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Comment on a universal upper bound on the entropy-to-energy ratio for bounded systems

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Some difficulties are noted with Bekenstein's arguments that a system with energy E and effective radius R has an entropy bound $S \leq S_B \equiv 2\pi ER/\hbar c$. The inequality appears to be valid only for large S_B. For that regime an upper bound which is generally stronger is proposed.

Bekenstein has made the interesting suggestion[1] that the entropy of an arbitrary system cannot exceed the ratio of its effective circumference to its reduced Compton wavelength:

$$S \leq S_B \equiv 2\pi ER/\hbar c \ . \tag{1}$$

However, there are several problems with the present formulation of this idea.

First, the argument[1-3] for the minimum entropy increase of a large black hole absorbing a system is not valid. To attain the minimum, it is assumed that the system is released from a zero-momentum state when it just touches the hole. But if the center of the system is at proper distance R from the horizon before being released, it must be supported against a gravitational acceleration of c^2/R (independent of the size of the hole in the limit that it is much bigger than R). In fact, the tension in the system would diverge at the horizon if it were to be at rest just touching the hole. Such forces would greatly distort the system and alter the numerical analysis in a system-dependent way.

Second, the argument from quantum statistics breaks down if the vacuum energy $E_0 \leq 0$, for then clearly a statistical state with probabilities p_0 to be in the ground state and p_1 to be in an excited state with energy $E_1 > 0$ can give arbitrarily large $S/\bar{E}$ if $\bar{E} = p_0 E_0 + p_1 E_1$ is made small enough. Bekenstein gives reasons for assuming $E_0 > 0$, but the Casimir[4] effect is a counterexample which can give negative vacuum energy even in a *finite* cavity (if sufficiently nonspherical or noncubical).[5] It might be supposed that one could avoid this problem simply by excluding all states with $E \leq 0$, but this *ad hoc* procedure does not suffice, for one can choose the system (e.g., the Casimir cavity shape) so that the lowest positive energy is arbitrarily small.

A formulation that seems more likely to work for a small number of free fields is to redefine the energy of the ground state as zero and then exclude that state from consideration. However, one can imagine cases with interacting fields (e.g., a Higgs field with the appropriate self-interaction potential) in which the first excited state E_1 has only slightly higher energy and hence could be incorporated into a statistical state to violate the bound on S/E. Even for free fields the bound could be violated if there is a sufficiently large number N of species, by using a density matrix with N equal diagonal entries, each representing a state with only one field excited. The mean energy would be independent of N if the excitation energy were the same for all species, but the entropy would grow as $\ln N$. It is not correct to use this result to deduce a limit on N,[6] for it simply shows that the assumed bound would not hold in this case. Thus it does not seem possible to obtain a valid universal bound on S/E simply in terms of the size of the system.

Nevertheless, the fact that Bekenstein's bound works for black holes much bigger than the Planck mass and is saturated for Schwarzschild holes suggests that it may have some domain of validity. These holes have S_B much larger than unity, whereas the violations of the bound occur when other dimensionless quantities (such as $2\pi E_1 R/\hbar c$ or $\ln N$) dominate this quantity. Thus it seems plausible that the entropy of a system cannot increase faster than S_B as this quantity is made large, so S/S_B cannot exceed unity in the limit of large S_B.

Now we may ask how closely Bekenstein's bound S_B may be approached when it is large. For a stationary system of energy E in a spherical box of radius $R >> GE/c^4$ and volume $V=4\pi R^3/3$, the maximum entropy state is presumably thermal radiation with a possible black hole at the center, with total entropy[7–10]

$$S \approx 4\pi GM^2/\hbar c + \tfrac{4}{3}(aV)^{1/4}(E-Mc^2)^{3/4} . \quad (2)$$

