each amplitude.

Both reconstructions involve large errors and the mean values obtained in the two cases are in poor agreement with each other. The influence of the experimental errors is large. On the other hand, the mean values of the input observables used in the semidirect reconstruction are actually incompatible with each other. Indeed, if we replace the uniform distributions for the data discussed in Sec. VI A by distributions peaked at the mean experimental values, we find that all our simulations are rejected if σ is to be reconstructed within its experimental error of 5%.

TABLE VII. Determination of C_{anc0} and C_{nb0d} (equivalently M_{a0cn} and N_{0bnd}) from an optimal set known at the point ($E_{lab}=425$ MeV, $\theta=65°$).

$C_{s'ns0}$	0.028 ± 0.273	$C_{ns''0s}$	-0.088 ± 0.178
$C_{k'ns0}$	-0.067 ± 0.209	$C_{nk''0s}$	-0.019 ± 0.266
$C_{s'nk0}$	0.130 ± 0.231	$C_{ns''0k}$	0.012 ± 0.301
$C_{k'nk0}$	-0.008 ± 0.278	$C_{nk''0k}$	-0.098 ± 0.226

The amplitude f, violating isospin invariance, was assumed to vanish in the phase-shift analysis.[23] Consequently, we find $\text{Re}f=\text{Im}f=0$ with high accuracy in the reconstruction from phase-shift values in Table VIII. In the semidirect reconstruction we find that $\text{Re}f=0.114\pm0.304$, i.e., this amplitude does not necessarily vanish, however no conclusions on isospin violation can be drawn here.

A discussion of the actual limits acceptable for the observables in order to permit a "reasonably" accurate determination of the amplitudes goes beyond the scope of this paper. The experimental situation for *np* scattering is at this stage much less favorable than for *pp* scattering, where ambiguities are also encountered in direct reconstructions.[14–16]

VII. CONCLUSIONS

This paper consists of essentially two parts. In the first we derive a variety of nonlinear relations between experimental quantities in the elastic scattering of two nonidentical spinor particles. The relations can be used in order to determine whether a given set of experiments is complete, sufficient, or neither. They can also be used to calculate the values of certain observables on the basis of the measurement of other ones, or to put limits on the possible values of certain variables.

The second part (Secs. V and VI) is concerned with a reconstruction of the *np* scattering amplitudes without assuming isospin invariance. In Sec. V we discuss the completeness of various specific sets of 12 experiments. In Sec. VI we actually perform an amplitude reconstruction at $E_{lab}=425$ MeV and $\theta=65°$. We call the reconstruction a "semidirect" one, since it was necessary to add the values of the asymmetry tensor A_{00cd}, calculated on the basis of a phase-shift analysis, to the measured observables σ, A_{00n0}, A_{000n}, A_{00nn}, K_{0nn0}, $K_{0s''s0}$, $K_{0s''k0}$, and $K_{0k''s0}$. The reconstruction is meant more as an illustration of the use of the nonlinear relations than anything else. It shows how these relations help to distinguish between ambiguities inherent in the formalism (incomplete sets of data) and ambiguities due to experimental errors. The fact that at a given energy and angle the nonlinear relations predict some observables with good accuracy but others with very large error bars, should be used to plan experiments resolving ambiguities.

A combination of a direct reconstruction with other methods of extracting amplitudes from data seems to be a fruitful approach. At energies up to about 1 GeV a phase-shift analysis is quite feasible. At higher energies we plan to apply analyses involving simultaneous expansions in terms of func-

TABLE VIII. Reconstruction of the *np* amplitudes at the point $E_{lab}=425$ MeV, $\theta_{c.m.}=65°$.

Amplitudes	Semidirect reconstruction		Reconstruction from phase-shift values	
Re*a*	0.286 ± 0.327	[−0.929,1.118]	-0.002 ± 0.032	[−0.092,0.137]
Im*a*	0.645 ± 0.486	[−0.939,1.086]	0.351 ± 0.244	[0.088,0.841]
Re*b*	-0.581 ± 0.220	[−1.284,−0.242]	-0.735 ± 0.403	[−1.395,−0.224]
Im*b*	-0.343 ± 0.780	[−1.133,1.140]	-0.074 ± 0.930	[−1.205,1.187]
Re*c*	0.424 ± 0.132	[0.153,0.815]	0.513 ± 0.327	[0.175,1.077]
Im*c*	0	0	0	0
Re*d*	-0.721 ± 0.496	[−1.148,0.874]	-0.319 ± 0.509	[−1.098,0.324]
Im*d*	0.259 ± 0.343	[−1.059,1.116]	-0.213 ± 0.109	[−0.374,−0.059]
Re*e*	0.274 ± 0.299	[−0.915,1.008]	-0.176 ± 0.873	[−1.219,1.189]
Im*e*	0.123 ± 0.165	[−0.592,1.097]	0.737 ± 0.399	[0.217,1.417]
Re*f*	0.114 ± 0.304	[−1.058,1.108]	-0.003 ± 0.102	[−0.395,0.534]
Im*f*	0.000 ± 0.012	[−0.059,0.049]	0.000 ± 0.014	[−0.047,0.050]

tions of the scattering angle and the energy[24,25] and to combine these with a direct reconstruction.

Such an approach should make it possible to combine the advantages of a direct reconstruction, namely the freedom from theoretical bias, with the advantages of partial wave and other expansions, namely the possibility of simultaneously using data measured at different energies and angles.

We plan to return to the problem of the reconstruction of *np* scattering amplitudes and in particular the distinction between algebraic ambiguities occurring even for precise data and statistical ambiguities due to experimental errors, in the future, once more data become available.

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