Rescattering effects and the Schmid theorem

C. J. Goebel and S. F. Tuan*
Department of Physics, University of Wisconsin—Madison, Madison, Wisconsin 53706

W. A. Simmons
Department of Physics and Astronomy, University of Hawaii at Manoa, Honolulu, Hawaii 96822
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We point out here the importance of the 1967 paper by Christoph Schmid in clarifying recent studies concerning the existence (or nonexistence) of peaks in three-body hadron systems. We further point out that the Schmid theorem left open the possibility that a residual interference term involving the singularity of a triangle diagram *can* in principle at least lead to observable effects in the differential cross section. Some numerical examples are given.

There has been a spate of papers, both recent[1] and not so recent,[2–4] discussing mechanisms for possible peaks in three-body hadron systems—mostly associated with the singularities of the triangle diagram. We wish to point out here the relevance of an earlier (largely unnoticed) paper by Schmid[5] which might have shed a great deal of light on the current confusion of claims[2–4] and counterclaims.[1]

Following Schmid,[5] we first take up once more the mechanism proposed by Peierls[6] twenty years back. The simplest manifestation is via triangle diagram Fig. 1(a), where W is the invariant mass in the initial two-body system and Δ' and Δ are resonances. The singularities of this diagram (triangle singularities) are at energies which allow all three internal lines to be on their respective mass shells and the rescattering angle to be at an extreme value $z=\pm 1$ (hence all internal momentum vectors are parallel in the c.m. frame); for the Peierls case $\Delta'=\Delta$ and $a=c$ ($a\equiv m_a$, $\Delta=M_\Delta$, etc.) these energies are given by

$$W_l^{\,2}=2(\Delta^2+c^2)-b^2\,,$$
$$W_u^{\,2}=(\Delta^2-c^2)^2/b^2\,. \tag{1}$$

On which sheet are the triangle singularities? Coleman and Norton and Bronzan[7] gave a simple and general answer. The singularity of a Feynman amplitude is on the physical boundary if, and only if, the diagram can be interpreted as a classical process in space-time. For the rescattering process Fig. 1(a) (triangle amplitude), this means: All three internal particles must be on their mass shells, the decaying Δ' must emit the b in the direction of c, and b and c must come together with the correct relative velocity to form Δ. In the case $\Delta=\Delta'$ these conditions are not satisfied at either W_l or W_u, as is well known.[8] In passing we mention that a recent treatment of pion-deuteron scattering in a relativistic three-body model[4] bears *less* on the existence of rescattering [e.g., Fig. 1(a)] but rather on a resonance-production-type zeroth-order graph [Fig.

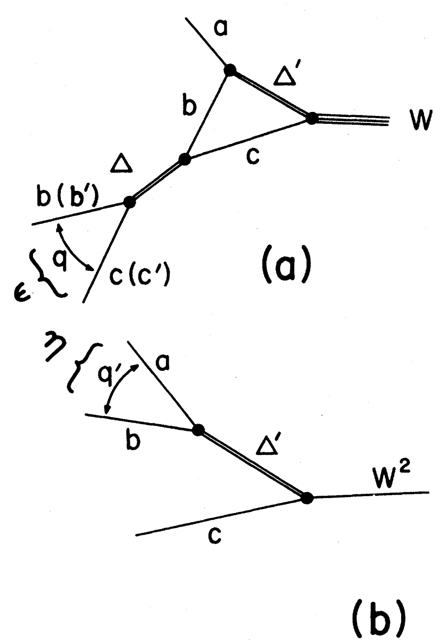

FIG. 1. (a) Triangle graph for three particles a, b, and c. Δ' and Δ are resonances formed from a, b, and b, c, respectively. Final-state (or charge-exchange) scattering $b+c\to b(b')+c(c')$ occurs with subenergy ϵ and momentum q. (b) Zeroth-order graph for three particles a, b, and c in which resonance Δ' decays into a,b with subenergy η and momentum q'.

