Photon-photon scattering at high energy and fixed angle

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In this paper, we study the photon-photon elastic scattering amplitude in the limit of the incident energy $\omega \to \infty$ with the scattering angle θ fixed. In the lowest (4th) order, this amplitude is a function of θ only. We calculate explicitly these scaling functions of θ and find that they are surprisingly simple. The higher-order photon-photon scattering amplitudes have $\ln s$ factors. Thus scaling is violated. These $\ln s$ factors come from self-energy diagrams and vertex diagrams only and can therefore be absorbed into the running coupling constant. The photon-photon scattering amplitude of the nth perturbative order is hence of the form of $e^4[e(t)]^{n-4}$ times a function of θ.

I. INTRODUCTION

In recent years, many authors have discussed the scaling of elastic scattering amplitudes at high energy and fixed angle.[1] Unfortunately, this scaling law has never been correctly proved on a field-theoretic basis. Even worse, the conclusions are often erroneous. In this paper, we show that, to the lowest order of perturbation,

$$\mathscr{M}(s,t) \simeq F(\theta) , \tag{1.1}$$

where θ is the scattering angle, $s = 4\omega^2$ is the square of the energy of the system, $t = -2\omega^2(1-\cos\theta)$ is the negative of the momentum transfer squared (all of them measured in the center-of-mass frame), and m is the electron mass. It follows from (1.1) that the lowest-order differential cross section for photon-photon scattering in the limit $\omega/m \to \infty$ with θ fixed is given by

$$\lim_{\omega/m \to \infty} \omega^2 d\sigma/d\Omega = \frac{1}{(16\pi)^2} |F(\theta)|^2 . \tag{1.2}$$

In this paper, we give explicitly the scaling function $F(\theta)$.

We would like to clarify at this point the difference between our work and those in the literature. In the literature, the scaling law was often stated as a hypothesis. In the instances when it was discussed on the basis of gauge field theories, the proof is either wrong or irrelevant. Consequently, there was a great deal of misconception about what is true. For example, it is commonly believed that the scaling law holds for the scattering of neutral particles or colorless composites. To some degree, this is due to the work of Cornwall and Tiktopoulos,[1] who showed, correctly, that the leading logarithms cancel for the scattering amplitude of colorless composites. However, the nonleading logarithms do not cancel. Another paper by Tiktopoulos[1] asserts that the scattering amplitudes of neutral particles scale. This is again not true, as can be seen by the example of the "Delbrück" scattering of a neutral scalar meson which turns into a π^+-π^- pair. We believe that our work is the first one establishing scaling for an exclusive process in gauge field theories. Furthermore, this process is a physical one and the scaling behavior can be experimentally observed.

The asymptotic form of the photon-photon scattering amplitude in the limit $s \to \infty$ with t fixed (high-energy and small-angle scattering) was studied by Cheng and Wu,[2] who found that the lowest contributing order is the eighth. This is because the fourth- and sixth-order diagrams of photon-photon scattering give amplitudes that are smaller than those of the eighth order by a factor of t/s or m^2/s. The situation is completely different in the realm of $s \to \infty$ with t/s or, equivalently, θ fixed. Since the coupling constant is dimensionless, the scaling formula (1.1) is satisfied by the amplitude of any perturbative order,[1,3] provided that the amplitude is finite as we set $m = 0$.[4] For experimental purposes, we need to calculate only fourth-order diagrams—a relatively minor task.

A very elegant treatment of the fourth-order diagrams for photon-photon scattering has been given by Karplus and Neuman.[5,6] A presentation of their results can also be found in the classic textbook by Jauch and Rohrlich, who states that "in the extreme relativistic limit ($\omega \gg m$) the angular distributions has a different energy dependence for different angles and does not factor into an angular function and energy function.[7]" We claim that exactly the opposite is true. Since the calculations are not too difficult, we have decided to go through them ourselves, instead of relying on the works of Karplus and Neuman.