Stefan's constant for n_b massless boson helicity states and n_f massless fermion helicity states is

$$a=\pi^2(8n_b+7n_f)/240\hbar^3c^3 , \quad (3)$$

and it is assumed all particle species either can be counted as massless or else make a negligible contribution to the entropy. The black-hole mass M is chosen to maximize (2). For $R \gtrsim 12.6(G^4E^5/a\hbar^4c^{20})^{1/3}$, the maximum occurs for $M=0$ (no hole present), but if this inequality is violated, the maximum entropy occurs when M is the larger root of $E-Mc^2=aV(8\pi GM/\hbar c^3)^{-4}$. An upper bound to the entropy may be obtained by replacing M by E/c^2 in the first term of (1) and by 0 in the second term, giving

$$S \lesssim 4\pi GE^2/\hbar c^5 + \tfrac{4}{3}(a\hbar^3c^3/6\pi^2)^{1/4}S_B{}^{3/4} . \quad (4)$$

This bound may be improved by extremizing (2) more carefully and by analyzing the self-gravitation of the radiation,[11,12] but it is an order-of-magnitude estimate of the attainable entropy and is not exceeded even when these effects are considered. The formula should apply to a confined stationary system whenever it gives a number large compared with the theory-dependent maximum entropy for S_B of order unity.

It is obvious that (4) gives a much lower bound on the entropy than (1) when S_B is large, unless $R \sim GE/c^4$ so that the system contains a black hole and does not extend far outside the horizon. In particular, the entropy of a stationary system not containing a black hole is much smaller than (1) if S_B is large. This is true even if the system is self-gravitating and is confined to a radius not much bigger than its Schwarzschild radius.[10,11] Thus there *is* a large gap between black-hole entropy and the entropy of other stationary systems of comparable size and mass much bigger than the Planck mass, though of course the gap may be removed by going to the Planck mass.

For nonstationary self-gravitating systems, it is much more uncertain what the entropy limits are. A Friedmann region joined by a narrow throat to a Schwarzschild exterior can have an arbitrarily large entropy for a given mass, but it must have evolved from a white-hole singularity.[11] Even excluding this by restricting consideration to nonsingular initial data with no past trapped surfaces, Sorkin, Wald, and Jiu find a configuration which has a long throat just outside its Schwarzschild radius and an arbitrarily large entropy-to-energy ratio.[11] However, they argue that the configuration is unphysical, because it is based upon a semiclassical analysis and ignores large quantum fluctuations. When the configuration is restricted so that the quantum fluctuations are small, it obeys the bounds (1) and (4).

Another way to get an arbitrarily large entropy into a region of finite mass and size in a semiclassical analysis is to form a black hole and then feed it for a sufficiently long time with incoming radiation at just the rate to balance the mass loss to outgoing Hawking radiation.[13] Then consider a nearly null spatial hypersurface which starts at the center when the horizon forms there and spreads outward just barely outside the horizon, forming a sort of sheath around the horizon in a spacetime diagram. If the horizon is made to persist an arbitrarily long time and this hypersurface stays very near it for an arbitrarily large amount of radiation to cross it inside some radius slightly outside the horizon, the entropy on the hypersurface within this region will be arbitrarily large. This example avoids the difficulties of assembly[11] of the previous configuration, but it is also based upon the semiclassical approximation which will measurably break down in black-hole evaporation.[14] There will not be a definite horizon or spatial hypersurface having the properties above, because of the Brownian motion of the hole. Even if one considers a collection of hypersurfaces in the different quantum geometries, it is possible that correlations in the Hawking radiation with what has been fed into the hole would keep the entropy finite.

These considerations suggest that it may not be possible to derive an entropy bound such as (1) or (4) for nonstationary self-gravitating systems until we have a better understanding of quantum gravity. However, semiclassical analyses of stationary systems support the upper limit (4) when S_B is large, generally a stronger upper limit than Bekenstein's bound (1). When S_B is small, the maximum entropy depends upon the system and is not rigorously bounded by either (1) or (4).

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