1(b)] projected out to a given partial wave—hence the interpretation is not really a test of triangle singularity[9] $W_u{}^2$ as suggested by Brayshaw *et al.*[2]

Another interesting facet of Schmid's work concerns the question of overlapping resonance bands. It has been known for a long time[10] that in general a Dalitz plot of the kinetic energies T_{π_2} vs T_{π_3} for the final-state products from, say, $\pi N \to \pi_2 + \pi_3 + N'$ shows bands $T_{\pi_2} = \text{constant}$ and $T_{\pi_3} = \text{constant}$, corresponding to resonance (isobar) formation. The triangle singularities [e.g., Eq. (1) with $\Delta = \Delta(1238)$, $c = a = \pi$, $b = N$] correspond kinematically to the crossing point of the above-mentioned bands entering and leaving the physical region of the Dalitz plot. Chang[11] proposed that, irrespective of the sheet difficulties of the Peierls singularity, one could hope for production of peaks by interference of overlapping resonance bands as they enter the physical region of Dalitz plot. However, Schmid[5] has shown quite generally that interference of overlapping resonance bands for the commonly discussed cases[10,11] *does not produce peaks in either total transition rate or two-body mass plots.*

It has been argued, especially by Brayshaw and Peierls[3] that even if one such graph by itself [e.g., Fig. 1(a)] is regular on the physical boundary, the sum of multiple-rescattering graphs (in particular, the inclusion of the "box graph") might sum to a singularity as a truly dynamical effect, e.g., to a true three-body resonance pole on the *correct sheet* at an energy *near* the appropriate Peierls singularity. This was first conjectured by Hwa[12] essentially treating the triangle singularity as a kind of dynamical driving force, but the claim of a "nearby" resonance pole is *not* borne out in the strong-coupling static model.[8] Another class of calculations has been to examine soluble models less physical than strong-coupling theory to examine their solutions for three-body resonances or bound states. The complete calculation of Srivastava[13] in the Lee model found *no* resonance in the vicinity of the Peierls singularity. Hence, in the absence of a detailed calculation of the touted importance of an off-shell analysis with respect to the singularity of their "box graph" (amongst other higher-order diagrams), the Brayshaw-Peierls[3] suggestion remains an *unproven* conjecture.

Next, we take up the work of Polykarpov and van der Velde.[1] Their point is well taken that the Brayshaw mechanism,[2] except for the emphasized importance of final charge exchange with $b+c \to b'+c'$ and $b'+c'$ at threshold in Fig. 1(a) (where Δ is replaced by some effective S-wave scattering at that vertex), is the same mechanism already noted by Aitchison and Kacser,[14] by Anisovich *et al.*,[15] and by Valuev,[15] but probably first noticed by Landshoff.[16] In the language of Schmid[5] this is the modified (and weak) Peierls mechanism. As for the final-charge-exchange mechanism of Brayshaw,[2] we agree with Polykarpov and van der Velde[1] that the Brayshaw singularity which moves upward towards the real axis, ostensibly "very close" to the physical region due to $b+c \to b'+c'$ charge exchange, actually lies on a distant sheet. This is most transparently seen by examining Eq. (5) of Brayshaw, Simmons, and Tuan[2] for Im s where s is the (square of) three-body energy of the system. There is nothing to prevent Im s here from going positive if κ_B (in their notation) is increased enough (see also Ref. 1, Fig. 2). Since nature will not allow this disaster on the physical sheet, it must mean that the singularity whose position is given by their Eq. (5) is not near the physical sheet. *Hence the Brayshaw mechanism is invalid.*