Since the fine-structure constant is small, the higher-order diagrams of photon-photon scattering are of no experimental interest in the foreseeable future. However, a knowledge of their behavior may throw some light on the behavior of hadron-hadron scattering. We have therefore examined the photon-photon scattering amplitude to all orders. We have found that in the limit $s\to\infty$ with θ fixed, the higher-order amplitudes have $\ln s$ factors. Thus scaling is broken. It is significant that these $\ln s$ factors come from self-energy diagrams and vertex diagrams only, and can hence be absorbed into the coupling constant. More precisely, the photon-photon scattering amplitude of the nth perturbative order is of the form

$$e^4[e(t)]^{n-4}F(\theta)\ , \tag{1.3}$$

where $e(t)$ is the running coupling constant evaluated at the momentum transfer t. Physically, (1.3) just demonstrates the well-known fact that the relevant coupling constant approaches the bare charge for small-distance (large-momentum-transfer) reactions. Mathematically, it is interesting that the $\ln s$ factors for reactions of large s and fixed θ and the $\ln s$ factors for reactions of large s and fixed t have completely different origins. In the latter case, it is well known that such $\ln s$ factors come from closed fermion loops attached to four or more photon lines. However, such loops give no $\ln s$ factors in the region of large s and fixed θ.

II. THE ASYMPTOTIC AMPLITUDE

We choose to study the scattering in the center-of-mass frame. We shall designate the momenta of the incoming photons as k_1 and k_2, and those of the outgoing photons as k_1' and k_2'. Their polarization indices are called μ, ν, ρ, and σ, respectively. We shall also call

$$\Delta=k_1'-k_1=k_2-k_2'\ . \tag{2.1}$$

Since the polarization vectors of the photons are transverse, and since $\vec{k}_1=-\vec{k}_2$, we have

$$k_{1\mu}=k_{2\mu}=k_{1\nu}=k_{2\nu}=k'_{1\rho}=k'_{2\rho}=k'_{1\sigma}=k'_{2\sigma}=0 \tag{2.2}$$

and

$$k'_{1\mu}=-k'_{2\mu}=\Delta_\mu,\quad k'_{1\nu}=-k'_{2\nu}=\Delta_\nu\ ,$$
$$k_{1\rho}=-k_{2\rho}=-\Delta_\rho,\quad k_{1\sigma}=-k_{2\sigma}=-\Delta_\sigma\ . \tag{2.3}$$

In the above, $k_{1\mu}$, for example, is the abbreviation for $k_1\cdot\epsilon_1$, where ϵ_1 is the polarization vector for the photon of momentum k_1. As usual, we define

$$s=(k_1+k_2)^2=4\omega^2\ ,$$
$$t=(k_1'-k_1)^2=-4\omega^2\sin^2\frac{\theta}{2}\ , \tag{2.4}$$
$$u=(k_2'-k_1)^2=-4\omega^2\cos^2\frac{\theta}{2}\ .$$

Three of the fourth-order diagrams for photon-photon scattering are illustrated in Fig. 1. There are three more fourth-order diagrams which differ from the three drawn in Fig. 1 only by the arrow direction in the closed loop. They can be taken care of by multiplying the amplitudes for the diagrams in Fig. 1 by a factor 2. The photon-photon scattering amplitude is therefore equal to

$$\mathcal{M}=\mathcal{M}_a+\mathcal{M}_b+\mathcal{M}_c+\frac{e^4}{12\pi^2}(g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho})\ , \tag{2.5}$$

