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Delbrück scattering

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We calculate the lowest-order Delbrück scattering amplitude without approximations. We also show that this amplitude scales in the form of $\omega^{-1}f(\theta)$ as $\omega/m \to \infty$ with θ fixed, where ω is the energy of the photon, m is the electron mass, and θ is the scattering angle. In addition, we prove that this scaling behavior is obeyed by the Delbrück amplitude even if the higher-order diagrams of multiphoton exchanges are taken into account. Existing experimental data appear to be in conflict with the scaling behavior, and we suggest that the data be reanalyzed or (and) additional experiments be performed.

I. INTRODUCTION

Delbrück scattering is a classic problem in QED which remains to be calculated. The existence of Delbrück scattering (that of a photon by a static Coulomb field Ze^2/r) was first proposed almost 50 years ago.[1] On the experimental side, people have measured its cross sections for various energies and scattering angles.[2] However, on the theoretical side, its exact calculation has never been successfully completed. There is no problem in principle, for we may apply the Feynman rules and express it in the form of multiple integrals in the momentum space. However, the numerators of the integrands contain so many terms that the process of its Feynman parametrization and momentum-space integration has discouraged even the hardiest souls.

Theoretical results on Delbrück scattering have therefore been limited to special regions. For example, the imaginary part of the lowest-order Delbrück scattering amplitude at the forward direction is related to pair production by the optical theorem, and can therefore be more easily computed. This was done by Jost, Luttinger, and Slotnik.[3] The real part of the forward Delbrück amplitude can then be deduced by the dispersion relation.[4,5] In 1969, Cheng and Wu[6] succeeded in calculating the Delbrück amplitude in the limit $\omega/m \gg 1$ with $|\vec{\Delta}| \ll \omega$, where ω, $\vec{\Delta}$, and m are the incident photon energy, the momentum transfer, and the electron mass, respectively. This is the region of high energies and small scattering angles, where most scatterings occur. Their calculation also includes the higher-order effects of multiphoton exchanges, and therefore is valid even when $Z\alpha = O(1)$, where α is the fine-structure constant.

We have finally carried out the exact calculation

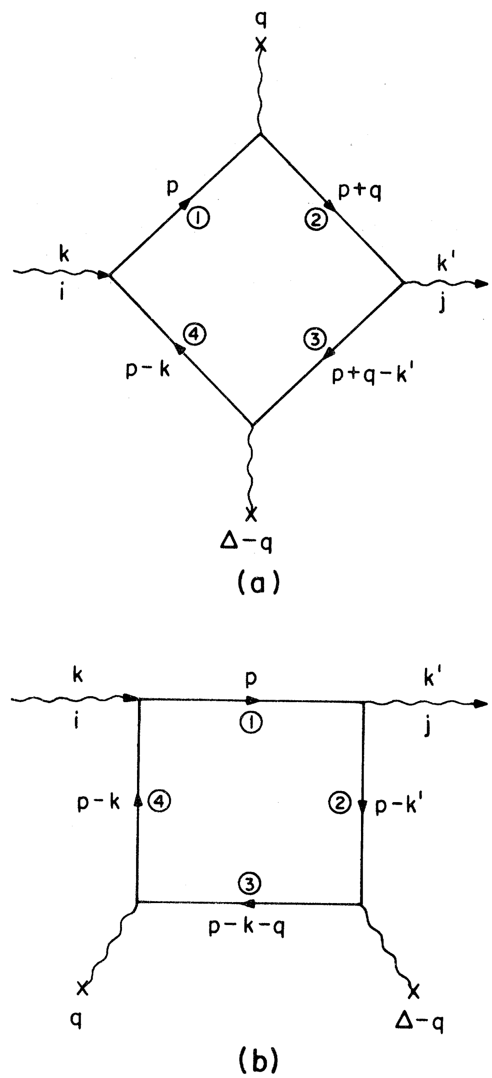

FIG. 1. Lowest-order diagrams for Delbrück scattering.

of the lowest-order Delbrück scattering amplitude. The result is reported in this paper. We also find[7] that this amplitude is asymptotically equal to

$$\omega^{-1}f(\theta) \tag{1.1}$$

in the limit $\omega/m \gg 1$, $|\vec{\Delta}|/m \gg 1$, where θ is the scattering angle related to ω and $|\vec{\Delta}|$ by

$$|\vec{\Delta}| = 2\omega \sin\frac{\theta}{2} .$$

This is the region of high energies with fixed scattering angles. The scaling formula (1.1) is a consequence of the finiteness of the Delbrück amplitude at $m=0$, and is obtained by setting $m=0$ in the Delbrück amplitude.

Although we are as yet unable to calculate the corrections of multiphoton exchanges to the Delbrück amplitude, we have proved that this amplitude, with the multiphoton-exchange corrections included, still satisfies (1.1). This will be discussed in Appendix B. It follows from (1.1) that the differential cross section multiplied by ω^2 is a function of θ only, independent of ω:

$$\omega^2 \frac{d\sigma_D}{d\Omega} \simeq (4\pi)^{-2}|f(\theta)|^2, \quad \omega \gg m . \tag{1.2}$$

The observation that the Delbrück amplitude is finite in the limit $m \to 0$ helps to simplify the exact calculation of the lowest-order Delbrück amplitude for $m \neq 0$. This is because terms in the numerators which diverge as $m \to 0$ must cancel one another. It is therefore helpful to group the terms into sums, each of which is convergent at $m=0$. There are then extensive cancellations among terms in each sum and the final expressions are simpler than what we naively expected.

II. CALCULATIONS

In this section, we give the details of the exact calculation of the lowest-order Delbrück amplitude. The reader who is not interested in such details is advised to skip to the end of the section.

The Delbrück amplitude of Fig. 1 is given by

$$\mathscr{M}_0^{(D)} = ie^6 Z^2 \int \frac{d^3q}{(2\pi)^3} \frac{M_0(k,k',q)}{(\vec{q}^{\,2})(\vec{\Delta}-\vec{q})^2} , \tag{2.1}$$

where M_0 is the lowest-order amplitude for the scattering of a real photon from a Coulomb photon:

$$\begin{aligned} M_0(k,k',q) = \frac{i\delta_{ij}}{24\pi^2} + \int \frac{d^4p}{(2\pi)^4} \Bigg\{ & \frac{\mathrm{Tr}[\gamma_i(\not p+m)\gamma_0(\not p+\not q+m)\gamma_j(\not p+\not q-\not k'+m)\gamma_0(\not p-\not k+m)]}{(p^2-m^2)[(p+q)^2-m^2][(p+q-k')^2-m^2][(p-k)^2-m^2]} \\ & + \frac{\mathrm{Tr}[\gamma_i(\not p+m)\gamma_j(\not p-\not k'+m)\gamma_0(\not p-\not k-\not q+m)\gamma_0(\not p-\not k+m)]}{(p^2-m^2)[(p-k')^2-m^2][(p-k-q)^2-m^2][(p-k)^2-m^2]} \\ & + \text{preceding term with } q \to \Delta - q \Bigg\} . \end{aligned} \tag{2.2}$$

In (2.2) the constant $i\delta_{ij}/24\pi^2$ is a subtraction term which is not given by the Feynman rules but is required to ensure the condition of gauge invariance of M_0 (0,0,0)=0. Also, the contribution of diagram 1(b) to M_0 is expressed as the sum of two terms: the last two in (2.2). Upon integration over $\vec{q}$, these two terms actually contribute equally to $\mathscr{M}_0^{(D)}$. We choose the present expression in order to retain the symmetry of M_0 with respect to the two momenta q and $\Delta - q$.

