Multidimensional WKB approach to high-energy elastic scattering at fixed angle

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We discuss here the application of a multidimensional WKB method to the Schrödinger equation and the Klein-Gordon equation. This method enables us to calculate scattering amplitudes in the limit of large incident momentum p and fixed scattering angle θ. As an application, we calculate the scattering amplitude for the Klein-Gordon equation with the potential $\alpha e^{-\lambda r}/r$, and find that in the limit of large p and fixed θ, it is equal to the scattering amplitude for a Coulomb potential multiplied by $(\lambda e^{\gamma}/2p)^{2i\alpha}$, where $\gamma=0.577\,21\ldots$. Our result differs from the eikonal formula, in the high-energy fixed-angle region. We also show that our formula agrees with the eikonal formula for small scattering angles satisfying $1 \gg \theta \gg \lambda/p$. Thus our formula, together with the eikonal formula, give complete information on the high-energy amplitude for all angles. A side result of the application of the WKB method is a generalized eikonal formula for the case of the Klein-Gordon equation with a four-potential $A_\mu(x)$ which depends not only on space but also on time. This formula holds for the scattering amplitude in the region $p \to \infty$ with the momentum transfer fixed.

I. INTRODUCTION

Despite a great amount of work devoted to high-energy scattering at fixed angles in quantum field theories, very few firm results have been established. In this paper, we try to resolve one elementary issue which, to the authors' knowledge, has never been settled.

It has been known for more than a decade that the elastic scattering amplitude of multiphoton exchange in QED with massive photons is given by the eikonal formula,[1] in the limit in which the incident c.m. momentum $p \to \infty$ with the momentum transfer fixed. However, no one has yet succeeded in calculating this amplitude in the limit $p \to \infty$ with the scattering angle fixed. One may try to calculate it by summing the leading terms of each perturbative order. However, in this limit, the amplitude for an n-photon-exchange diagram is of the order of $(\ln p)^{2n}$, while the sum of the leading terms is of magnitude $O(1)$. This means that there is a great deal of cancellation in the summation. Therefore, there is no way to justify that the nonleading terms are negligible. Alternatively, one might conjecture that the amplitude in this limit is obtained from the eikonal formula by taking $\Delta \to \infty$. A related conjecture is that the correct answer is given by the eikonal formula plus the Saxon-Schiff correction.[2–4]

The main thrust of this paper is to study high-energy potential scattering with large momentum transfers. We shall pay special attention to the Klein-Gordon equation with the static potential $\alpha e^{-\lambda r}/r$. The amplitude in this special case is of course interesting in its own right, but the reason we choose to study it here is that it somewhat resembles the multiphoton scattering amplitude in QED. For example, in the high-energy limit with the momentum transfer fixed, both amplitudes are given by the same eikonal formula. We shall give the former amplitude in the high-energy limit with the scattering angle fixed. We show that it is equal to the scattering amplitude of the Klein-Gordon equation in the Coulomb potential, multiplied by $(\lambda e^{\gamma}/2p)^{2i\alpha}$, where $\gamma=0.577\,21\cdots$ is Euler's constant. We also conclude that, in the region $\theta=O(1)$, it disagrees with both the sum of leading terms and the eikonal formula, with or without the Saxon-Schiff correction. Furthermore, it reduces to the eikonal form in the region $\lambda/p \ll \theta \ll 1$, and there are no intermediate regions in which the high-energy amplitude takes different forms. Our results in the fixed-angle regime are for the Klein-Gordon equation and are not field-theoretic results.

II. THE WKB METHOD FOR SCHRÖDINGER'S EQUATION

The high-energy potential scattering problem is best handled by the WKB approximation.[5] In standard textbooks on quantum mechanics, the application of the WKB method is usually restricted to one-dimensional problems. Actually, there is little difficulty in generalizing it to multidimensional problems.[5] For the purpose of completeness, we shall give a presentation of such a generalization here.

The Schrödinger equation in a potential $V(\vec{x},t)$ is given by

$$i\hbar\frac{\partial\psi(\vec{x},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+V(\vec{x},t)\right]\psi(\vec{x},t)\ . \tag{2.1}$$

Putting

$$\psi(\vec{x},t)=\exp[i\mathscr{S}(\vec{x},t)/\hbar]\ , \tag{2.2}$$

and substituting (2.2) into (2.1), we get

$$-\frac{\partial\mathscr{S}}{\partial t}=\frac{1}{2m}(\vec{\nabla}\mathscr{S})^2-\frac{\hbar i}{2m}\nabla^2\mathscr{S}+V(\vec{x},t)\ . \tag{2.3}$$

If

$$\left|\frac{\hbar\nabla^2\mathscr{S}}{(\vec{\nabla}\mathscr{S})^2}\right|\ll 1\ , \tag{2.4}$$

(2.3) is reduced to

$$-\frac{\partial\mathscr{S}}{\partial t}=\frac{1}{2m}(\vec{\nabla}\mathscr{S})^2+V\ . \tag{2.5}$$

Equation (2.5) is the Hamilton-Jacobi equation. At first sight, it may appear that this equation, being nonlinear, is more difficult to solve than the Schrödinger equation. However, it is well known that (2.5) is satisfied by the classical action. (Some modification is necessary to incorporate the initial condition, which is imposed on the momentum, not the position. We shall discuss this in a moment.) The test of validity of this semiclassical approximation is simple, and is given by (2.4). The condition (2.4) is valid either in the limit $\hbar\to 0$, or $\nabla^2\mathscr{S}/(\vec{\nabla}\mathscr{S})^2\to 0$, the latter happening at infinite energy.