Coming back to our central theme, the crucial Schmid observation[5] is that when the triangle *amplitude* B of Fig. 1(a), a first-order rescattering correction, is added to the zeroth-order amplitude A of Fig. 1(b), the effect of rescattering is seen to be nothing more than a multiplication of part of the zero-order amplitude by a phase factor $e^{2i\delta}$ [see below, Eq. (12)]. Explicitly in the nonrelativistic limit, the total amplitude F is, up to a constant factor,

$$F \cong A(\eta) + B(E,\epsilon) \,, \tag{2}$$

where η and ϵ are, respectively, ab and bc subenergies (cf. Fig. 1) and E is the total energy. (All energies are relative to the relevant sum of rest masses.) Following the methods of Aitchison and Kacser,[17] A and B are given by

$$A(\eta) = 1/(\eta - \Delta') \,, \tag{3}$$

$$B(E,\epsilon) = -if(q)\frac{(a+b)(b+c)}{(a+b+c)p}\langle \ln\beta \rangle \,, \tag{4a}$$

where $f(q)$ is the b-c scattering amplitude, and

$$\langle \ln\beta \rangle = \int_0^\infty d\eta' \frac{\Gamma'}{2\pi}\left[\frac{1}{|\eta' - \Delta'|^2}\right] \ln\beta(E,\epsilon,\eta') \,, \tag{4b}$$

$$\beta = \frac{(b+c)[q+p'(\eta')] + cp + i0^+}{(b+c)[q+p'(\eta')] - cp + i0^+} \,. \tag{4c}$$

The kinematic variables, cf. Fig. 1, are

$$q=\left[\frac{2bc\epsilon}{b+c}\right]^{1/2},$$

$$q'(\eta')=\left[\frac{2ab\eta'}{a+b}\right]^{1/2},$$

$$p=\left[\frac{2a(b+c)(E-\epsilon)}{a+b+c}\right]^{1/2}, \tag{5}$$

$$p'(\eta')=\left[\frac{2c(a+b)(E-\eta')}{a+b+c}\right]^{1/2}.$$

Δ is the complex resonance energy $\Delta=\bar{\Delta}-i\Gamma/2$, where Γ is the energy-dependent width $\Gamma=\gamma q$, and $\Gamma'=\gamma q'$. The once-integrated differential cross section $d\sigma/d\epsilon$ is

$$d\sigma/d\epsilon \cong qp\int_{-1}^{+1} d\cos\theta\,|F|^2 = qp\sum_{l=0}^{\infty}(2l+1)|F_l|^2, \tag{6}$$

where $\cos\theta=\vec{p}\cdot\vec{q}/pq$ and the F_l are the partial-wave projections of F with respect to the relative direction of $\vec{p}$ and $\vec{q}$. Because the rescattering is pure S wave, $F_0=A_0+B$, $F_l=A_l$ $(l\neq 0)$, and so

$$d\sigma/d\epsilon=d\sigma/d\epsilon\Big|_{0\text{ order}}+\Delta\frac{d\sigma}{d\epsilon}, \tag{7}$$

where

$$\frac{d\sigma}{d\epsilon}\Big|_{0\text{ order}}\cong\frac{qp}{2}\int_{-1}^{1}d\cos\theta\,|A|^2 \tag{8}$$

and

$$\Delta\frac{d\sigma}{d\epsilon}=qp(|A_0+B|^2-|A_0|^2). \tag{9}$$

For constant Γ,

$$A_0=\frac{(a+b)(b+c)}{2(a+b+c)qp}\ln\left[\frac{\alpha_{\Delta'}}{\beta_{\Delta'}}\right], \tag{10}$$

where $\alpha_{\Delta'}$ and $\beta_{\Delta'}$ are α and β, respectively, evaluated at $\eta'=\Delta'$, i.e.,

$$\alpha_{\Delta'}=\frac{(b+c)[-q+p'(\Delta')]+cp}{(b+c)[-q+p'(\Delta')]-cp},$$
$$\beta_{\Delta'}=\frac{(b+c)[q+p'(\Delta')]+cp}{(b+c)[q+p'(\Delta')]-cp}. \tag{11}$$