A. Traces

The numerators in (2.2) are evaluated in a straightforward way. We have

$$\begin{aligned} &\mathrm{Tr}[\gamma_i(\not p_1+m)\gamma_0(\not p_2+m)\gamma_j(\not p_3+m)\gamma_0(\not p_4+m)] \\ &\quad = 8E_2E_4[p_{1i}p_{3j}+p_{3i}p_{1j}+\delta_{ij}(p_1\cdot p_3-m^2)] + 8E_1E_3[p_{2i}p_{4j}+p_{4i}p_{2j}+\delta_{ij}(p_2\cdot p_4-m^2)] \\ &\quad\quad + 8E_1E_2[p_{4i}p_{3j}-p_{3i}p_{4j}-\delta_{ij}(p_3\cdot p_4-m^2)] + 8E_3E_4[p_{1i}p_{2j}-p_{2i}p_{1j}-\delta_{ij}(p_1\cdot p_2-m^2)] \\ &\quad\quad - 4(p_{1i}p_{2j}-p_{2i}p_{1j})(p_3\cdot p_4-m^2) - 4(p_{4i}p_{3j}-p_{3i}p_{4j})(p_1\cdot p_2-m^2) \\ &\quad\quad - 4(p_{1i}p_{3j}+p_{3i}p_{1j})(p_2\cdot p_4-m^2) - 4(p_{2i}p_{4j}+p_{4i}p_{2j})(p_1\cdot p_3-m^2) \\ &\quad\quad + 4(p_{1i}p_{4j}+p_{4i}p_{1j})(p_2\cdot p_3-m^2) + 4(p_{2i}p_{3j}+p_{3i}p_{2j})(p_1\cdot p_4-m^2) \\ &\quad\quad + 4\delta_{ij}[(p_1\cdot p_2-m^2)(p_3\cdot p_4-m^2)+(p_1\cdot p_4-m^2)(p_2\cdot p_3-m^2)-(p_1\cdot p_3-m^2)(p_2\cdot p_4-m^2)] \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\mathrm{Tr}[\gamma_i(\not p_1+m)\gamma_j(\not p_2+m)\gamma_0(\not p_3+m)\gamma_0(\not p_4+m)]\\&\quad=8E_3E_4[p_{1i}p_{2j}+p_{2i}p_{1j}+\delta_{ij}(p_1\cdot p_2-m^2)]+8E_2E_3[p_{1i}p_{4j}+p_{4i}p_{1j}+\delta_{ij}(p_1\cdot p_4-m^2)]\\&\qquad+8E_1E_3[p_{4i}p_{2j}-p_{2i}p_{4j}-\delta_{ij}(p_2\cdot p_4-m^2)]-4(p_{1i}p_{2j}+p_{2i}p_{1j})(p_3\cdot p_4-m^2)]\\&\qquad-4(p_{1i}p_{4j}+p_{4i}p_{1j})(p_2\cdot p_3-m^2)+4(p_{1i}p_{3j}+p_{3i}p_{1j})(p_2\cdot p_4-m^2)\\&\qquad-4(p_{4i}p_{2j}-p_{2i}p_{4j})(p_1\cdot p_3-m^2)+4(p_{3i}p_{2j}-p_{2i}p_{3j})(p_1\cdot p_4-m^2)\\&\qquad+4(p_{4i}p_{3j}-p_{3i}p_{4j})(p_1\cdot p_2-m^2)\\&\qquad-4\delta_{ij}[(p_1\cdot p_2-m^2)(p_3\cdot p_4-m^2)+(p_1\cdot p_4-m^2)(p_2\cdot p_3-m^2)-(p_1\cdot p_3-m^2)(p_2\cdot p_4-m^2)]\,. \qquad (2.4)\end{aligned}$$

In (2.3) and (2.4), E_n is the time component of p_n, $n=1,2,3,4$. By identifying p_n with the momentum of the nth line in the diagrams (see Fig. 1 for the number designation for the lines), we obtain the explicit forms for the numerators in (2.2).

B. Cancellation of trace terms

There are terms of diagram 1(a) which cancel terms of diagram 1(b). This comes about because diagram 1(a) with line 1 fused is the same as diagram 1(b) with line 4 fused, if we identify lines 2, 3, and 4 in diagram 1(a) with lines 1, 2, and 3 in diagram 1(b), respectively. Since a factor $(p_i{}^2-m^2)$ in the numerator cancels the propagator of the ith line and has the effect of fusing line i, we get, for example, the following relations between terms of the numerators:

$$\begin{aligned}&p_{2j}(p_1{}^2-m^2)\ \text{for diagram 1(a)}\\&\quad=p_{1j}(p_4{}^2-m^2)\ \text{for diagram 1(b)}\,. \qquad (2.5\text{a})\end{aligned}$$

Similar relations arise from fusing other internal lines. For example,

$$\begin{aligned}&p_{1i}(p_3{}^2-m^2)\ \text{for diagram 1(a)}\\&\quad=-p_{4i}(p_2{}^2-m^2)\ \text{for diagram 1(b)}\,. \qquad (2.5\text{b})\end{aligned}$$

The numerators become simpler if we take advantage of relations such as (2.5). We shall express $2p_n\cdot p_m$ in the numerator by

$$2p_n\cdot p_m=p_n{}^2+p_m{}^2-(p_n-p_m)^2\,. \qquad (2.6)$$

A term proportional to $(p_i{}^2-m^2)$ in the numerator of diagram 1(a) can be used to cancel a corresponding term in diagram 1(b). Care must be exercised, however, in choosing such terms for cancellation. This is because canceling terms which are divergent in the limit $m\to 0$ may complicate the expressions instead. According to the discussion in Sec. III, the following terms in (2.3) are convergent:

$$\begin{aligned}&2\delta_{ij}(p_1{}^2-m^2)(p_3{}^2-m^2)+2\delta_{ij}(p_2{}^2-m^2)(p_4{}^2-m^2)\\&\quad-4(p_1{}^2-m^2)(\Delta-q)_ip_{2j}-4(p_3{}^2-m^2)p_{4i}q_j+4(p_2{}^2-m^2)p_{1i}(\Delta-q)_j+4(p_4{}^2-m^2)q_ip_{3j}\,. \qquad (2.7)\end{aligned}$$

We shall therefore subtract the terms in (2.7) from (2.3) as well as the following terms from (2.4):

$$-2\delta_{ij}(p_2{}^2-m^2)(p_4{}^2-m^2)+4(p_4{}^2-m^2)(\Delta-q)_ip_{1j}-4(p_2{}^2-m^2)p_{1i}q_j\,. \qquad (2.8)$$

These terms, together with the ones obtained from (2.8) with $q\to\Delta-q$, cancel one another.

After making such cancellations, the two traces of (2.3) and (2.4) become, respectively,

$$\begin{aligned}&8E_2E_4[p_{1i}p_{3j}+p_{3i}p_{1j}+\delta_{ij}(p_1\cdot p_3-m^2)]+8E_1E_3[p_{2i}p_{4j}+p_{4i}p_{2j}+\delta_{ij}(p_2\cdot p_4-m^2)]\\&\quad+8E_1E_2[p_{4i}p_{3j}-p_{3i}p_{4j}-\delta_{ij}(p_3\cdot p_4-m^2)]+8E_3E_4[p_{1i}p_{2j}-p_{2i}p_{1j}-\delta_{ij}(p_1\cdot p_2-m^2)]\\&\quad-2(p_1{}^2-m^2)\Delta_i(\Delta-q)_j-2(p_2{}^2-m^2)(\Delta-q)_i\Delta_j-2(p_3{}^2-m^2)q_i\Delta_j-2(p_4{}^2-m^2)\Delta_iq_j\\&\quad-2p_{1i}p_{2j}(A+B+C+D)+2p_{3i}p_{4j}A+2p_{2i}p_{1j}B-2p_{3i}p_{1j}C-2p_{2i}p_{4j}D\\&\quad+\delta_{ij}[(p_1{}^2-m^2)(B-C)+(p_2{}^2-m^2)(B-D)+(p_3{}^2-m^2)(A-C)+(p_4{}^2-m^2)(A-D)+AB-CD] \qquad (2.9)\end{aligned}$$

and

$$-4\delta_{ij}E_2[(p_2{}^2+p_4{}^2-2m^2)\omega+E_1\vec{\Delta}^2]+8E_2[\omega(\Delta_i p_{1j}-p_{1i}\Delta_j)+E_1\Delta_i\Delta_j]+4p_{1i}p_{1j}(8E_2{}^2-C-D)$$
$$+2[p_{1i}\Delta_j(A-D)+\Delta_i p_{1j}(A-C)-(p_{1i}q_j+q_i p_{1j})\vec{\Delta}^2-C\Delta_i\Delta_j]$$
$$+2[(p_1{}^2-m^2)(q_i\Delta_j-\Delta_i q_j)+(p_2{}^2-m^2)q_i\Delta_j-(p_3{}^2-m^2)\Delta_i\Delta_j+(p_4{}^2-m^2)\Delta_i(\Delta-q)_j]$$
$$-\delta_{ij}[(p_1{}^2-m^2)(C+D)+(p_2{}^2-m^2)(A-C)-(p_3{}^2-m^2)\vec{\Delta}^2+(p_4{}^2-m^2)(B-C)-C\vec{\Delta}^2]$$
$$-8(p_3{}^2-m^2)p_{1i}p_{1j}+8E_2{}^2(p_1{}^2-m^2)\delta_{ij}-2(p_1{}^2-m^2)(p_3{}^2-m^2)\delta_{ij}\ . \tag{2.10}$$

In (2.9) and (2.10) ω is the energy of the incident photon,

$$A=-q^2=\vec{q}^{\,2},\quad B=-(\Delta-q)^2=(\vec{\Delta}-\vec{q})^2\ ,$$
$$C=-(k+q)^2=2\vec{k}\cdot\vec{q}+\vec{q}^{\,2},\quad D=-(k'-q)^2=-2\vec{k}'\cdot\vec{q}+\vec{q}^{\,2}\ .$$

Note that

$$A+B-C-D=\vec{\Delta}^2\ .$$

In deriving (2.9) and (2.10), we have made use of the transversality condition for the photons $k_i=k_j'=0$, which leads to $p_{1i}=p_{4i}$, $p_{2j}=p_{3j}$ for diagram 1(a) and $p_{1i}=p_{4i}$, $p_{1j}=p_{2j}$ for diagram 1(b).