At the initial time T_i in the distant past when the potential is negligible, let the wave function be

$$\exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-ET_i)\right]\ .$$

Then

$$\mathscr{S}(\vec{x},T_i)=\vec{p}\cdot\vec{x}-ET_i\ . \tag{2.6}$$

The solution of the Hamilton-Jacobi equation satisfying the initial condition (2.6) is

$$\mathscr{S}(\vec{x},t)=\int_{T_i}^{t}\left[\frac{m}{2}\dot{\vec{x}}_c{}^2-V(\vec{x}_c)\right]dt' +\vec{p}\cdot\vec{x}_c(T_i)-ET_i\ , \tag{2.7}$$

where $\vec{x}_c(t')$ obeys Newton's equation of motion with the boundary conditions

$$\vec{x}_c(t)=\vec{x} \tag{2.8a}$$

and

$$m\dot{\vec{x}}_c(T_i)=\vec{p}\ . \tag{2.8b}$$

Note that (2.8a) and (2.8b) say that while we specify at the final time t the position of the particle, we specify at the initial time T_i the momentum, not the position, of the particle. This is because the initial state is an eigenstate of the momentum, not that of the position. Thus, $\vec{x}_c(T_i)$ is not specified *a priori*, but is determined by Newton's equation and (2.8). This means that, as we vary x or t, $\vec{x}_c(T_i)$ also varies. Such a variation of $\vec{x}_c(T_i)$ causes the term $\int L_c dt'$ in (2.7) to vary whenever we vary x or t, where

$$L_C=\frac{m}{2}\dot{\vec{x}}_c{}^2-V(\vec{x}_c)\ .$$

However, this variation of $\int L_c dt'$ is equal to

$$-\frac{\delta L_c}{\delta\dot{x}_c}\delta x_c\bigg|_{t'=T_i}$$

which cancels exactly the variation of $\vec{p}\cdot\vec{x}_c(T_i)$, the second term in the right side of (2.7). Thus, as is the case with the usual classical action, $\mathscr{S}$ as given by (2.7) has the partial derivatives

$$\mathscr{S}_t=-\left[\frac{m}{2}\dot{\vec{x}}_c^{\,2}+V(\vec{x}_c)\right] \tag{2.9}$$

and

$$\vec{\nabla}\mathscr{S}=m\dot{\vec{x}}_c \ . \tag{2.10}$$

Consequently, $\mathscr{S}$ obeys the Hamilton-Jacobi equation. Furthermore, at $t=T_i$, we have $\vec{x}_c(T_i)=\vec{x}$, thus the initial condition (2.6) is also satisfied.

Equation (2.7) can be reduced by replacing $V(\vec{x}_c)$ with

$$V(\vec{x}_c)=E-\frac{m}{2}\dot{\vec{x}}_c^{\,2} \ .$$

After this replacement and some algebra, we get

$$\mathscr{S}(\vec{x},t)=\int_{\vec{x}_c(T_i)}^{\vec{x}}[\vec{p}(\vec{x}_c)-\vec{p}]\cdot d\vec{x}_c +\vec{p}\cdot\vec{x}-Et \ , \tag{2.11}$$

where the line integral is along the classical path satisfying (2.8) and where $\vec{p}(\vec{x}_c)$ is the classical momentum of the particle at $\vec{x}_c$. Thus we get

$$\psi_{\vec{p}}^{(-)}(\vec{x},t)=\exp\left\{\frac{i}{\hbar}\int_{\vec{x}'_c(-\infty)}^{\vec{x}}[\vec{p}(\vec{x}\,')-\vec{p}]\cdot d\vec{x}\,' +\frac{i\vec{p}\cdot\vec{x}}{\hbar}-\frac{iEt}{\hbar}\right\}, \tag{2.12}$$

where the contour of integration is the classical path of the particle which has incident momentum $\vec{p}$ and passes through the point $\vec{x}$. Similarly,

$$\psi_{\vec{p}}^{(+)}(\vec{x},t)=\exp\left\{\frac{i}{\hbar}\int_{\vec{x}'_c(\infty)}^{\vec{x}}[\vec{p}(\vec{x}\,')-\vec{p}]\cdot d\vec{x}\,' +\frac{i\vec{p}\cdot\vec{x}}{\hbar}-\frac{iEt}{\hbar}\right\}, \tag{2.13}$$

where the integral is the classical path of the particle which has outgoing momentum $\vec{p}$ and passes through the point $\vec{x}$.

The three-dimensional case differs from the one-dimensional case in one important aspect: There may exist more than one classical path satisfying (2.8). Let us consider, for example, the scattering by a potential which is finite everywhere except at the origin, where it is infinite. Then, as $E\to\infty$, $V(\vec{x})$ is much smaller than E except in a small neighborhood of the origin. Thus, the motion of a high-energy classical particle with finite impact parameter is unaffected by the potential and its trajectory is a straight line. However, if the impact parameter is very small, the classical particle would approach the origin and be deflected by an angle. This is illustrated in Fig. 1. Thus there are two classical paths which are of the same initial momentum and final position. The high-energy scattering amplitude in quantum mechanics is equal to the sum of contributions from these two classical paths. More precisely, the straight-line path contributes to near-forward scattering of fixed momentum transfer, while the deflected path contributes to fixed-angle scattering.

For near-forward scattering, the contributing path is given by the straight-line path

$$\vec{x}_c(t')=\vec{x}_\perp+z_c(t')\vec{e}_z \ ,$$

where

$$\vec{x}_\perp=x\vec{e}_x+y\vec{e}_y \ ,$$

with the incident momentum in the z direction. Thus

$$\vec{p}(\vec{x}\,')=\{2m[E-V(\vec{x}\,')]\}^{1/2}\vec{e}_z \ ,$$

and (2.12) becomes

$$\psi_{\vec{p}}^{(-)}(\vec{x},t)=\exp\left[\frac{1}{\hbar}\int_{-\infty}^{z}(\{2m[E-V(\vec{x},z')]\}^{1/2}-p)dz'\right]\exp\left[\frac{ipz}{\hbar}-\frac{iEt}{\hbar}\right] . \tag{2.14}$$

Similarly,

$$\psi_{\vec{p}\,'}^{(+)}(\vec{x},t)=\exp\left[\frac{i}{\hbar}\int_{\infty}^{z}(\{2m[E-V(\vec{x},z')]\}^{1/2}-p)dz'\right]\exp\left[\frac{i\vec{p}\,'\cdot\vec{x}}{\hbar}-\frac{iEt}{\hbar}\right] . \tag{2.15}$$

Equation (2.15) holds for all $\vec{p}\,'=\vec{p}+\vec{\Delta}$, with $\vec{\Delta}$ fixed and $|\vec{p}|\to\infty$. From (2.14) and (2.15) we get the eikonal formula for the S matrix,

$$S(p',p)=\int d^2x_\perp \exp\left\{\frac{1}{\hbar}\int_{-\infty}^{\infty}[p(\vec{x}_\perp,z)-p)]dz-\frac{i}{\hbar}\vec{\Delta}\cdot\vec{x}_\perp\right\}2\pi\delta(p-p') , \quad (2.16)$$

valid for $p\rightarrow\infty$ with $\vec{\Delta}$ fixed.