An approximation to $\langle\ln\beta\rangle$, Eq. (4b), which is good when the width of the resonance is small, and in any case has the triangle singularity, is $\ln\beta_{\Delta'}$. Using this, the S-wave amplitude to first order in the rescattering is approximately

$$A_0+B\approx\frac{(a+b)(b+c)}{2(a+b+c)qp}[\ln(\alpha_{\Delta'}/\beta_{\Delta'})-2iqf(q)\ln\beta_{\Delta'}] = \frac{(a+b)(b+c)}{2(a+b+c)qp}(\ln\alpha_{\Delta'}-e^{2i\delta}\ln\beta_{\Delta'}). \tag{12}$$

Notice that, as emphasized by Schmid,[5] the partial-wave projection of the zeroth-order amplitude A has singularities where $\alpha_{\Delta'}$ or $\beta_{\Delta'}$ equal 0 or ∞ which are "triangle singularities" corresponding to energy-conserving and collinear intermediate states. Further, the effect of the rescattering amplitude B is to multiply the term $\ln\beta_{\Delta'}$ of the zeroth-order amplitude by a phase factor $e^{2i\delta}$. The rescattering correction to the cross section is

$$\Delta\frac{d\sigma}{d\epsilon}\cong(|A_0+B|^2-|A_0|^2)qp.$$

Using Eq. (12), we have

$$\Delta\frac{d\sigma}{d\epsilon}\cong\frac{1}{qp}(|\ln\alpha_{\Delta'}-e^{2i\delta}\ln\beta_{\Delta'}|^2-|\ln\alpha_{\Delta'}-\ln\beta_{\Delta'}|^2) \cong-\frac{2}{qp}\mathrm{Re}[(e^{2i\delta}-1)\ln\alpha^*_{\Delta'}\ln\beta_{\Delta'}]. \tag{13}$$

Schmid's point is that $\Delta d\sigma/d\epsilon$ has no $|\ln\beta_{\Delta'}|^2$ term, even though $|B|^2$, the contribution of the rescattering term alone (neglecting interference) does. Thus a large value of $\ln\beta_{\Delta'}$ (e.g., due to a nearby "triangle singularity") does not necessarily result in a large value of $\Delta d\sigma/d\epsilon$. However, as Eq. (13) shows, there is still the possibility that the rescattering contributes interesting structure to $d\sigma/d\epsilon$ from either the factor $\ln\alpha_{\Delta'}$ or the factor $\ln\beta_{\Delta'}$.

Clearly, explicit calculations of $d\sigma/d\epsilon$ are called for. For numerical work, we use $a=c$, and $f=$same resonant scattering as in the triangle diagram, namely,

$$e^{2i\delta}=1+2iqf(q)=\frac{\Delta^*-\epsilon}{\Delta-\epsilon},$$

i.e.,

$$2iqf(q)=\frac{\Delta^*-\Delta}{\Delta-\epsilon}. \tag{14}$$

To make a comparison with Polykarpov and van der Velde,[1] we shall illustrate with their parameters for the $N\Delta$ system:

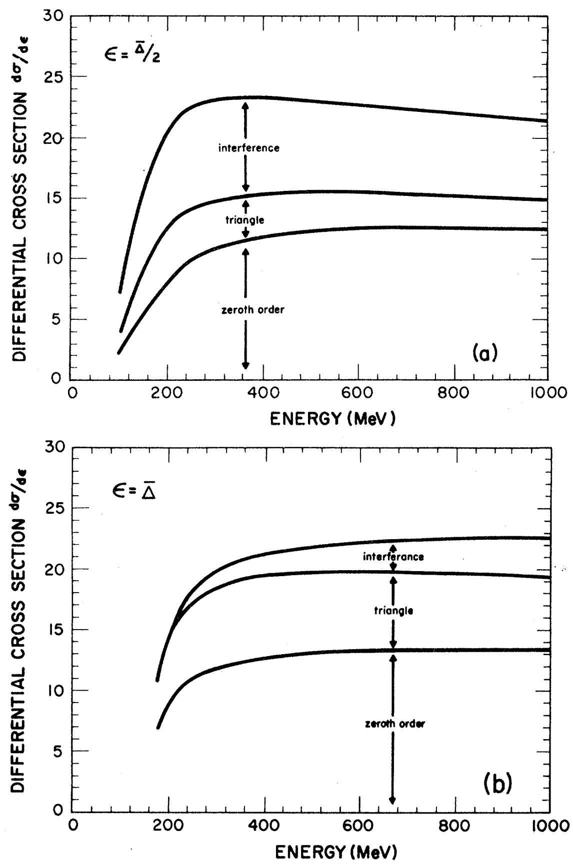

FIG. 2. (a) The differential cross sections $d\sigma/d\epsilon$ (units are arbitrary) vs total energy E with a breakdown of the contributions from zeroth-order, triangle, and interference terms at $\epsilon=\bar{\Delta}/2$ computed from Eq. (7). (b) The same entries as in (a) but for $\epsilon=\bar{\Delta}$.

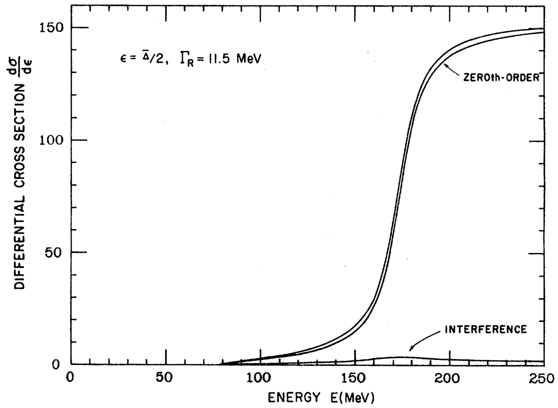

FIG. 3. The differential cross section $d\sigma/d\epsilon$ (arbitrary units) vs total energy E for the case $\epsilon=\bar{\Delta}/2$ and the width Γ_R reduced to 11.5 MeV. Also shown is the interference-term contribution alone; the triangle-diagram- (rescattering) amplitude squared is less than 10% of the interference term plotted here.

$$
\begin{aligned}
& b/a=m_\pi/m_N=0.145\ ,\\
& \Delta=\bar{\Delta}-i\Gamma/2=(157-i\Gamma/2)\ \text{MeV}\ ,\\
& \Gamma(\epsilon)=\gamma q=\Gamma_R(\epsilon/\bar{\Delta})^{1/2}\ ,\\
& \text{where } \Gamma_R=115\ \text{MeV}\ .
\end{aligned}
\tag{15}
$$

However, we shall not use their form factors $g^2(\epsilon)$. For consistency, we shall compute $d\sigma/d\epsilon$ from (7) with

$$\Delta'=\Delta'(\eta)=\bar{\Delta}-i\Gamma'(\eta)/2$$

and

$$\Gamma'(\eta)=\Gamma_R(\eta/\bar{\Delta})^{1/2}\ .$$

The A_0 contribution to $d\sigma/d\epsilon$ is numerically calculated from

$$A_0=\frac{1}{2}\int_{-1}^{+1} d\cos\theta\frac{1}{\eta(p,q,\theta)-\Delta'(\eta)}$$

rather than from the constant-width formula (10), although (10) proved to be a *good* approximation to the exact integral expression for A_0 with $\Gamma'(\eta)$ dependence. The total cross section is then

$$\sigma(E)=\int\frac{d\sigma}{d\epsilon}d\epsilon\ . \tag{16}$$

In the numerical evaluation of Eq. (4b) the phase choice is as follows. In the region $0<\epsilon<E$ and $E>\eta'$, β is positive only if

$$E<\frac{(a+b)(b+c)}{b(a+b+c)}\eta'$$

and

$$\epsilon<\frac{1}{(a+b)(b+c)}\times\{(ac\eta')^{1/2}-[b(a+b+c)(E-\eta')]^{1/2}\}^2\ .$$

Otherwise, β is negative and $\ln\beta=\ln|\beta|-i\pi$.