C. Feynman parameters

Next we introduce Feynman parameters α_n, $n=1,2,3,4$ for the nth internal line. We then make, for diagram 1(a), the change of variables

$$p=l+(\alpha_3+\alpha_4)k+\alpha_3\Delta-(\alpha_2+\alpha_3)q\ , \tag{2.11a}$$

and for diagram 1(b), we make the change of variables

$$p=l+(\alpha_2+\alpha_3+\alpha_4)k+\alpha_2\Delta+\alpha_3 q\ . \tag{2.11b}$$

Then (2.2) with the traces given by (2.9) and (2.10) becomes

$$M_0(k,k',q)=6\int d^4\alpha\,\delta\left[1-\sum_1^4\alpha_n\right]\int\frac{d^4l}{(2\pi)^4}\left[\frac{N_a}{(l^2-\mathscr{D}_a)^4}+\frac{N_b}{(l^2-\mathscr{D}_b)^4}+\frac{N_c}{(l^2-\mathscr{D}_c)^4}\right]$$
$$+8\int d\alpha_1 d\alpha_2 d\alpha_4\delta(1-\alpha_1-\alpha_2-\alpha_4)\int\frac{d^4l}{(2\pi)^4}\frac{l^2\delta_{ij}+4\alpha_2\alpha_4\Delta_i\Delta_j}{(l^2-\alpha_2\alpha_4\vec{\Delta}^2-m^2)^3}$$
$$+\int d\alpha_2 d\alpha_3 d\alpha_4\delta(1-\alpha_2-\alpha_3-\alpha_4)\int\frac{d^4l}{(2\pi)^4}\frac{8l^2\delta_{ij}}{(l^2-\alpha_3\alpha_4 A-\alpha_2\alpha_3 B-\alpha_2\alpha_4\vec{\Delta}^2-m^2)^3}$$
$$-\int_0^1 d\alpha\int\frac{d^4l}{(2\pi)^4}\frac{4\delta_{ij}}{[l^2-\alpha(1-\alpha)\vec{\Delta}^2-m^2]^2}+\frac{i\delta_{ij}}{24\pi^2}\ . \tag{2.12}$$

In (2.12)

$$N_a=-\tfrac{4}{3}\delta_{ij}(l^2)^2+Pl^2+Q \tag{2.13}$$

with

$$P=-2\Delta_i\Delta_j+8[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[(\alpha_1+\alpha_4)q-\alpha_4\Delta]_j$$
$$+\delta_{ij}(C+D+\vec{\Delta}^2)-4\delta_{ij}\omega^2[(\alpha_1+\alpha_2)^2+(\alpha_3+\alpha_4)^2]\ , \tag{2.14}$$

$$Q=4[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[\alpha_4\Delta-(\alpha_1+\alpha_4)q]_j[8\omega^2(\alpha_3+\alpha_4)(\alpha_1+\alpha_2)+C+D]$$
$$+4\omega^2(\alpha_1\alpha_3-\alpha_2\alpha_4)[\delta_{ij}(D-C)+2(\Delta_i q_j-q_i\Delta_j)]+[(2\Delta^2-8\omega^2)\delta_{ij}-4\Delta_i\Delta_j](\mathscr{D}_a-2m^2)\ , \tag{2.15}$$

and

$$\mathscr{D}_a=\alpha_1\alpha_2 A+\alpha_3\alpha_4 B+\alpha_2\alpha_4 C+\alpha_1\alpha_3 D+m^2 . \tag{2.16}$$

Also,

$$N_b=-\tfrac{4}{3}\delta_{ij}(l^2)^2+Rl^2+S , \tag{2.17}$$

$$R=\delta_{ij}(12\alpha_1\omega^2+2C-A-B-8\alpha_1{}^2\omega^2)+8(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j+4(q_i\Delta_j-\Delta_i q_j)+2\Delta_i\Delta_j , \tag{2.18}$$

$$\begin{aligned}S=&[4(q_i\Delta_j-\Delta_i q_j)+\delta_{ij}(8\alpha_1\omega^2-2D)](\mathscr{D}_b-2m^2)+[2AB-4\alpha_1\omega^2(C+D)]\alpha_3\delta_{ij}+8\alpha_1\alpha_3\omega^2(q_i\Delta_j-\Delta_i q_j)\\&+4(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j(8\alpha_1{}^2\omega^2-C-D)-4\alpha_3[\Delta_i(\Delta-q)_j A+q_i\Delta_j B-q_i(\Delta-q)_j\vec{\Delta}^2] ,\end{aligned} \tag{2.19}$$

with

$$\mathscr{D}_b=\alpha_3\alpha_4 A+\alpha_2\alpha_3 B+\alpha_1\alpha_3 C+\alpha_2\alpha_4\vec{\Delta}^2+m^2 \tag{2.20}$$

and

$$N_c=N_b(q\leftrightarrow\Delta-q),\quad D_c=D_b(q\leftrightarrow\Delta-q) .$$

In deriving the above, we have made use of the formula

$$(\delta p_i)^2=\mathscr{D}-\frac{\partial\mathscr{D}}{\partial\alpha_i}-m^2 , \tag{2.21}$$

where $\delta p_i=p_i-l$, $i=1,2,3,4$. This formula is derived in Appendix A. We have also made the following replacements:

$$l_\mu l_\nu\to\tfrac{1}{4}g_{\mu\nu}l^2,\quad l_\mu l_\nu l_\rho l_\sigma\to\frac{(l^2)^2}{24}(g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}) .$$

We remark that a great amount of cancellation has occurred in the derivation of (2.13)–(2.20).

Next we carry out the integration over l. We shall need to use the following formulas:

$$\int\frac{d^4p}{(p^2-\mathscr{D})^4}\begin{bmatrix}1\\p^2\\(p^2)^2\end{bmatrix}=\frac{i\pi^2}{6\mathscr{D}^2}\begin{bmatrix}1\\-2\mathscr{D}\\6\mathscr{D}^2\ln\left[\dfrac{\Omega}{\mathscr{D}}\right]-11\mathscr{D}^2\end{bmatrix} , \tag{2.22}$$

$$\int\frac{d^4p}{(p^2-\mathscr{D})^3}\begin{bmatrix}1\\p^2\end{bmatrix}=-\frac{i\pi^2}{2\mathscr{D}}\begin{bmatrix}1\\-2\mathscr{D}\ln\left[\dfrac{\Omega}{\mathscr{D}}\right]+3\mathscr{D}\end{bmatrix} , \tag{2.23}$$

and

$$\int\frac{d^4p}{(p^2-\mathscr{D})^2}=i\pi^2\left[\ln\frac{\Omega}{\mathscr{D}}-1\right] , \tag{2.24}$$

where Ω is an ultraviolet cutoff which we introduce for the convenience of handling divergent integrals. Since the Delbrück amplitude is ultraviolet finite, this cutoff parameter will disappear from the final expression. With (2.22)–(2.24), we may carry out the integration over l in (2.12) and get

$$M_0(k,k',q)=\frac{i}{16\pi^2}\int d^4\alpha\,\delta\left[1-\sum_1^4\alpha_n\right]\left[\frac{P'}{\mathscr{D}_a}+\frac{Q'}{\mathscr{D}_a{}^2}+8\delta_{ij}\ln\mathscr{D}_a+\frac{R'}{\mathscr{D}_b}+\frac{S'}{\mathscr{D}_b{}^2}+8\delta_{ij}\ln\mathscr{D}_b\right.$$