Next we consider fixed-angle scattering. We shall assume that the potential is finite everywhere except at the origin. Then at any point away from the origin, ψ is equal to the sum of two WKB solutions, each the contribution of a classical path as depicted in Fig. 1. The WKB solution corresponding to the path parallel to the z axis is given by Eq. (2.14), while the WKB solution corresponding to the deflected path can be obtained by the following considerations. We notice that, away from the origin, this path is a straight line with fixed angular coordinates θ and ϕ. Thus, the phase angle of the solution is of the form

$$\frac{1}{\hbar}\int^r\{2m[E-V(r',\theta,\phi)]\}^{1/2}dr' ,$$

where $V(r,\theta,\phi)$ is just another notation for $V(\vec{x})$. The phase angle is determined only up to an additive function independent of r, and we shall choose it to be

$$-\frac{1}{\hbar}\int_r^{\infty}(\{2m[E-V(r',\theta,\phi)]\}^{1/2}-p)dr'+\frac{pr}{\hbar} . \quad (2.17)$$

Thus the WKB solution corresponding to the deflected path is

$$\frac{1}{r}\exp\left[\frac{-i}{\hbar}\int_r^{\infty}(\{2m[E-V(r',\theta,\phi)]\}^{1/2}-p)dr'+\frac{ipr}{\hbar}\right] . \quad (2.18)$$

The factor $1/r$ in (2.18) is inserted because (2.18) describes a spherically outgoing wave. Alternatively, we may think of it as coming from the higher-order WKB correction, which ensures that the probability current is conserved. The wave function ψ is, therefore, of the form

$$\psi(\vec{x})\simeq\exp\left[\frac{i}{\hbar}\int_{-\infty}^{z}(\{2m[E-V(\vec{x},z')]\}^{1/2}-p)dz'+\frac{ipz}{\hbar}\right]$$
$$+\frac{f(\theta,\phi)}{r}\exp\left[-\frac{i}{\hbar}\int_r^{\infty}(\{2m[E-V(r',\theta,\phi)]\}^{1/2}-p)dr'+\frac{ipr}{\hbar}\right] , \quad (2.19)$$

where a factor $\exp(-iET/\hbar)$ has been omitted. As $r\rightarrow\infty$,

$$\psi(\vec{x})\simeq\exp\left[\frac{ipz}{\hbar}\right]+f(\theta,\phi)\frac{\exp(ipr/\hbar)}{r} , \quad (2.20)$$

thus $f(\theta,\phi)$ is the scattering amplitude.

To determine the scattering amplitude, we must solve the Schrödinger equation approximately near the turning point $r=0$, and match the solution with (2.20). We shall demonstrate this with an explicit example in Sec. IV.

If the potential has n singular points, then there exist n deflected classical paths and there are $(n+1)$ WKB solutions. This can be understood from the viewpoint of the path-integral formulation. The wave function is equal to the sum of contributions from all possible paths. When the energy is large, the contribution from a path oscillates rapidly as we vary the path, and the sums of contributions cancel. Therefore, the dominant contribution comes from the paths which give stationary actions. Such paths are the classical paths.

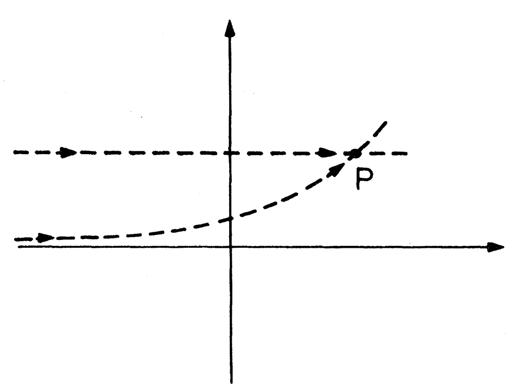

FIG. 1. Two classical paths with the same initial momentum and the same final position. Note that the curved path has a smaller impact distance.

When there is more than one classical path, we must add up the contributions from all of them. To obtain the scattering amplitude, we solve the Schrödinger equation approximately near all turning points and match them with a superposition of all WKB solutions.

What happens if the potential is finite everywhere? In this case, a high-energy particle in classical mechanics cannot be deflected by a large angle. Thus there is no classical path contributing to the amplitude of high-energy scattering at a fixed angle. In such a case, the amplitude is exponentially small. It is possible to calculate this small amplitude if $V(\vec{x})$ can be analytically continued to complex values of $\vec{x}$. Scattering occurs at the turning points in the complex plane, and the scattering amplitude can be obtained by following the analytically continued classical path in the complex plane.[6,7]

III. THE WKB METHOD FOR THE KLEIN-GORDON EQUATION

In this section, we extend the WKB method to the Klein-Gordon equation.

The Klein-Gordon equation with the four-vector potential $A_\mu(x)$ is

$$-D_\mu D^\mu \phi(x)=m^2\phi(x) , \tag{3.1}$$

where

$$D_\mu \equiv \partial_\mu + ieA_\mu(x) ,$$

and we have set $\hbar=c=1$. The incident wave is assumed to be

$$\phi_{\rm inc}(x)=\exp(-ip\cdot x), \;\; p^2=m^2 . \tag{3.2}$$

Unlike the Schrödinger equation, (3.1) is a second-order differential equation with respect to time. Therefore, to obtain a unique solution, we need to impose one more boundary condition on ϕ. For relativistic potential scattering, the solution we want is the one which satisfies Feynman's boundary condition, i.e., no negative-energy waves as $t\to\infty$. At first sight, this last condition makes it more difficult to apply the WKB method. We shall overcome this difficulty by introducing Feynman's fifth variable[8] which we denote by τ. Thus we consider the equation

$$D_\mu D^\mu \Psi(x,\tau)=2i\frac{\partial}{\partial \tau}\Psi(x,\tau) \tag{3.3}$$

with the initial condition

$$\Psi(x,\tau)\to\exp[i(\tfrac{1}{2}m^2\tau-p\cdot x)], \;\; \tau\to-\infty . \tag{3.4}$$

We shall show in Appendix A that $\Psi(x,\tau)$ is related to $\phi(x)$ by

$$\Psi(x,0)=\phi(x) . \tag{3.5}$$

(We assume that A_μ is adiabatically switched off as $|\tau|\to\infty$.) Thus, instead of solving (3.1) together with (3.2) and Feynman's boundary condition, we shall solve (3.3) together with (3.4). Since (3.3) is in the same form as the Schrödinger equation, the formalism developed in the previous section can be directly applied.