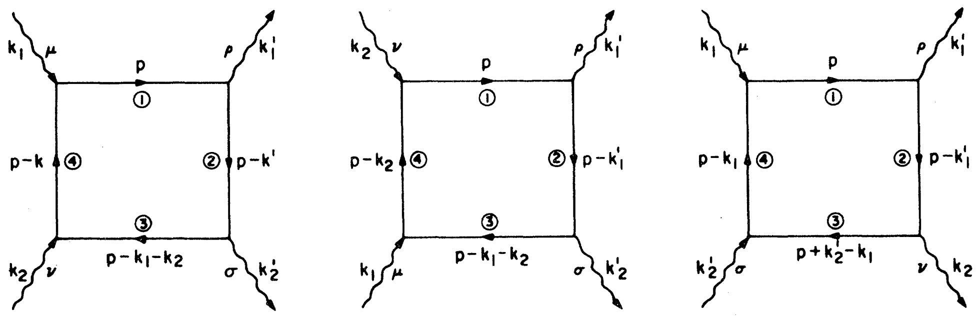
where

$$\mathcal{M}_a=2ie^4\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma_\mu(\not p+m)\gamma_\rho(\not p-\not k_1'+m)\gamma_\sigma(\not p-\not k_1-\not k_2+m)\gamma_\nu(\not p-\not k_1+m)]}{(p^2-m^2)[(p-k_1')^2-m^2][(p-k_1-k_2)^2-m^2][(p-k_1)^2-m^2]}\ , \tag{2.6}$$

$$\mathcal{M}_b=\mathcal{M}_a(k_1\leftrightarrow k_2,\mu\leftrightarrow\nu)\ , \tag{2.7}$$

and

$$\mathcal{M}_c=\mathcal{M}_a(k_2\leftrightarrow -k_2',\sigma\leftrightarrow\nu)\ . \tag{2.8}$$



(a) (b) (c)

FIG. 1. Lowest-order diagrams for photon-photon scattering.

The last term in (2.5) is a subtraction term to ensure that $\mathcal{M}=0$ at $k_1=k_2=\Delta=0$, a condition required by gauge invariance.

As was discussed elsewhere,[1,3] the asymptotic form of $\mathcal{M}$ in the limit $s \gg m^2$ and $|t| \gg m^2$ is obtained by setting $m=0$. Thus we have

$$\mathcal{M}_a \simeq 2ie^4 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu \not{p} \gamma_\rho (\not{p}-\not{k}_1')\gamma_\sigma(\not{p}-\not{k}_1-\not{k}_2)\gamma_\nu(\not{p}-\not{k}_1)]}{(p^2)[(p-k_1')^2][(p-k_1-k_2)^2][(p-k_1)^2]} . \tag{2.9}$$

The right-hand side of (2.9) is the quantity we need to calculate.

Since a presentation for the calculations of the photon-photon scattering amplitude already exists,[4,5] we shall give only the results, not the details, of our calculations here. In contrast to the exact formulas of Karplus and Neuman, which are quite elaborate, our asymptotic formulas are surprisingly simple. There are five independent helicity amplitudes. Asymptotically, two of them are just constants:

$$F_{++--} = -F_{++-+} = 8\alpha^2 , \tag{2.10}$$

where the plus- and the minus-helicity vectors are defined by $(\vec{e}_1+i\vec{e}_2)/\sqrt{2}$ and $(\vec{e}_1-i\vec{e}_2)/\sqrt{2}$, respectively, with $\vec{e}_2$ perpendicular to the scattering plane, for a photon with a momentum in the plus z direction, and the first two subscripts refer to the helicities of the incoming photons. The other three helicity amplitudes are

$$F_{+--+} = -4\alpha^2 \left[\frac{u^2+s^2}{t^2} \left[\ln^2\frac{u}{s} + \pi^2 \right] + \frac{2(u-s)}{t} \ln\frac{u}{s} + 2 \right] , \tag{2.11}$$

$$F_{++++} = F_{+--+}(t \leftrightarrow s) , \tag{2.12}$$

and

$$F_{+-+-} = F_{+--+}(t \leftrightarrow u) . \tag{2.13}$$

In the above, u and t are negative and their phase angles are defined to be π. We shall, by using (2.4), express the helicity amplitudes as functions of θ. We shall also exhibit the real and the imaginary parts of these amplitudes explicitly. We get