$$\left.+\text{ the three preceding terms with } q\to\Delta-q\right]$$

$$-\frac{i\delta_{ij}}{2\pi^2}\int d\alpha_2 d\alpha_3 d\alpha_4\delta(1-\alpha_2-\alpha_3-\alpha_4)\ln(\alpha_3\alpha_4 A+\alpha_2\alpha_3 B+\alpha_2\alpha_4\vec{\Delta}^2+m^2)$$

$$+\frac{i}{4\pi^2}\frac{m^2}{\vec{\Delta}^2}\left[2\frac{\Delta_i\Delta_j}{\vec{\Delta}^2}-\delta_{ij}\right]\int_0^1\frac{d\alpha\ln[1+\alpha(1-\alpha)\vec{\Delta}^2/m^2]}{\alpha(1-\alpha)}-\frac{i\Delta_i\Delta_j}{2\pi^2\vec{\Delta}^2}+\frac{i\delta_{ij}}{4\pi^2}\,, \tag{2.25}$$

$$P'=-16[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[(\alpha_1+\alpha_4)q-\alpha_4\Delta]_i-2\delta_{ij}[C+D+8\omega^2(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)]\,, \tag{2.26}$$

$$Q'=4[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[\alpha_4\Delta-(\alpha_1+\alpha_4)q]_j[8\omega^2(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)+C+D]$$
$$+4\omega^2(\alpha_1\alpha_3-\alpha_2\alpha_4)[\delta_{ij}(D-C)+2(\Delta_i q_j-q_i\Delta_j)]-4m^2[(\vec{\Delta}^2-4\omega^2)\delta_{ij}-2\Delta_i\Delta_j]\,, \tag{2.27}$$

$$R'=2\delta_{ij}[\vec{\Delta}^2-C-8\alpha_1(1-\alpha_1)\omega^2]-4q_i\Delta_j-4\Delta_i(\Delta-q)_j-16(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j\,, \tag{2.28}$$

and

$$S'=2[\alpha_3 AB-2\alpha_1\alpha_3\omega^2(C+D)-m^2(8\alpha_1\omega^2-2D)]\delta_{ij}+8(\alpha_1\alpha_3\omega^2-m^2)(q_i\Delta_j-\Delta_i q_j)$$
$$+4(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j(8\alpha_1{}^2\omega^2-C-D)$$
$$-4\alpha_3[\Delta_i(\Delta-q)_j A+q_i\Delta_j B-q_i(\Delta-q)_j\vec{\Delta}^2]\,. \tag{2.29}$$

We make two remarks about Eq. (2.25):

(i) Tests on the correctness of the algebra up to this point are provided by the conditions

$$M_0(k,k',0)=M_0(k,k',\Delta)=0$$

which are consequences of gauge invariance. Equation (2.25) satisfies these conditions.

(ii) The expression (2.25) is finite at $m=0$. For example, at $m=0$, $\mathscr{D}_a$ vanishes at $\alpha_1=\alpha_4=0$. Thus, the integral of $(\mathscr{D}_a)^{-2}$ over the two-dimensional plane normal to the line of singularity $\alpha_1=\alpha_4=0$ is logarithmically divergent. However, all terms of Q' vanish at $\alpha_1=\alpha_4=0$, therefore the integrals in (2.25) have no infinities caused by the singularity at $\alpha_1=\alpha_4=0$. Similarly for the singularity of $(\mathscr{D}_a)^{-2}$ at $\alpha_1=\alpha_3=0$ and the singularity of $(\mathscr{D}_b)^{-2}$ at $\alpha_2=\alpha_3=0$ and $\alpha_4=\alpha_3=0$. Thus M_0 is finite at $m=0$. This implies that $\mathscr{M}_0^{(D)}$ is finite at $m=0$, as the integration over $\vec{q}$ in (2.1) also gives no divergences.

The logarithmic functions in the integrals in (2.25) can be eliminated by making use of the following identities:

$$\int d^4\alpha\,\delta\left[1-\sum_1^4\alpha_k\right]\ln\mathscr{D}=\tfrac{1}{3}\int d^4\alpha\,\delta\left[1-\sum_1^4\alpha_k\right]\left[\frac{\partial\mathscr{D}/\partial\alpha_n+2m^2}{\mathscr{D}}-2\right]$$
$$+\tfrac{1}{3}\int d^3\alpha\,\delta\left[1-\sum_{k\neq n}\alpha_k\right]\ln\mathscr{D}\Big|_{\alpha_n=0}\,,\quad n=1,2,3,4 \tag{2.30}$$

where $\mathscr{D}$ can be either $\mathscr{D}_a$, $\mathscr{D}_b$, or $\mathscr{D}_c$. A proof of (2.30) will be given in Appendix A. We express the logarithmic functions in the first integral in (2.25) as

$$8(\ln\mathscr{D}_a+\ln\mathscr{D}_b+\ln\mathscr{D}_c)=4\ln(\mathscr{D}_a{}^2/\mathscr{D}_b\mathscr{D}_c)+12\ln(\mathscr{D}_b\mathscr{D}_c)\,. \tag{2.31}$$

For the first term in (2.31), we apply (2.30) in such a way that the integrated parts cancel. This is possible because $\mathscr{D}_a\,|_{\alpha_1=0}$ is equal to $\mathscr{D}_b\,|_{\alpha_4=0}$ with $\alpha_n\to\alpha_{n+1}$, $n=1,2,3$, and similarly for $\mathscr{D}_a\,|_{\alpha_n=0}$, $n=2,3,4$. For $12\ln(\mathscr{D}_b\mathscr{D}_c)$ in (2.31) we integrate by parts with respect to α_1. We then get

$$M_0 \simeq \frac{i}{16\pi^2}\int d^4\alpha\,\delta\left[1-\sum_1^4 \alpha_n\right]\left\{\frac{P'+\frac{2}{3}\delta_{ij}[8m^2+(\alpha_1+\alpha_2)A+(\alpha_3+\alpha_4)B+(\alpha_2+\alpha_4)C+(\alpha_1+\alpha_3)D]}{\mathscr{D}_a}\right.$$
$$+\frac{Q'}{\mathscr{D}_a{}^2}+\frac{R'-\frac{2}{3}\delta_{ij}[\alpha_3(A+B)+(\alpha_2+\alpha_4)\vec{\Delta}^2-8m^2]+4\alpha_3 C\delta_{ij}}{\mathscr{D}_b}+\frac{S'}{\mathscr{D}_b{}^2}$$
$$\left.+\text{ the two preceding terms with } q\to\Delta-q\right\}$$
$$-\frac{i\Delta_i\Delta_j}{2\pi^2\vec{\Delta}^2}+\frac{i}{4\pi^2}\frac{m^2}{\vec{\Delta}^2}\left[\frac{2\Delta_i\Delta_j}{\vec{\Delta}^2}-\delta_{ij}\right]\int_0^1\frac{d\alpha\ln[1+\alpha(1-\alpha)\vec{\Delta}^2/m^2]}{\alpha(1-\alpha)}+\frac{i\delta_{ij}}{12\pi^2}\,. \tag{2.32}$$

D. Final integration

It remains to substitute (2.32) into (2.1) and carry out the integration over $\vec{q}$. We shall introduce the Feynman parameters β_5, β_6, and ρ for the factors $\vec{q}^{\,2}$, $(\vec{\Delta}-\vec{q})^2$, and $\mathscr{D}_a$ (or $\mathscr{D}_b$), respectively. Calling

$$\rho\alpha_i=\beta_i,\quad i=1,2,3,4$$

we get

$$\rho=(\beta_1+\beta_2+\beta_3+\beta_4) \tag{2.33}$$

and

$$d\beta_5 d\beta_6 d\rho\,\delta(1-\beta_5-\beta_6-\rho)d^4\alpha\,\delta\left[1-\sum_1^4\alpha_n\right]=\rho^{-3}d^6\beta\,\delta\left[1-\sum_1^6\beta_i\right].$$