Before we go on to present the WKB method for (3.3), let us make a side remark. Let us imagine that we did not know of the existence of the time-dependent Schrödinger equation and just wanted to solve the time-independent Schrödinger equation. The introduction of t and the time-dependent Schrödinger equation would then be regarded as a purely mathematical device to facilitate the calculation. Indeed, in our WKB treatment of the time-independent Schrödinger equation in the last section, we found it convenient to introduce the variable t, although the final formulas do not involve t. The introduction of the fifth variable to facilitate the solving of the Klein-Gordon equation is of the same motivation.

Let us now apply the WKB method to (3.3). We put

$$\Psi(x,\tau)=e^{i\mathscr{S}(x,\tau)} , \tag{3.6}$$

then (3.3) gives

$$-2\frac{\partial \mathscr{S}}{\partial \tau}=-(\partial_\mu \mathscr{S}+eA_\mu)(\partial^\mu \mathscr{S}+eA^\mu) + i(\partial_\mu\partial^\mu \mathscr{S}+e\partial_\mu A^\mu) \tag{3.7}$$

with the initial condition

$$\mathscr{S}(x,\tau_i)=\tfrac{1}{2}m^2\tau_i-p\cdot x , \tag{3.8}$$

where τ_i is eventually taken to be $-\infty$. Equation (3.7) is exact. In the WKB approximation, we neglect the last term in (3.7). Then (3.7) becomes

$$-\frac{\partial \mathscr{S}}{\partial \tau}\simeq-\tfrac{1}{2}(\partial_\mu \mathscr{S}+eA_\mu)(\partial^\mu \mathscr{S}+eA^\mu) . \tag{3.9}$$

Equation (3.9) is the Hamilton-Jacobi equation of a particle with the Hamiltonian

$$H=-\tfrac{1}{2}(P_\mu+eA_\mu)(P^\mu+eA^\mu) , \tag{3.10}$$

where P_μ is the four-"momentum" of the particle. The Lagrangian of this particle is found by noting that

$$\dot{x}_\mu = \frac{\delta H}{\delta P^\mu} = -P_\mu - eA_\mu \ . \tag{3.11}$$

Thus

$$L = P\cdot\dot{x} - H = -\tfrac{1}{2}\dot{x}^2 - eA\cdot\dot{x} \ , \tag{3.12}$$

where x_μ are the four-coordinates of the particle and

$$\dot{x}_\mu \equiv \frac{dx_\mu}{d\tau}, \quad \dot{x}^2 \equiv \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} \ . \tag{3.13}$$

Note that in the present treatment, space and time are treated on equal footing. Also, H in (3.12) is not related to the energy and its identification with the Hamiltonian is merely formal.

Using the same arguments we presented in the last section, we may show that Eq. (3.9) together with the initial condition (3.8) are solved by

$$S(x,\tau) = \int_{\tau_i}^{\tau} [-\tfrac{1}{2}\dot{x}_c^{\,2} - eA(x_c)\cdot\dot{x}_c] d\tau' + \tfrac{1}{2}m^2\tau_i - p\cdot x_c(\tau_i) \ . \tag{3.14}$$

In (3.14), x_c is the abbreviated notation for $x_c(\tau')$ which obeys the classical equations of motion

$$\ddot{x}_\mu = e\left[\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}\right]\dot{x}^\nu \ , \tag{3.15}$$

with the boundary conditions

$$x_c^\mu(\tau) = x^\mu \tag{3.16}$$

and

$$\dot{x}_c^\mu(\tau_i) = p^\mu \ . \tag{3.17}$$

We have assumed that A_μ has been adiabatically switched off at τ_i. Otherwise, a term eA_μ should be inserted at the left side of (3.17).

Equation (3.14) can be simplified by making use of the fact that x_c satisfies the classical equation of motion, hence the Hamiltonian is independent of τ. Since at τ_i, H is equal to $-m^2/2$, we get from (3.10) and (3.11)

$$\tfrac{1}{2}\dot{x}_c^{\,2} = \tfrac{1}{2}m^2 \ .$$

Therefore,

$$\mathcal{S}(x,\tau) = -\int_{x_c(\tau_i)}^{x} eA(x_c)\cdot dx_c - \tfrac{1}{2}m^2(\tau - 2\tau_i) - p\cdot x_c(\tau_i) \ . \tag{3.18}$$

Equation (3.18) can be simplified further by utilizing the fact that p is large. Let the range of $A(x)$ be L, then the τ duration required for the particle with velocity p_μ to pass through the potential is of the order of L/p. In the limit $p\to\infty$ with L fixed, L/p is very small. Thus the particle essentially travels in a straight line and

$$x_c^\mu(\tau') = x^\mu - (\tau - \tau')p^\mu \ . \tag{3.19}$$

Since

$$p_- \simeq 0, \quad \vec{p}_\perp = 0 \ ,$$

where $p_\pm \equiv p_0 \pm p_3$ and $\vec{p}_\perp \equiv p_1\vec{e}_1 + p_2\vec{e}_2$, (3.19) is, more explicitly,

$$\begin{aligned} (x_c)_+ &= x_+ - (\tau - \tau')p_- \ , \\ (x_c)_- &= x_- \ , \\ (\vec{x}_c)_\perp &= \vec{x}_\perp \ . \end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.18) and taking $\tau_i \to -\infty$, we get

$$\mathcal{S}(x,\tau) = -\tfrac{1}{2}\int_{-\infty}^{x_+} eA_-(x'_+, x_-, \vec{x}_\perp)dx'_+ + \tfrac{1}{2}m^2\tau - p\cdot x \ . \tag{3.21}$$

From (3.21), (3.6), and (3.5), we find the WKB solution of the Klein-Gordon equation (3.1) as

$$\phi(x) \simeq \exp\left[-ip\cdot x - \frac{i}{2}\int_{-\infty}^{x_+} eA_-(x'_+, x_-, \vec{x}_\perp)dx'_+\right] \ . \tag{3.22}$$

We see from (3.22) that the effect of the potential is to add a phase shift $-\frac{1}{2}\int eA_- dx_+$ to the wave function.

