# Vacuum polarization near the event horizon of a charged rotating black hole

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The renormalized value of $\langle \Phi^2(x) \rangle$ near the pole of the event horizon of the charged rotating black holes is obtained.

Considering the problem of vacuum polarization in Schwarzschild spacetime, Candelas[1] obtained a remarkably simple explicit expression for the Hartle-Hawking propagator $G_H(x|x')$ of the massless scalar field in the case wherein one of the points lies on the bifurcation two-sphere of the horizon. This expression was found by summing the series representation for $G_H(x|x')$. In this paper we generalize this result to the case of charged rotating black holes.

The Kerr-Newman metric describing the gravitational field of a charged rotating black hole can be presented in the form

$$ds^2 = P_{AB}\, d\eta^A d\eta^B + Q_{XY}\, d\xi^X d\xi^Y , \tag{1}$$

where $A,B=0,1$, $X,Y=2,3$, and $\partial P_{AB}/\partial\eta_C = 0$. $\partial Q_{XY}/\partial\eta_C=0$. In particular, in Boyer-Lindquist coordinates one has $(\eta^0=t, \eta^1=\phi, \xi^2=r, \xi^3=\theta)$

$$\begin{aligned} ds_1{}^2 &\equiv P_{AB}\, d\eta^A d\eta^B \\ &= -\left(1-\frac{Z}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta}{\Sigma} Z\, dt\, d\phi \\ &\quad + \sin^2\theta\left(r^2+a^2+a^2\sin^2\theta\frac{Z}{\Sigma}\right)d\phi^2 , \\ ds_2{}^2 &\equiv Q_{XY} d\xi^X d\xi^Y = \Sigma(\Delta^{-1}dr^2+d\theta^2) , \end{aligned} \tag{2}$$

where $\Sigma=r^2+a^2\cos^2\theta$, $\Delta=r^2-2Mr+a^2+Q^2$, and $Z=2Mr-Q^2$. This metric possesses two Killing vectors $\vec{m}=m^\alpha\partial_\alpha=\partial_\phi$ and $\vec{k}=k^\alpha\partial_\alpha=\partial_t$. Their linear combination, $\vec{l}=\vec{k}+\Omega\vec{m}$, is null at the event horizon $r=r_+\equiv M+(M^2-Q^2-a^2)^{1/2}$ if $\Omega=a/(r_+{}^2+a^2)$. If we perform the coordinate transformation to coordinates $\tilde{\eta}^A=(t, \tilde{\phi}=\phi-\Omega t)$ which corotate with the hole and then perform the Wick rotation $t=-i\tau$ we obtain the complex Riemannian metric

$$\begin{aligned} &ds^2=d\tilde{s}_1{}^2+ds_2{}^2, \quad d\tilde{s}_1{}^2=\tilde{P}_{AB}d\tilde{\eta}^A d\tilde{\eta}^B , \\ &\tilde{P}_{\tau\tau}=-(P_{tt}+2\Omega P_{t\phi}+\Omega^2 P_{\phi\phi}), \quad \tilde{P}_{\tau\tilde{\phi}}=-i(P_{t\phi}+\Omega P_{\phi\phi}) , \\ &\tilde{P}_{\tilde{\phi}\tilde{\phi}}=P_{\phi\phi}, \quad \det\tilde{P}=-\det P . \end{aligned} \tag{3}$$

This metric is analytic if $\tau$ is an angular parameter with period $2\pi/\kappa$, where $\kappa=\frac{1}{2}(r_+-r_-)/(r_+{}^2+a^2)$ is the surface gravity of the black hole.[2] In $(t,\tilde{\phi})$ coordinates one has $\vec{m}=\partial_{\tilde{\phi}}$ and $\vec{l}=\partial_t$. The norm of these vectors is equal to zero at the pole $\xi_0=(r=r_+, \theta=0)$ of the event horizon. It means that after the Wick rotation the point $\vec{x}_0$ lying at the pole of the bifurcation two-surface of the event horizon becomes a stable point of the symmetry group whose generators are $\partial_{\tilde{\phi}}$ and $\partial_\tau$.

The Hartle-Hawking propagator $G_H(x|x')$ for a scalar massless field is defined as a symmetric solution of the equation

$$\begin{aligned} \Box G_H(-i\tau,\tilde{\phi},\vec{\xi}\,|-i\tau',\tilde{\phi}',\vec{\xi}') &\equiv \left[{}^{(2)}\Delta+(\tilde{P}^{-1})^{AB}\frac{\partial^2}{\partial\tilde{\eta}^A\partial\tilde{\eta}^B}\right] G_H(-i\tau,\tilde{\phi},\vec{\xi}\,|-i\tau',\tilde{\phi}',\vec{\xi}') \\ &= -i\,\frac{\delta(\tau-\tau')\delta(\tilde{\phi}-\tilde{\phi}')\delta(\vec{\xi},\vec{\xi}')}{(\det\tilde{P}\det Q)^{1/2}} , \end{aligned} \tag{4}$$

which is periodic in $\tau$ with the period $2\pi/\kappa$. [Here ${}^{(2)}\Delta \equiv (|\det P \det Q|)^{-1/2}(\partial/\partial\xi^X)(|\det P\det Q|^{1/2}(Q^{-1})^{XY}\partial/\partial\xi^Y)$.]