Whenever we encounter a term of the numerator with a factor $\vec{q}^{\,2}$ or $(\vec{\Delta}-\vec{q})^2$, we shall use this factor to cancel the same factor in the denominator. Thus we get

$$\mathscr{M}_0^{(D)}=-\frac{\alpha^3Z^2}{\pi^2}\int\frac{d^6\beta\,\delta\left[1-\sum_1^6\beta_i\right]}{\rho^3}\int d^3q\left[\frac{P_1}{\mathscr{E}_a{}^3}+\frac{\rho Q_1}{\mathscr{E}_a{}^4}+\frac{R_1}{\mathscr{E}_b{}^3}+\frac{\rho S_1}{\mathscr{E}_b{}^4}\right]$$
$$-\frac{\alpha^3Z^2}{2\pi^2}\int\frac{d\beta_6\prod_1^4 d\beta_i\delta\left[1-\beta_6-\sum_1^4\beta_i\right]}{\rho^3}\int d^3q\left[\frac{P_2}{\mathscr{E}_a{}^2}+\frac{\rho Q_2}{\mathscr{E}_a{}^3}+\frac{R_2}{\mathscr{E}_b{}^2}+\frac{\rho S_2}{\mathscr{E}_b{}^3}\right]\Bigg|_{\beta_5=0}$$
$$-\frac{\alpha^3Z^2}{2\pi^2}\int\frac{d\beta_5\prod_1^4 d\beta_i\delta\left[1-\beta_5-\sum_1^4\beta_i\right]}{\rho^3}\int d^3q\left[\frac{P_3}{\mathscr{E}_a{}^2}+\frac{\rho Q_3}{\mathscr{E}_a{}^3}+\frac{R_3}{\mathscr{E}_b{}^2}+\frac{\rho S_3}{\mathscr{E}_b{}^3}\right]\Bigg|_{\beta_6=0}$$
$$-\frac{2\alpha^3Z^2\delta_{ij}}{\pi^2}\int d^4\alpha\,\delta\left[1-\sum_1^4\alpha_n\right]\int d^3q\frac{\alpha_3}{\mathscr{D}_b{}^2}+\frac{4\pi\alpha^3Z^2}{|\vec{\Delta}|}F(\vec{\Delta}^2,m^2)\,, \tag{2.34}$$

where

$$P_1=\tfrac{1}{3}\delta_{ij}[5\vec{\Delta}^2-48\omega^2(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)+16m^2+2(\alpha_2+\alpha_4-\alpha_1-\alpha_3)(\vec{k}+\vec{k}')\cdot\vec{q}]$$
$$-16[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[(\alpha_1+\alpha_4)q-\alpha_4\Delta]_j\,, \tag{2.35}$$

$$P_2=-\delta_{ij}[1+\tfrac{2}{3}(\alpha_3+\alpha_4)]\ , \tag{2.36}$$

$$P_3=-\delta_{ij}[1+\tfrac{2}{3}(\alpha_1+\alpha_2)]\ , \tag{2.37}$$

$$\begin{aligned}Q_1=12\{\,&[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[\alpha_4\Delta-(\alpha_1+\alpha_4)q]_j[8\omega^2(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)-\vec{\Delta}^2]\\&+2\omega^2(\alpha_1\alpha_3-\alpha_2\alpha_4)[\Delta_i q_j-q_i\Delta_j-(\vec{k}+\vec{k}')\cdot\vec{q}\delta_{ij}]-m^2[(\vec{\Delta}^2-4\omega^2)\delta_{ij}-2\Delta_i\Delta_j]\}\ ,\end{aligned} \tag{2.38}$$

$$Q_2=8[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[\alpha_4\Delta-(\alpha_1+\alpha_4)q]_j\ , \tag{2.39}$$

$$Q_3=8[\alpha_3\Delta-(\alpha_2+\alpha_3)q]_i[\alpha_4\Delta-(\alpha_1+\alpha_4)q]_j\ , \tag{2.40}$$

$$\begin{aligned}R_1=4\delta_{ij}&\left[(2\alpha_3-1)(\vec{k}+\vec{k}')\cdot\vec{q}-8\alpha_1(1-\alpha_1)\omega^2+\frac{7+2\alpha_1-4\alpha_3}{6}\vec{\Delta}^2+\tfrac{8}{3}m^2\right]\\&-8q_i\Delta_j-8\Delta_i(\Delta-q)_j-32(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j\ ,\end{aligned} \tag{2.41}$$

$$R_2=2\delta_{ij}\left[\frac{4\alpha_3}{3}-1\right]\ , \tag{2.42}$$

$$R_3=2\delta_{ij}\left[\frac{4\alpha_3}{3}-1\right]\ , \tag{2.43}$$

$$\begin{aligned}S_1=24\{\,&\alpha_1\alpha_3\omega^2\vec{\Delta}^2\delta_{ij}-m^2[4\alpha_1\omega^2+\tfrac{1}{2}\vec{\Delta}^2+(\vec{k}+\vec{k}')\cdot\vec{q}]\delta_{ij}+(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j(8\alpha_1{}^2\omega^2+\vec{\Delta}^2)\\&+\alpha_3 q_i(\Delta-q)_j\vec{\Delta}^2+2(\alpha_1\alpha_3\omega^2-m^2)(q_i\Delta_j-\Delta_i q_j)\}\ ,\end{aligned} \tag{2.44}$$

$$S_2=-16\{\delta_{ij}(\alpha_1\alpha_3\omega^2-\tfrac{1}{2}m^2)+(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j+\alpha_3\Delta_i(\Delta-q)_j\}\ , \tag{2.45}$$

$$S_3=-16\{\delta_{ij}(\alpha_1\alpha_3\omega^2-\tfrac{1}{2}m^2)+(\alpha_2\Delta+\alpha_3 q)_i[\alpha_3 q-(\alpha_3+\alpha_4)\Delta]_j+\alpha_3 q_i\Delta_j\}\ , \tag{2.46}$$

$$\mathscr{E}_a=\beta_5\vec{q}^2+\beta_6(\vec{\Delta}-\vec{q})^2+\rho m^2+\rho^{-1}[(\beta_1+\beta_4)(\beta_2+\beta_3)\vec{q}^2+2(\beta_2\beta_4\vec{k}-\beta_1\beta_3\vec{k}'-\beta_3\beta_4\vec{\Delta})\cdot\vec{q}+\beta_3\beta_4\vec{\Delta}^2]\ , \tag{2.47}$$

$$\mathscr{E}_b=\beta_5\vec{q}^2+\beta_6(\vec{\Delta}-\vec{q})^2+\rho m^2+\rho^{-1}[\beta_3(\beta_1+\beta_2+\beta_4)\vec{q}^2+2\beta_3(\beta_1\vec{k}-\beta_2\vec{\Delta})\cdot\vec{q}+\beta_2(\beta_3+\beta_4)\vec{\Delta}^2]\ , \tag{2.48}$$

and

$$F(\vec{\Delta}^2,m^2)=\frac{\Delta_i\Delta_j}{\vec{\Delta}^2}-\tfrac{1}{6}\delta_{ij}-\frac{m^2}{\vec{\Delta}^2}\left[\frac{2\Delta_i\Delta_j}{\vec{\Delta}^2}-\delta_{ij}\right]\int_0^1\frac{d\alpha\ln[1+\alpha(1-\alpha)\vec{\Delta}^2/m^2]}{\alpha}\ . \tag{2.49}$$

To carry out the integration over $\vec{q}$, we make, for diagram 1(a), the change of variable

$$\vec{q}=\vec{l}+\delta\vec{q}_a \tag{2.50}$$

and, for diagram 1(b), the change of variable

$$\vec{q}=\vec{l}+\delta\vec{q}_b\ , \tag{2.51}$$

where

$$\delta\vec{q}_a=[(\beta_6\rho+\beta_3\beta_4)\vec{\Delta}-\beta_2\beta_4\vec{k}+\beta_1\beta_3\vec{k}']/\Lambda_a\ , \tag{2.52}$$

$$\delta\vec{q}_b=[(\beta_6\rho+\beta_2\beta_3)\vec{\Delta}-\beta_1\beta_3\vec{k}]/\Lambda_b\ , \tag{2.53}$$

with

$$\Lambda_a=(\beta_5+\beta_6)\rho+(\beta_1+\beta_4)(\beta_2+\beta_3)\ , \tag{2.54}$$

$$\Lambda_b=(\beta_5+\beta_6)\rho+\beta_3(\beta_1+\beta_2+\beta_4)\ . \tag{2.55}$$