It remains to verify that the WKB solution is a good approximation. This means that we have to check if $|\partial_\mu\partial^\mu\mathcal{S} + e\partial_\mu A^\mu|$ is much smaller than the terms we keep in (3.9). Now $|\partial_\mu\partial^\mu\mathcal{S} + e\partial_\mu A^\mu|$ is of the order of unity, while the term $(\partial_\mu\mathcal{S})A^\mu$ is of the order of E. Thus we have justified the WKB approximation in the region where A^μ is not singular.

From (3.22), we may derive the scattering amplitude for small scattering angle θ. We have

$$S = \int d^2x_\perp dx_- \exp\left\{i\left[\frac{\Delta_+ x_-}{2} - \vec{\Delta}_\perp\cdot\vec{x}_\perp - \frac{e}{2}\int_{-\infty}^{\infty} A_-(x_+, x_-, \vec{x}_\perp)dx_+\right]\right\} \ , \tag{3.23}$$

where Δ is the momentum transfer. Equation (3.22) is the eikonal formula valid in the limit $E\to\infty$ with the momentum transfer fixed. It is a generalization of the usual eikonal formula which holds for the special case in which $A_\mu(x)$ is independent of time.

As in the case of the Schrödinger equation, the amplitude for high-energy fixed-angle scattering in the present case is obtained by joining the WKB solutions with the approximate solutions of (3.1) near the singular points of $A_\mu(x)$. We shall treat the example of $A_\mu(x)=\delta_{\mu 0}\alpha e^{-\lambda r}/r$ in the next section.

IV. FIXED-ANGLE SCATTERING IN THE KLEIN-GORDON EQUATION

In this section, we shall treat fixed-angle scattering in the Klein-Gordon equation for a static scalar potential $V(\vec{x})$. In this special case, the Klein-Gordon equation is

$$\{[E-V(\vec{x})]^2+\nabla^2-m^2\}\psi(\vec{x})=0\ . \tag{4.1}$$

In (4.1), E and m are the energy and the mass of the particle, respectively. We shall solve (4.1) with the condition that the incident wave is e^{ipz}, where $p=(E^2-m^2)^{1/2}$.

Equation (4.1) can be rewritten as

$$[\nabla^2+p^2-U(\vec{x})]\psi(\vec{x})=0\ , \tag{4.2}$$

where

$$U(\vec{x})=2EV(\vec{x})-V^2(\vec{x})\ . \tag{4.3}$$

Equation (4.2) is of the same form as the time-independent Schrödinger equation. Therefore, we need to use not the formalism developed in the last section but the simpler formalism of Sec. II. Indeed, the formulas in Sec. II directly apply if we make the replacement

$$[2m(E-V)]^{1/2}\to p-V\ . \tag{4.4}$$

Thus, instead of (2.19), we have, for the wave function away from the origin

$$\psi(\vec{x})\simeq\exp\left[-i\int_{-\infty}^{z}V(\vec{x}_\perp,z')dz'+ipz\right]+\frac{f(\theta,\phi)}{r}\exp\left[i\int_{r}^{\infty}V(r',\theta,\phi)dr'+ipr\right]\ . \tag{4.5}$$

To be specific, we shall choose

$$V(\vec{x})=\alpha e^{-\lambda r}/r\ , \tag{4.6}$$

where $r=|\vec{x}|$. Extension to a superposition of Yukawa potentials is not difficult and we shall discuss it briefly at the end of this section. For the potential in (4.6), the WKB approximation holds if

$$(p-V)^2\gg\left|\frac{\partial V}{\partial r}\right|$$

or (4.7)

$$pr\gg O(1)\ .$$

In the region

$$\frac{1}{\lambda}\gg r\gg\frac{1}{p}\ ,$$

the WKB solution (4.5) becomes, after some algebra,

$$\psi(\vec{x})\simeq\exp\left\{ipz+i\alpha\ln\left[\frac{\lambda(r-z)e^{\gamma}}{2}\right]\right\}$$

$$+\frac{f(\theta,\phi)}{r}\exp[ipr-i\alpha\ln(\lambda re^{\gamma})]\ . \tag{4.8}$$

The detailed derivation of (4.8) is given in Appendix B.

The WKB solutions do not hold in the neighborhood of the origin, where we may solve the Klein-Gordon equation approximately in another way. We note that, for $\lambda r\ll 1$,

$$V(\vec{x})\simeq\frac{\alpha}{r}\ . \tag{4.9}$$

Thus, for $\lambda r\ll 1$, and $p\gg m$, the Klein-Gordon equation (3.1) is approximately

$$\left[\nabla^2+p^2-\frac{2\alpha p}{r}+\frac{\alpha^2}{r^2}\right]\psi\simeq 0\ . \tag{4.10}$$

This equation can be easily solved by the separation in spherical coordinates. It suffices to give the solution for $pr\gg 1$. We have

$$\psi\simeq\frac{1}{2ipr}\sum_{l=0}^{\infty}A_l(2l+1)P_l(\cos\theta)\{\exp[i(pr-\alpha\ln 2pr-\tfrac{1}{2}l'\pi+\delta_l')]-\exp[-i(pr-\alpha\ln 2pr-\tfrac{1}{2}l'\pi+\delta_l')]\}\ , \tag{4.11}$$

where

$$l' = [(l+\tfrac{1}{2})^2 - \alpha^2]^{1/2} - \tfrac{1}{2} \tag{4.12}$$

and

$$e^{2i\delta'_l} = \frac{\Gamma(l'+1+i\alpha)}{\Gamma(l'+1-i\alpha)} .$$

The constants A_l are determined in such a way that (4.11) is equal to an incident wave plus a spherically outgoing wave. According to (4.8) the incident wave is equal to

$$\left[\frac{\lambda e^{\gamma}}{2p}\right]^{i\alpha} \exp[ipz + i\alpha \ln p(r-z)] . \tag{4.13}$$