$$F_{+--+} = -16\alpha^2 \left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} \left[\ln^2\cos\frac{\theta}{2} + i\pi \ln\cos\frac{\theta}{2} \right] + \frac{1+\cos^2(\theta/2)}{\sin^2(\theta/2)} \left[\ln\cos\frac{\theta}{2} + \tfrac{1}{2} i\pi \right] + \frac{1}{2} \right] , \tag{2.14}$$

$$F_{++++} = -16\alpha^2 \left\{ \left[\sin^4\frac{\theta}{2} + \cos^4\frac{\theta}{2} \right] \left[\ln^2\cot\frac{\theta}{2} + \frac{\pi^2}{4} \right] - \cos\theta \ln\cot\frac{\theta}{2} + \frac{1}{2} \right\} , \tag{2.15}$$

and

$$F_{+-+-}=-16\alpha^2\left[\frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\left[\ln^2\sin\frac{\theta}{2}+i\pi\ln\sin\frac{\theta}{2}\right]+\frac{1+\sin^2(\theta/2)}{\cos^2(\theta/2)}\left[\ln\sin\frac{\theta}{2}+\tfrac{1}{2}i\pi\right]+\frac{1}{2}\right]. \tag{2.16}$$

All other helicity amplitudes are related to the five amplitudes above by parity conservation, time-reversal invariance, and Bose statistics:

$$\begin{aligned}&F_{--++}=F_{++--},\quad F_{-++-}=F_{+--+},\\ &F_{----}=F_{++++},\quad F_{+-+-}=F_{-+-+},\\ &F_{--+-}=F_{---+}=F_{+---}=F_{-+--}=F_{+++-}=F_{-+++}=F_{+-++}=F_{++-+}.\end{aligned} \tag{2.17}$$

With (2.17) and (1.2), we obtain the differential cross section for unpolarized photons as

$$\lim_{\omega/m\to\infty}\omega^2 d\sigma/d\theta=\frac{\sin\theta}{256\pi}(|F_{++--}|^2+|F_{+--+}|^2+|F_{++++}|^2+|F_{+-+-}|^2+4|F_{++-+}|^2). \tag{2.18}$$

The right-hand side of (2.18) divided by α^4 is plotted in Fig. 2. We illustrate this function only for $0\leq\theta\leq\pi/2$ as it is invariant under $\theta\to\pi-\theta$. There are, unfortunately, no experimental data to compare with (2.18).

The scaling functions at $\theta=\pi/2$ have already been given by Karplus and Neuman.[6] Our results disagree slightly with theirs. However, the results agree completely if we change the plus helicities of the outgoing photons into the minus helicities and vice versa.

The asymptotic formula (1.1) actually holds as long as $|t|\gg m^2$ and $s\gg m^2$. Furthermore, although (1.1) does not hold at the forward direction $\theta=0$, we may obtain the leading terms (terms of the order of $\ln^2\omega$ and $i\ln\omega$) of the forward amplitude by setting $t=-m^2$ in (2.14)–(2.16). These leading terms agree with those given by Karplus and Neuman[6] if we make the change of $M_{++++}\leftrightarrow M_{+-+-}$ in their paper.

III. ARBITRARY PERTURBATIVE ORDERS

In this section, we study the behavior of the photon-photon scattering amplitude of higher perturbative orders in the limit of $\omega/m\to\infty$ with θ fixed. The higher-order diagrams can be obtained from those in Fig. 1 by adding more photon lines or (and) electron loops. Since the added photon lines and fermion loops are always attached to lines which are off the mass shell, the addition of lines and loops does not induce any mass divergences.