Then we have

$$\mathscr{E}_a = \frac{\Lambda_a}{\rho}(\vec{l}^{\,2} + C_a) \tag{2.56}$$

and

$$\mathscr{E}_b = \frac{\Lambda_b}{\rho}(\vec{l}^{\,2} + C_b) , \tag{2.57}$$

where

$$C_a = \Lambda_a^{-2}[\vec{\Delta}^2\rho(\rho\beta_5\beta_6 + \beta_3\beta_4\beta_5 + \beta_1\beta_2\beta_6) - \omega^2(\beta_1\beta_3 - \beta_2\beta_4)^2 + m^2\rho^2\Lambda_a] \tag{2.58}$$

and

$$C_b = \Lambda_b^{-2}\{\vec{\Delta}^2\rho[\rho\beta_5\beta_6 + \beta_2\beta_5(\beta_3+\beta_4) + \beta_4\beta_6(\beta_2+\beta_3) + \beta_2\beta_3\beta_4] - \omega^2\beta_1^2\beta_3^2 + m^2\rho^2\Lambda_b\} . \tag{2.59}$$

We make use of the formulas

$$\int \frac{d^3 l}{(\vec{l}^{\,2}+C)^4}\begin{bmatrix} 1 \\ \vec{l}^{\,2} \\ (\vec{l}^{\,2})^2 \\ (\vec{l}^{\,2}+C) \\ (\vec{l}^{\,2}+C)^2 \\ \vec{l}^{\,2}(\vec{l}^{\,2}+C) \end{bmatrix} = \frac{\pi^2}{8C^{5/2}}\begin{bmatrix} 1 \\ C \\ 5C^2 \\ 2C \\ 8C^2 \\ 6C^2 \end{bmatrix} .$$

Then (2.34) becomes

$$\begin{aligned}\mathscr{M}_0^{(D)} \simeq{}& \alpha^3 Z^2 \int d^6\beta\, \delta\left[1 - \sum_1^6 \beta_i\right]\left[\frac{N_1}{\rho\Lambda_a^4 C_a^{3/2}} + \frac{N_2}{\Lambda_a^4 C_a^{5/2}} + \frac{N_3}{\rho\Lambda_b^4 C_b^{3/2}} + \frac{N_4}{\Lambda_b^4 C_b^{5/2}}\right] \\ &+ \alpha^3 Z^2 \int \prod_1^5 d\beta_i\, \delta\left[1 - \sum_1^5 \beta_i\right]\left[\frac{N_5}{\Lambda_a^2\rho C_a^{1/2}} + \frac{N_6}{\Lambda_a^3 C_a^{3/2}} + \frac{N_7}{\Lambda_b^2\rho C_b^{1/2}} + \frac{N_8}{\Lambda_b^3 C_b^{3/2}}\right]\Bigg|_{\beta_6=0} \\ &- 2\alpha^3 Z^2 \delta_{ij} \int \frac{d^4\alpha\, \delta\left[1 - \sum_1^4 \alpha_n\right]\alpha_3^{-1/2}}{(\alpha_1+\alpha_2+\alpha_4)[\vec{\Delta}^2\alpha_2\alpha_4 - \alpha_1^2\alpha_3\omega^2 + m^2(\alpha_1+\alpha_2+\alpha_4)]^{1/2}} + \frac{4\pi\alpha^3 Z^2}{|\vec{\Delta}|}F(\vec{\Delta}^2, m^2) ,\end{aligned} \tag{2.60}$$

where

$$\begin{aligned} N_1 ={}& \frac{8\beta_2\beta_6(\beta_4\beta_5 - \beta_1\beta_6)\rho}{\Lambda_a}\Delta_i\Delta_j - 8\delta_{ij}\beta_2(\beta_1+\beta_4)C_a\Lambda_a/\rho \\ &+ \delta_{ij}[8(\beta_1+\beta_2)(\beta_3+\beta_4)\beta_5 - \tfrac{2}{3}(\beta_2+\beta_4)(\beta_1\beta_3 - \beta_2\beta_4)](\omega^2 - \tfrac{1}{4}\vec{\Delta}^2) \\ &- \tfrac{1}{4}\delta_{ij}[4(\beta_5+\beta_6)(\beta_1+\beta_2)^2 - \tfrac{1}{3}\rho\Lambda_a]\vec{\Delta}^2 - \tfrac{4}{3}\delta_{ij}m^2\Lambda_a\rho ,\end{aligned} \tag{2.61}$$

$$\begin{aligned} N_2 ={}& \frac{6\beta_2\beta_6(\beta_4\beta_5 - \beta_1\beta_6)}{\Lambda_a^2}[4(\beta_1+\beta_2)(\beta_3+\beta_4)\omega^2 - \tfrac{1}{2}\vec{\Delta}^2\rho^2]\Delta_i\Delta_j \\ &+ 3\left[\frac{(\beta_1\beta_3 - \beta_2\beta_4)^2\omega^2}{\Lambda_a} - m^2\rho^2\right]\left[\Delta_i\Delta_j + (2\omega^2 - \tfrac{1}{2}\vec{\Delta}^2)\delta_{ij}\right] ,\end{aligned} \tag{2.62}$$

$$N_3=\left[2\rho(\Lambda_b-\beta_1\beta_3)-8\Lambda_b\left[\beta_2+\beta_3\frac{\beta_6\rho+\beta_2\beta_3}{\Lambda_b}\right]\left[\beta_4+\beta_3\frac{\beta_5\rho+\beta_3\beta_4}{\Lambda_b}\right]\Big/\rho\right]\Delta_i\Delta_j$$

$$+\delta_{ij}[(2\beta_2+\beta_1-\beta_3)\beta_1\beta_3(\tfrac{1}{2}\vec{\Delta}^2-2\omega^2)+8\beta_1(2\beta_2+\beta_3)\Lambda_b\omega^2/\rho-8\beta_1{}^2\beta_3{}^2\omega^2/\rho-2\rho^2\beta_5\vec{\Delta}^2$$

$$-(\tfrac{1}{3}\beta_2+\tfrac{1}{2}\beta_1-\tfrac{1}{2}\beta_3)\Lambda_b\vec{\Delta}^2-8\rho\Lambda_b m^2/3+8\beta_3{}^2\Lambda_b C_b/\rho]\ , \tag{2.63}$$

$$N_4=3\Delta_i\Delta_j\left[\left[\beta_2+\beta_3\frac{\beta_6\rho+\beta_2\beta_3}{\Lambda_b}\right]\left[\beta_4+\beta_3\frac{\beta_5\rho+\beta_3\beta_4}{\Lambda_b}\right](8\beta_1{}^2\omega^2/\rho^2+\vec{\Delta}^2)\right.$$
$$\left.+\frac{2\beta_1\beta_3(\beta_1\beta_3\omega^2-\rho^2m^2)}{\Lambda_b}-\beta_3\rho\frac{(\beta_6\rho+\beta_2\beta_3)(\beta_5\rho+\beta_3\beta_4)}{\Lambda_b{}^2}\vec{\Delta}^2\right]$$
$$-3\delta_{ij}\beta_i\beta_3\vec{\Delta}^2\omega^2+\tfrac{3}{2}\delta_{ij}m^2\rho\left[8\beta_1\omega^2+\rho\vec{\Delta}^2-\frac{\rho\beta_1\beta_3(4\omega^2-\vec{\Delta}^2)}{\Lambda_b}\right]\ , \tag{2.64}$$

$$N_5=\delta_{ij}\left[\frac{2\rho\beta_5}{\Lambda_a}-1+\frac{2}{3}\frac{\beta_1+\beta_2}{\rho}\right]\ , \tag{2.65}$$

$$N_6=-\frac{2\rho\beta_3\beta_4\beta_5{}^2}{\Lambda_a{}^2}\Delta_i\Delta_j\ , \tag{2.66}$$

$$N_7=-2\delta_{ij}\left[\frac{4\beta_3}{3\rho}-1-2\beta_3{}^2/\Lambda_b\right]\ , \tag{2.67}$$

$$N_8=4\delta_{ij}(\beta_1\beta_3\omega^2/\rho-\tfrac{1}{2}m^2\rho)+4\beta_3{}^2\beta_2\Delta_i\Delta_i/\Lambda_b-4\beta_2\left[\beta_4+\beta_3\frac{\beta_5\rho+\beta_4\beta_3}{\Lambda_b}\right]\left[1+\frac{\beta_3{}^2}{\Lambda_b}\right]\Delta_i\Delta_j/\rho\ . \tag{2.68}$$

In deriving the above, we have made use of the invariance property of C_a and Λ_a under the transformation

$$\beta_5\leftrightarrow\beta_6,\ \ \beta_1\leftrightarrow\beta_4,\ \ \beta_2\leftrightarrow\beta_3\ ,$$

and the invariance property of C_b and Λ_b under the transformation

$$\beta_5\leftrightarrow\beta_6,\ \ \beta_2\leftrightarrow\beta_4\ .$$

By this invariance, we may, for example, ignore the second integral in (2.34) and multiply the third integral in (2.34) by two, or make the following replacement in N_1:

$$(\beta_2\beta_6-\beta_3\beta_5)(\beta_4\beta_5-\beta_1\beta_6)\rightarrow 2\beta_2\beta_6(\beta_4\beta_5-\beta_1\beta_6)\ .$$

III. THE SCALING BEHAVIOR

The lowest-order Delbrück amplitude has a very interesting property: It is finite in the limit of $m\rightarrow 0$. As we shall explain below, this means that the high-energy fixed-angle Delbrück amplitude scales in the form of (1.1).