Now, the well-known Coulomb wave function has the asymptotic form[9]

$$\exp\{i[pz + \alpha \ln p(r-z)]\} + \frac{f_C(\theta)}{r}\exp[i(pr - \alpha \ln 2pr)], \quad pr \gg 1 \tag{4.14a}$$

where $f_C(\theta)$ is the Coulomb scattering amplitude. Alternatively, we may express this same Coulomb wave function in the form of the partial-wave expansion[9]

$$\frac{1}{2ipr}\sum_l (2l+1)P_l(\cos\theta)\{\exp[i(pr - \alpha \ln 2pr + 2\delta_l)] - \exp[-i(pr - l\pi - \alpha \ln 2pr)]\}, \quad pr \gg 1 . \tag{4.14b}$$

Equating (4.14a) and (4.14b), we find that

$$\left[\frac{\lambda e^{\gamma}}{2p}\right]^{i\alpha} \exp\{i[pz + \alpha \ln p(r-z)]\}$$
$$\sim \frac{(\lambda e^{\gamma}/2p)^{i\alpha}}{2ipr}\sum_l (2l+1)P_l(\cos\theta)\{\exp[i(pr - \alpha \ln 2pr)] - \exp[-i(pr - l\pi - \alpha \ln 2pr)]\}, \quad pr \gg 1 \tag{4.15}$$

where a factor $(\lambda e^{\gamma}/2p)^{i\alpha}$ has been inserted for the convenience of later comparisons. Therefore, if the right-hand side of (4.11) is equal to the incident wave (4.13) plus a spherically outgoing wave, we must equate the coefficient of the spherically incoming wave in (4.11) with that in (4.15). We get

$$A_l = \left[\frac{\lambda e^{\gamma}}{2p}\right]^{i\alpha} e^{i\pi[l-(1/2)l']} e^{i\delta_{l'}} . \tag{4.16}$$

Substituting (4.16) into (4.11), we get

$$\psi \simeq \left[\frac{\lambda e^{\gamma}}{2}\right]^{i\alpha}\left[\exp[ipz + i\alpha \ln(r-z)] + \frac{F(\theta)}{r}\exp(ipr - i\alpha \ln 2pr)\right], \quad \frac{1}{\lambda} \gg r \gg \frac{1}{p} , \tag{4.17}$$

where

$$F(\theta) = \frac{1}{2ip}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\left[e^{i\pi(l-l')}\frac{\Gamma(l'+1+i\alpha)}{\Gamma(l'+1-i\alpha)} - 1\right] . \tag{4.18}$$

Note that $F(\theta)$ is the amplitude for relativistic Coulomb scattering. Comparing (4.17) with (4.8), we obtain the scattering amplitude as

$$f = \left[\frac{\lambda e^{\gamma}}{2p}\right]^{2i\alpha} F(\theta) . \tag{4.19}$$

This is the central result of this paper.

Equation (4.19) is easily generalized to the case of

$$V(r) = \int_{\mu_0}^{\infty} d\mu\, \sigma(\mu)\frac{e^{-\mu r}}{r} ,$$

where $\sigma(\mu)$ vanishes sufficiently rapidly as $\mu \to \infty$. In this case, the high-energy fixed-angle scattering amplitude is given by (4.19) with

$$\alpha = \int_{\mu_0}^{\infty} d\mu\, \sigma(\mu)$$

and

$$\ln\lambda = \int_{\mu_0}^{\infty} d\mu\, \sigma(\mu) \ln u \Big/ \int_{\mu_0}^{\infty} d\mu\, \sigma(\mu) \,.$$

In summary, we note the following properties of the scattering amplitude.

(i) It is in the form of $p^{-1-2i\alpha}$ times a function of θ.

(ii) This function of θ is a transcendental function which reduces to an elementary function in the limit $\theta \ll 1$. In this limit, the partial waves with $l \gg 1$ in (4.19) constitute the bulk of scattering. Thus we may make in (4.19) the approximation of

$$[(l+\tfrac{1}{2})^2 - \alpha^2]^{1/2} \simeq l + \tfrac{1}{2} \,.$$

Then the partial-wave amplitude in (4.18) is of the same form as that of nonrelativistic Coulomb scattering, and we get

$$f \simeq \frac{-2\alpha p \Gamma(1+i\alpha)}{(\Delta^2)^{1+i\alpha}\Gamma(1-i\alpha)} (\lambda e^{\gamma})^{2i\alpha}, \; \theta \ll 1 \,, \tag{4.20}$$

where

$$\Delta = 2p \sin\frac{\theta}{2} \,.$$

(iii) We may compare (4.20) with the eikonal formula. This formula for the potential $\alpha e^{-\lambda r}/r$ is

$$f = \frac{ip}{2\pi} \int d^2 b\, e^{-i\vec{\Delta}\cdot\vec{b}} \{1 - \exp[-2i\alpha K_0(\lambda b)]\} \,. \tag{4.21}$$

In the limit $\Delta/\lambda \gg 1$, (4.21) also gives (4.20). Therefore, (4.19) agrees with the eikonal formula in the region

$$\theta \ll 1, \;\; \Delta/\lambda \gg 1 \,. \tag{4.22}$$

However, for $\theta = O(1)$, the scattering amplitude is a transcendental function and differs from the eikonal formula. This remains to be the case even if the Saxon-Schiff correction is included. Numerical plots of (4.18) are shown in Fig. 2.

The sum of leading terms is $-2\alpha p/(\Delta^2)^{1+i\alpha}$, which differs from (4.20) by a phase angle even in the region $\theta \ll 1$.