There is only one exception: Mass divergences can occur as a result of renormalization. For example, while the unrenormalized vertex function has no mass divergences at $m=0$ if the external fermion lines are off the mass shell, it is ultraviolet divergent. To define this function, we must take a renormalization. The renormalized vertex function obtained by a subtraction at an on-the-mass-shell point has a mass divergence, since the subtraction

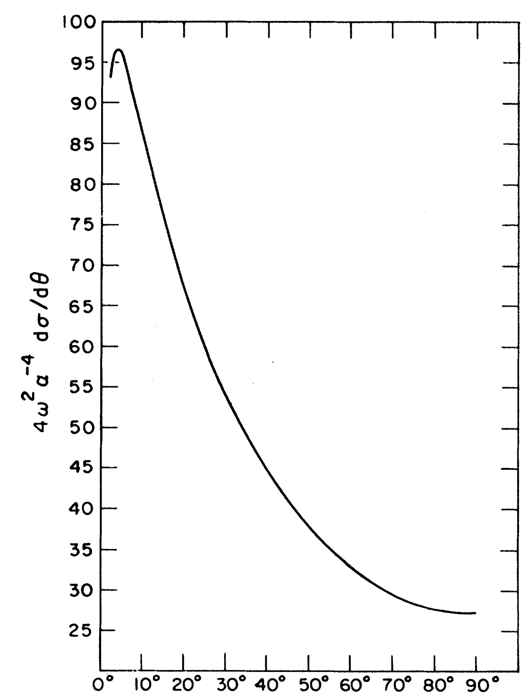


FIG. 2. The differential cross section for unpolarized photons. The function plotted is the energy-independent quantity $\alpha^{-4}\omega^2 d\sigma/d\theta$. This function is plotted only for $0\leq\theta\leq\pi/2$. The value of this function at θ in the range of $[\pi/2,\pi]$ is equal to that at $\pi-\theta$.

term does. Similarly, the renormalized photon propagator and the renormalized fermion propagator have mass divergences. This means that, while irreducible diagrams give amplitudes which scale in the form of (1.1), diagrams with self-energy insertions or (and) vertex insertions give, after renormalization, factors of $\ln(\omega/m)$.

Because of the Ward identity, the vertex renormalization constant and the fermion wave-function renormalization constant cancel each other. This means that the mass divergences obtained from the renormalization of the vertex and the fermion wave function cancel. This, in turn, implies that the $\ln(\omega/m)$ factors obtained from making such renormalizations cancel. Next, the mass divergences resulting from making mass renormalizations are of the form of m times a power of $\ln m$, as the self-mass has such divergences. Since such a form vanishes at $m=0$, mass renormalization does not give factors of $\ln(\omega/m)$. We have therefore shown that the factors of $\ln(\omega/m)$ from vertex insertions and fermion self-energy insertions completely cancel one another.

The uncanceled factors of $\ln(\omega/m)$ therefore come from renormalizing photon self-energy insertions. One way to deal with these logarithmic factors is to make the subtraction for the photon self-energy function at the point $k^2=t$ instead of at $k^2=0$, where k is the momentum of the photon. Since the subtraction point is off the mass shell, the renormalized photon self-energy function has no mass divergences at $m=0$, and yields no factors of $\ln(\omega/m)$. The logarithmic factors are absorbed into the renormalized coupling constant, which is the running coupling constant evaluated at t. However, since the external photons are on the mass shell, the coupling constant at their vertices is $e(t)$ evaluated at $t=0$. The photon-photon scattering amplitude of the nth perturbative order is therefore of the form of (1.3). The violation of scaling is therefore given by the dependence of the running coupling constant on t. (Another way to see this is to invoke the familiar renormalization-group argument. The amplitude has no divergence at $\lambda=m=0$, and, after factoring out its dimension, is a function of the dimensionless variables t/μ^2 and θ only, where μ is the point of subtraction. If we choose $\mu^2=t$, then the amplitude is independent of ω.)

ACKNOWLEDGMENT

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