We shall give a discussion of this scaling behavior in a more general context. An elastic scattering amplitude is a function of the variables ω, $|\vec{\Delta}|$ and the masses μ_i, $i=1,2,\ldots$, of the particles involved. Out of these variables, we may construct the dimensionless variables θ and μ_i/ω. Thus we may write

$$\mathscr{M}/\omega^d=g(\theta,\mu_i/\omega)\ , \tag{3.1}$$

where d is the dimension of $\mathscr{M}$ (for example, the photon-photon scattering amplitude is dimensionless, while the dimension of the Delbrück amplitude is the inverse of a mass). For the right-hand side of (3.1), the high-energy limit $\omega\rightarrow\infty$ with θ fixed is the same as the infrared limit $\mu_i\rightarrow 0$. If

$\mathscr{M}$ has no mass divergence as $\mu_i \to 0$, i.e.,

$$\lim_{\mu_i \to 0} g\left[\theta, \frac{\mu_i}{\omega}\right] = g(\theta), \text{ a finite function },$$

then we have

$$\mathscr{M} \simeq \omega^d g(\theta), \quad \omega \to \infty . \tag{3.2}$$

Equation (3.2) is a generalization of (1.1).

In QED, the only particle with a mass is the electron: We shall prove that the lowest-order Delbrück scattering amplitude has no divergence as $m \to 0$. Let us set $m = 0$ and examine if the scattering amplitude is singular. According to the Coleman-Norton theorem,[8] a singularity of the scattering amplitude for a diagram may occur if this diagram or one of its reduced diagrams (a diagram obtained from it by fusing one or more internal lines) can kinematically represent a classical process, in which all particles are on the mass shell. It is easy to verify that a massless classical particle is kinematically allowed to turn into two massless classical particles only if the momenta of all three particles are parallel. It turns out that, in QED with $m = 0$, the vertex function vanishes at such a point. For example, consider diagram 1(a). According to the Feynman rules, there is, associated with the vertex involving the incoming photon, a numerator factor $\not{p}\gamma_i(\not{p} - \not{k})$, where i denotes the polarization of the photon. When $p_\mu \propto k_\mu$, we have

$$\not{p}\gamma_i(\not{p} - \not{k}) \propto \not{k}\gamma_i\not{k} = 2k_i\not{k} - \gamma_i k^2 = 0 .$$

Because of this vanishing of the numerator at the singularity surface in the p space, the integral in (2.2) is convergent at $m = 0$. Therefore, the Delbrück amplitude corresponding to the diagrams in Fig. 1 is finite in the limit $m \to 0$. Consequently, the scaling formulas (1.1) and (1.2) hold. A more detailed discussion of the application of the Coleman-Norton theorem to the Delbrück amplitude can be found in Appendix B.

The fact that the lowest-order Delbrück amplitude is finite at $m = 0$ can also be seen directly from (2.60). It is not difficult to prove that all integrals in (2.60) remain convergent if we set $m = 0$. For example, if we set $m = 0$, we find from (2.58) that $\Lambda_a{}^2 C_a$ vanishes quadratically in the neighborhood of $\beta_1 = \beta_4 = \beta_5 = \beta_6 = 0$. Therefore, if N_2 did not vanish at $\beta_1 = \beta_4 = \beta_5 = \beta_6 = 0$, then the first integral in (2.60) would be logarithmically divergent. However, if we set $m = 0$, N_2 vanishes linearly in the neighborhood of $\beta_1 = \beta_4 = \beta_5 = \beta_6 = 0$. Thus, there is no divergence. Similar arguments hold for other singular surfaces of the integrand, and (2.60) is finite at $m = 0$. The scaling function $f(\theta)$ in (1.1) can therefore be obtained from (2.60) by setting $m = 0$, $\omega = 1$, and $|\vec{\Delta}| = 2\sin\theta/2$.

To compare our results quantitatively with experiments, numerical evaluation of (2.60) must be performed. For nuclear targets with large Z, we need to calculate, in addition, the corrections due to multiphoton exchanges. While we are as yet unable to calculate such corrections, we have proven

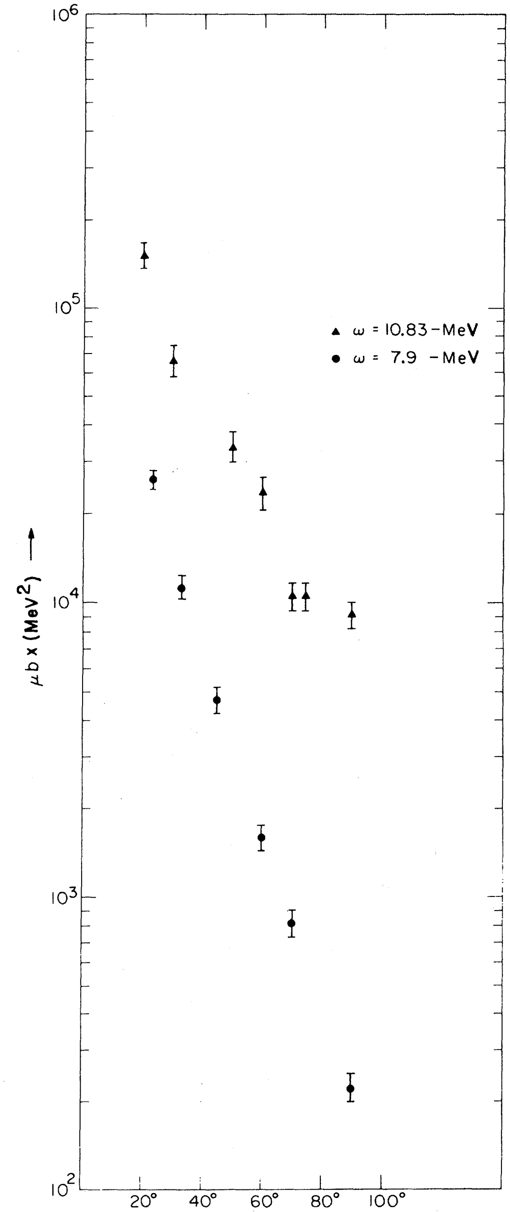

FIG. 2. Experimental values for the elastic scattering of 7.9- and 10.83-MeV photons. The function plotted is $\omega^2 d\sigma/d\Omega$.

that the Delbrück amplitude of multiphoton exchanges satisfies the scaling formula (1.1). This proof is sketched in Appendix B. Therefore, the qualitative feature of our results is of experimental relevance. There exist experimental data of 7.9-MeV photons[9] and 10.83-MeV photons[10] on ^{238}U. According to the analysis in Refs. 9 and 10, Delbrück scattering dominates over other coherent processes such as Thomson scattering, Rayleigh scattering, and nuclear resonance scattering, for $\theta \leq 75°$ with 7.9-MeV photons and for $\theta \leq 45°$ with 10.83-MeV photons. Since the energies of these photons divided by m are much larger than unity, Eq. (1.2) should hold approximately. Experimental values for the left side of (1.2) are reproduced in Fig. 2. We see that scaling is grossly violated. Therefore, we suggest that the data be reanalyzed or (and) additional experiments be performed (we must note in such analysis that ^{238}U is a complex nucleus and not a point charge. Therefore, the effects of the form factor must also be taken into account).

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APPENDIX A

In this appendix, we shall give proofs for (2.21) and (2.30).