We may incorporate the desired periodicity properties by setting

$$\delta(\tau-\tau')=\frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\kappa(\tau-\tau')}, \quad \delta(\tilde{\phi}-\tilde{\phi}')=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\tilde{\phi}-\tilde{\phi}')} . \tag{5}$$

If we substitute (5) into (4) we find that we can expand $G_H$ in the form

$$G_H(-i\tau,\tilde{\phi},\vec{\xi}\,|-i\tau',\tilde{\phi}',\vec{\xi}')=\frac{i\kappa}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{in\kappa(\tau-\tau')}e^{im(\tilde{\phi}-\tilde{\phi}')}G^{(n,m)}(\vec{\xi}\,|\,\vec{\xi}') , \tag{6}$$

where $G^{(n,m)}$ satisfies the equation

$$ {}^{(2)}\Delta G^{(n,m)}(\vec{\xi}\,|\,\vec{\xi}') - [(\tilde{P}^{-1})^{\tau\tau}n^2\kappa^2 + 2(\tilde{P}^{-1})^{\tau\tilde{\phi}}nm\kappa + (\tilde{P}^{-1})^{\tilde{\phi}\tilde{\phi}}m^2]G^{(n,m)}(\vec{\xi}\,|\,\vec{\xi}') = -\frac{\delta(\xi,\xi')}{|\det P \det Q|^{1/2}} \tag{7} $$

In the particular case when $\vec{\xi}' = \vec{\xi}_0$ and hence $\vec{x}'$ coincides with the stable point $\vec{x}_0$ of the symmetry group the Hartle-Hawking propagator does not depend on the coordinates $\eta^A$ and we have

$$ G_H(-i\tau, \tilde{\phi}, \vec{\xi}\,|-i\tau', \tilde{\phi}', \vec{\xi}_0) = \frac{i\kappa}{(2\pi)^2} G^{(0,0)}(\vec{\xi}\,|\,\vec{\xi}_0) . \tag{8} $$

Using Eq. (7) one can verify that $G^{(0,0)}(\vec{\xi}\,|\,\vec{\xi}')$ coincides with the expression for a static massless scalar field at the point $\xi$ created by the scalar charge $q = \frac{1}{2}[-P_{tt}(\vec{\xi}')]^{-1/2}$ located at the point $\xi'$ lying at the axis of the symmetry ($\theta' = 0$). Using the expression for this field obtained byLinet,[3]

$$ G^{(0,0)}(\vec{\xi}\,|\,\vec{\xi}') = \tfrac{1}{2}[(r-M)^2 + (r'-M)^2 - 2(r'-M)(r-M)\cos\theta - (M^2-a^2-Q^2)\sin^2\theta]^{-1/2} , \tag{9} $$

we finally obtain

$$ G_H(-i\tau, \tilde{\phi}, \vec{\xi}\,|-i\tau', \tilde{\phi}', \xi_0) = \frac{i\kappa}{8\pi^2}\,\frac{1}{r - M - (M^2-a^2-Q^2)^{1/2}\cos\theta} . \tag{10} $$

If $a=0$ then the metric is spherically symmetric and Eq. (10) gives the expression for the Hartle-Hawking propagator $G_H(x\,|x')$ for an arbitrary point $x'$ lying on the bifurcation two-sphere of the event horizons. In particular when $a=Q=0$, Eq. (10) coincides with the expression obtained by Candelas.[1]

Separating the points in the radial direction we can get the following finite expression for the renormalized Green's function for $x'$ lying on the bifurcation two-sphere:

$$ \begin{aligned} \langle H|\Phi^2(x)|H\rangle_{\rm ren} &= \lim_{x'\to x}\left[-iG_H(x\,|x') - \frac{1}{8\pi^2\sigma(x,x')} - \frac{1}{96\pi^2}R_{\alpha\beta}\frac{\sigma^{;\alpha}\sigma^{;\beta}}{\sigma}\right] \\ &= \frac{1}{24\pi^2(r_+^2+a^2)}\left(r_+\kappa - \frac{a^2}{r_+^2+a^2}\right). \end{aligned} \tag{11} $$

Here we used $R_{\alpha\beta}\sigma^{;\alpha}\sigma^{;\beta}/\sigma = 2R^r_r = -2Q^2/(r^2+a^2)^2$ and the expansion

$$ \sigma(x,x_0) \equiv \tfrac{1}{2}s^2(x,x_0) = \frac{1}{\kappa}(r-r_+)\left[1 + \frac{2}{3}\left(\frac{r_+}{r_+^2+a^2} - \frac{1}{2(r_+-r_-)}\right)(r-r_+) + O((r-r_+)^2)\right] $$

for the square line interval between the pole $x_0$ of the event horizon and the point $x$ lying on the axis of symmetry. For $a=Q=0$ we reproduce the Candelas result $\langle H|\Phi^2(x)|H\rangle_{\rm ren} = 1/192\pi^2M^2$. For the charged and rotating black hole the value $\langle H|\Phi^2(x)|H\rangle_{\rm ren}$ becomes negative if $M \leq (2a^2+Q^2)/(3a^2+Q^2)^{1/2}$. One can verify that this condition does not contradict the condition of the black-hole existence $M^2-a^2-Q^2 \geq 0$.

The properties of the stress-energy tensor near the event horizon of the charged rotating black hole will be discussed elsewhere.

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[1] P. Candelas, Phys. Rev. D 21, 2185 (1980).

[2] See, for example, G. W. Gibbons, in *General Relativity—An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, New York, 1979), p. 639.

[3] B. Linet, C. R. Acad. Sci. Ser. A 284, 215 (1977).