(iv) The fact that (4.19) agrees with the eikonal formula in the region $\theta \ll 1$, $p\theta \gg 1$ suggests that (4.19) holds not only in the region of fixed θ, but also down to the region of small angles as long as $\theta \gg 1/p$. It is not difficult to show that this is the case. The scattering amplitude is equal to

$$\int d^3 x\, e^{-i\vec{\Delta}\cdot\vec{x}} V(\vec{x}) F(\vec{x}) \,, \tag{4.23}$$

where

$$F(\vec{x}) \equiv e^{-ipz} \psi(\vec{x}) \,.$$

As $|\vec{\Delta}| \to \infty$, the dominant contribution to the integral (4.23) comes from the region $|x| = O(1/\Delta)$. For $|\vec{\Delta}| \ll p$, the region of contribution satisfies $|p\vec{x}| \gg 1$. In this region, the WKB method applies and the wave function is given accurately by the eikonal form of (2.14). Substituting (2.14) into (4.23), we get the eikonal formula (2.16). Therefore, the eikonal formula holds for $|\vec{\Delta}| \ll p$. Since (4.19) agrees with the eikonal formula in the region $p \gg |\vec{\Delta}| \gg \lambda$, it holds in this region as well. Together with the result established earlier that (4.19) holds for $\Delta = O(p)$, we conclude that (4.19) holds for all values $|\Delta| \gg \lambda$. There are no intermediate regions in which the scattering amplitude takes different asymptotic forms. The asymptotic high-energy amplitude is now known for all momentum transfers.

(v) As we have mentioned in Sec. I, the leading terms of perturbative calculations have powers of $\ln p$. The effects of such $(\ln p)^n$ terms can now be seen from (4.19): they modify greatly the phase angle but not the magnitude of f. As $p \to \infty$, the phase angle of f approaches infinity logarithmically.

(vi) Aside from the phase factor $(\lambda e^{\gamma}/2p)^{2i\alpha}$, (4.19) is the same as the scattering amplitude of the Klein-Gordon equation with the Coulomb potential α/r. This is easily understood: High-energy fixed-angle scattering is dependent on small-distance interactions only, and at small distances, the Yukawa potential $\alpha e^{-\lambda r}/r$ is equal to the Coulomb potential. The exponential decay of the Yukawa potential merely serves as a cutoff, and turns the well-known logarithmically divergent phase angle $\alpha \ln r$ in the Coulomb wave function into the phase angle $2\alpha \ln(\lambda e^{\gamma}/2p)$. It is also interesting to observe that the V^2 term in the Klein-Gordon equation (3.1) contributes to high-energy fixed-angle scattering, and cannot be ignored. This can also be understood without detailed calculations: at distance $r = O(1/p)$, V^2 of of the order of p^2. More generally, a potential p^{2-a}/r^a, $a \le 2$, cannot be considered to be small and must be treated to all orders. On the other hand, a poten-

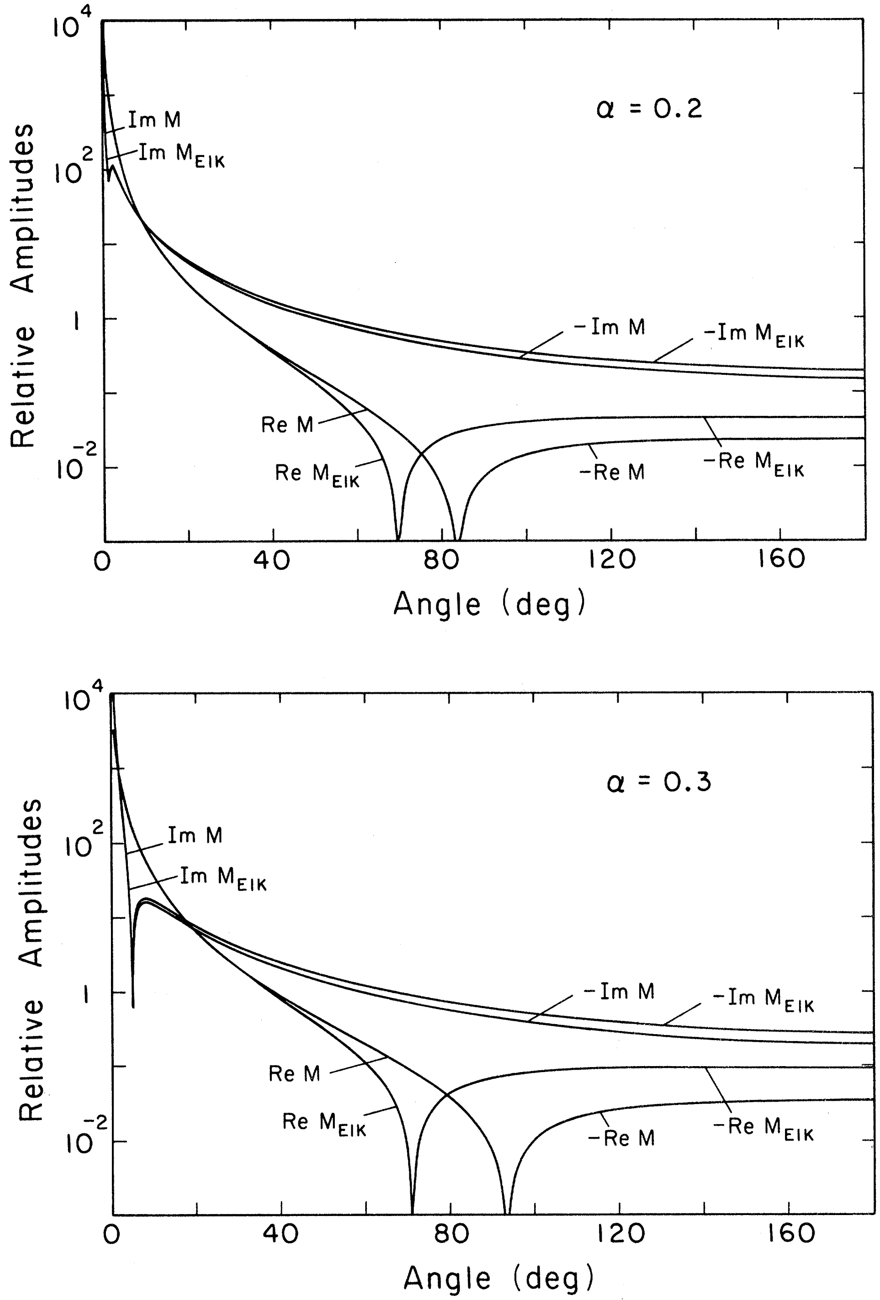

FIG. 2. Numerical values of the real and imaginary parts of $M=2ipF$. The scattering amplitude is proportional to M, a quantity which is independent of momentum. Results are shown for $\alpha=0.2$ and $\alpha=0.3$.