In the region $\eta'>E$, $p'=+i|p|$ and where

$$\epsilon>\frac{acE}{(a+b)(b+c)}$$

we have

$$\ln\beta=\ln|\beta|+i\left[\tan^{-1}\frac{|p'|}{q+[c/(b+c)]p}-\tan^{-1}\frac{|p'|}{q-[c/(b+c)]p}\right],$$

and where

$$\epsilon < \frac{acE}{(a+b)(b+c)}$$

we have

$$\ln\beta = \ln|\beta| + i\left[\tan^{-1}\frac{|p'|}{q+[c/(b+c)]p} - \pi + \tan^{-1}\frac{|p'|}{-q+[c/(b+c)]p}\right].$$

The dominant feature of $d\sigma/d\epsilon$ versus total energy E is the *large interference term* for $\epsilon=\overline{\Delta}/2$ throughout (note $\epsilon=\overline{\Delta}/2$ and $\eta=\overline{\Delta}$ bands intersect in the physical region of the Dalitz plot for $175 \leq E \leq 1810$ MeV). For $\epsilon=\overline{\Delta}$, the interference cross section becomes relatively unimportant *vis á vis* zeroth-order + triangle contributions. These are depicted in Figs. 2(a) and 2(b). The contrast for the two cases is due to the reality of $2iqf(q)$ in the interference term[18]

$$\left.\frac{d\sigma}{d\epsilon}\right|_{\text{int}} = -\left[\frac{(a+b)(b+c)}{2(a+b+c)}\right]^2 \frac{1}{qp}(2\,\mathrm{Re}\{[\ln(\alpha_{\Delta'}/\beta_{\Delta'})]^* 2iqf(q)\langle\ln\beta\rangle\}) \tag{17}$$

for $\epsilon=\overline{\Delta}$, and an enhanced value for $d\sigma/d\epsilon|_{\text{int}}$ at $\epsilon=\overline{\Delta}/2$ due in part to the well-known[1] fact that at $\epsilon=\overline{\Delta}/2$ the triangle singularity lies close to the resonance-particle threshold and is thus "near" the physical region. The dominant term in (17) for $\epsilon=\overline{\Delta}/2$ is proportional to $\mathrm{Im}(2iqf(q))$, absent in the $\epsilon=\overline{\Delta}$ case.

The full $d\sigma/d\epsilon$ for $\epsilon=\overline{\Delta}/100=1.57$ MeV (near the *b-c* threshold) and $\Gamma_R=115$ MeV is both small and dull (in terms of structure) reflecting the suppression due to q and $\Gamma(\epsilon)$ (Eq. 15) for the zeroth-order and triangle terms of Eq. (7). The total cross section as expected also shows no structure.

As for the cusp effects discussed above following Eq. (13), reducing Γ_R to 11.5 MeV does lead to small interference cusps at $E=553$ and 784 MeV where the intersection of $\epsilon=\overline{\Delta}/100$ and $\eta=\overline{\Delta}$ bands enter and leave the physical region of the Dalitz plot. Such effects are anticipated for $\epsilon\approx 0$ from Eqs. (13) and (17) in the narrow-width limit (in which $\langle\ln\beta\rangle \approx \ln\beta_{\Delta'}$) when $(\ln\alpha_{\Delta'})^*\ln\beta_{\Delta'}$ has a sizable and dominantly imaginary component; they are, however, totally swamped by the very much larger zeroth-order term and hence are unobservable in the complete expression for $d\sigma/d\epsilon$ [Eq. (7)]. For $\epsilon=\overline{\Delta}/2$, the rescattering contribution to $d\sigma/d\epsilon$ has a peak (half width ≈ 115 MeV) around $E=170$ MeV. However, it is only a less than 10% effect against a rapidly rising zeroth-order cross section; see Fig. 3.

Note added. After this work was submitted for publication we were informed by I. J. R. Aitchison of his work with C. Kacser [Phys. Rev 173, 1700 (1968)], which stressed a similar point regarding the interference term Eq. (17).

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*On faculty exchange from Department of Physics and Astronomy, University of Hawaii at Manoa, Honolulu, Hawaii 96822.

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