After introducing Feynman parameters, the product of propagators

$$\left[\prod_1^4 (p_i{}^2-m^2)\right]^{-1}$$

becomes

$$\left[\sum_1^4 \alpha_i(p_i{}^2-m^2)\right]^{-4},$$

where

$$\sum_{i=1}^4 \alpha_i=1 . \tag{A1}$$

Let

$$p_i=l+\delta p_i ,$$

then

$$\begin{aligned}\sum_{i=1}^4 \alpha_i(p_i{}^2-m^2)&=\sum_1^4 \alpha_i(l+\delta p_i)^2-m^2\\ &=l^2+2l\cdot\sum_{i=1}^4 \alpha_i\delta p_i\\ &\quad+\sum_{i=1}^4 \alpha_i(\delta p_i)^2-m^2\end{aligned} \tag{A2}$$

The variable l is chosen so that the term linear in l above vanishes:

$$\sum_{i=1}^4 \alpha_i\delta p_i=0 . \tag{A3}$$

The term in (A2) independent of l is defined to be $-\mathscr{D}$:

$$\mathscr{D}=-\sum_{i=1}^n \alpha_i(\delta p_i)^2+m^2 . \tag{A4}$$

Now we may prove that

$$\sum_{i,j} \alpha_i\alpha_j(\delta p_i-\delta p_j)^2=2\sum \alpha_i(\delta p_i)^2 , \tag{A5}$$

where (A1) and (A3) have been used. From (A4) and (A5) we get

$$\begin{aligned}\mathscr{D}&=-\tfrac{1}{2}\sum_{i,j} \alpha_i\alpha_j(\delta p_i-\delta p_j)^2+m^2\\ &=-\tfrac{1}{2}\sum_{i,j} \alpha_i\alpha_j(p_i-p_j)^2+m^2 .\end{aligned} \tag{A6}$$

Since p_i-p_j is dependent only on the external momenta but not on α_n, $n=1,2,3,4$, we have from (A6) that

$$\begin{aligned}\frac{\partial\mathscr{D}}{\partial\alpha_i}&=-\sum_j \alpha_j(p_i-p_j)^2\\ &=-\sum_j \alpha_j(\delta p_i-\delta p_j)^2\\ &=-(\delta p_i)^2-\sum_j \alpha_j(\delta p_i)^2 .\end{aligned} \tag{A7}$$

From (A4) and (A7), we get (2.21).

To prove (2.30) with $n=1$, let us make the change of variables

$$\alpha_k=\frac{x_k}{1+x_1}, \quad k=1,2,3,4 \tag{A8}$$

then it is easy to show that

$$d^4\alpha\,\delta\left[1-\sum_1^4 \alpha_k\right]=\frac{d^4x\,\delta(1-x_2-x_3-x_4)}{(1+x_1)^4} \tag{A9}$$

and

$$1-\alpha_1=\frac{1}{1+x_1},\quad \alpha_1=\frac{x_1}{1+x_1}\,. \tag{A10}$$

To be specific, we shall choose $\mathscr{D}=\mathscr{D}_a$. Then

$$\int d^4\alpha\,\delta\left[1-\sum_1^4\alpha_k\right]\ln\mathscr{D}_a$$
$$=\int dx_2dx_3dx_4\delta(1-x_2-x_3-x_4)$$
$$\times\int_0^\infty\frac{dx_1}{(1+x_1)^4}\ln\frac{\bar{D}_a}{(1+x_1)^2}\,, \tag{A11}$$

where

$$\bar{D}_a\equiv(1+x_1)^2\mathscr{D}_a\,. \tag{A12}$$

By writing $dx_1/(1+x_1)^4$ as $-\frac{1}{3}d[(1+x_1)^{-3}]$ and integrating by parts, we may get, after some algebra, Eq. (2.30).

APPENDIX B

In this appendix, we give a detailed proof of the finiteness at $m=0$ of the Delbrück amplitude of multiphoton exchanges.

We first consider the lowest-order diagrams in Fig. 1 and their reduced diagrams illustrated in Fig. 3. Since all classical massless particles travel with the velocity of light, it is not possible for two classical particles to begin at one point and meet again at another, if one particle travels freely while the other particle changes direction one or more times. Thus we eliminate diagrams 1(b), 3(a), 3(b), 3(c), and 3(g) as possible configurations for mass divergences, in accordance with the Coleman-Norton theorem.[8] Also, by observing that one of the particles in the loop of either diagram 3(h) or diagram 3(i) must have negative energy, we eliminate these two diagrams from consideration.

Thus we are left with diagrams 1(a), 3(d), 3(e), and 3(f). We shall show that although these diagrams may satisfy the condition of the Coleman-Norton theorem, they still do not cause $\mathscr{M}_0^{(D)}$ to diverge.

We begin by observing that diagram 3(f) does not satisfy the condition of the Coleman-Norton theorem if $(k+q)^2\neq0$ and $(k'-q)^2\neq0$. This is because in this case, classical kinematics dictates that the two massless particles in the loop travel in different directions, and therefore cannot meet twice. At $(k+q)^2=0$, the amplitude M_0 at $m=0$ has a divergence. However, the divergence is only logarithmic, and does not cause the integration over d^3q in (2.1) to diverge. Similarly for the

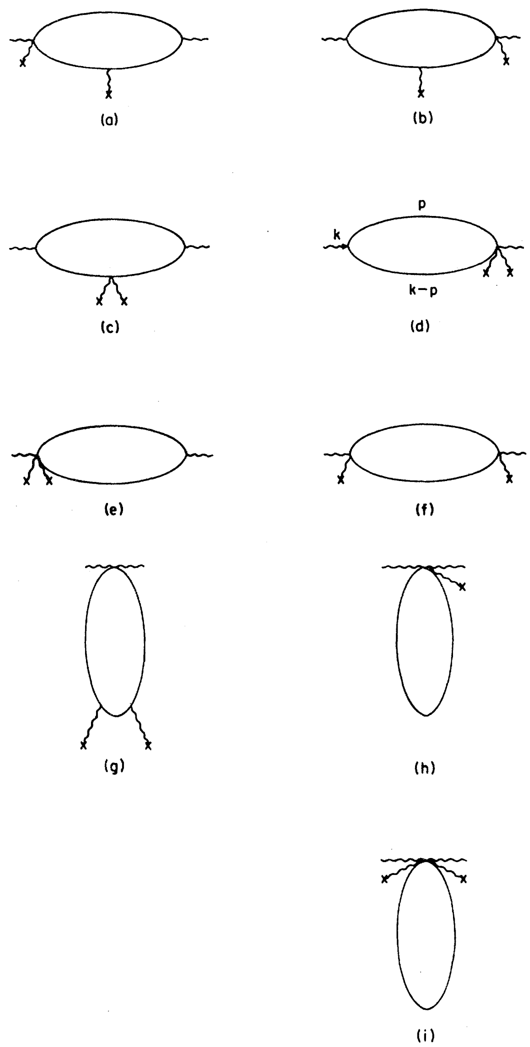

FIG. 3. The reduced diagrams for the diagrams in Fig. 1.

neighborhood of $(k'-q)^2=0$.

Next we turn to diagram 1(a). This diagram does not satisfy the condition of the Coleman-Norton theorem unless $\vec{q}=a\vec{\Delta}$, $0<a<1$. This is because classical kinematics dictates that $\vec{p}=a\vec{k}$ and $\vec{p}+\vec{q}=a\vec{k}'$, $0\leq a\leq1$. When $\vec{q}=a\vec{\Delta}$, the condition of the Coleman-Norton theorem is met. However, since this divergence is only logarithmic, integrating over the neighborhood of this one-dimensional surface of singularity does not cause $\mathscr{M}_0^{(D)}$ to become infinite.

Finally, we examine diagrams 3(d) and 3(e). As we have mentioned in the paragraph following Eq. (3.2), the vertex function has a zero on the surface of singularity in the p space, and the integration over d^4p is convergent. Thus diagrams 3(d) and 3(e) do not make M_{ij} infinite. We have therefore shown that $\mathscr{M}_0^{(D)}$ is finite at $m=0$.

The same arguments hold for all higher-order diagrams of multiphoton exchanges. We may prove that, at $m=0$, a configuration corresponding to a classical process either gives zero contribution because the numerator vanishes, or is realized under constraints of the variables. Hence the integrals for the amplitude are convergent at $m=0$. Therefore, the Delbrück amplitude of multiphoton exchanges satisfies (1.1).

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