tial which is equal to r^{-a} times a coefficient much smaller than p^{2-a} can be treated perturbatively. For example, in nonrelativistic Coulomb scattering, the potential is α/r, where α is independent of the energy and is hence much smaller than p as $p\rightarrow\infty$. Thus the nonrelativistic Coulomb scattering amplitude is asymptotically equal to its Born term in the high-energy limit. However, in relativistic Coulomb scattering, the effective potential, according to (4.3), is $2p\alpha/r-\alpha^2/r^2$, and neither term of this potential can be considered small.

(vii) We may extend the study to the Dirac equation. For this equation, the high-energy fixed-angle scattering amplitude in the potential $\alpha e^{-\lambda r}/r$

is also equal to that in the Coulomb potential α/r multiplied by $(\lambda e^{\gamma}/2p)^{2i\alpha}$.

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APPENDIX A

In this appendix, we shall prove (3.5).

Let us rewrite the Klein-Gordon equation as

$$(-\partial_\mu\partial^\mu - m^2)\phi(x) = U\phi(x) \; , \tag{A1}$$

where

$$U = ie(\partial_\mu A^\mu + A_\mu \partial^\mu) - e^2 A^2 \; .$$

Then the solution of (A1) satisfying Feynman's boundary conditions with the incident wave $e^{-ik\cdot x}$ is the solution of the integral equation

$$\phi(x) = e^{-ik\cdot x} + \int G(x,x')U(x')\phi(x')d^4x' \; , \tag{A2}$$

where

$$G(x,x') = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq\cdot(x-x')}}{q^2 - m^2 + i\epsilon} \; . \tag{A3}$$

Next we rewrite (3.3) as

$$\left[2i\frac{\partial}{\partial\tau} - \partial_\mu\partial^\mu\right]\Psi(x,\tau) = U\Psi(x,\tau) \; . \tag{A4}$$

The solution of Ψ satisfying the initial condition (3.4) is the solution of the integral equation

$$\Psi(x,\tau) = e^{i(m^2\tau/2 - p\cdot x)} + \int d^4x' d\tau' \, G(x,\tau;x',\tau')U(x')\Psi(x',\tau') \; , \tag{A5}$$

where

$$G(x,\tau;x',\tau') = \int \frac{d^4q}{(2\pi)^4} \frac{d\omega}{2\pi} \frac{\exp[i\omega(\tau-\tau') - iq\cdot(x-x')]}{-2\omega + q^2 + i\epsilon} \; . \tag{A6}$$

Since U is independent of τ, $\Psi(x,\tau)$ is of the form

$$\Psi(x,\tau) = f(x)e^{im^2\tau/2} \; . \tag{A7}$$

Substituting (A7) into (A5), we find that the integral equation satisfied by $f(x)$ is precisely (A2). Thus

$$\phi(x) = \psi(x,0) \; . \tag{A8}$$

APPENDIX B

In this appendix, we calculate, in the limit $\lambda r \to 0$, the integrals

$$I_1 \equiv \int_{-\infty}^{z} \frac{e^{-\lambda(b^2+z'^2)^{1/2}}}{(b^2+z'^2)^{1/2}} dz' \tag{B1}$$

and

$$I_2 \equiv \int_r^\infty \frac{e^{-\lambda r'}}{r'} dr' \; , \tag{B2}$$

where $b^2 + z^2 = r^2$. These two integrals appear in (3.5) with $V = \alpha e^{-\lambda r}/r$.

We have

$$I_1 = \int_{-\infty}^{0} \frac{e^{-\lambda(b^2+z'^2)^{1/2}}}{(b^2+z'^2)^{1/2}} dz' + \int_0^z \frac{e^{-\lambda(b^2+z'^2)^{1/2}}}{(b^2+z'^2)^{1/2}} dz' \; . \tag{B3}$$

The first integral in (B3) is equal to

$$K_0(\lambda b) = -\ln\frac{\lambda b}{2} - \gamma + O(\lambda^2 b^2 \ln(\lambda b)) \; , \tag{B4}$$

where $\gamma = 0.577\,21\ldots$ is Euler's constant. The second integral in (B3) is equal to

$$\int_0^z \left[\frac{1}{(b^2+z'^2)^{1/2}} - \lambda\right] dz' + O(\lambda^2 r^2) = \ln\frac{r+z}{b} = O(\lambda z) \; . \tag{B5}$$

Therefore,

$$I_1 \simeq -\ln\frac{\lambda(r-z)}{2} - \gamma \; . \tag{B6}$$

Next, we turn to I_2. We have

$$I_2=\int_{\lambda r}^{\infty}\rho^{-1}e^{-\rho}d\rho=\lim_{\epsilon\to 0}\left[\int_0^{\infty}\rho^{-1+\epsilon}e^{-\rho}d\rho-\int_0^{\lambda r}\rho^{-1+\epsilon}e^{-\rho}d\rho\right]$$

$$=\lim_{\epsilon\to 0}\left[\Gamma(\epsilon)-\int_0^{\lambda r}(\rho^{-1+\epsilon}-\rho^{\epsilon}+\cdots)d\rho\right]=\lim_{\epsilon\to 0}\left[\Gamma(\epsilon)-\frac{(\lambda r)^{\epsilon}}{\epsilon}+\frac{(\lambda r)^{1+\epsilon}}{1+\epsilon}+\cdots\right]$$

$$=-\gamma-\ln\lambda r+O(\lambda r)\ . \tag{B7}$$

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