Two separated SU(2) Yang-Mills-Higgs monopoles in the Atiyah-Drinfeld-Hitchin-Manin-Nahm construction

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We study two arbitrarily separated SU(2) Yang-Mills-Higgs monopoles in the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction. In particular, we obtain an exact analytical expression for the Higgs field on the axis connecting its two zeros which are defined to be the locations of the monopoles. From this expression we compute the zeros of the Higgs field for small and large separations.

I. INTRODUCTION

In the past year there has been rapid progress in determining exact (superimposed and separated) multimonopole solutions of the SU(2) Yang-Mills-Higgs theory in the limit of vanishing Higgs potential. Most of the explicit results have been obtained by means of the Atiyah-Ward (AW) ansatz.[1-6] For a recent review of exact results in the theory of magnetic monopoles in non-Abelian gauge theories see Ref. 7. Wherever possible, these solutions have been shown to be regular, but a general proof of this is still an open problem in the AW approach. In particular, one has to study the two-separated-monopole solution either perturbatively[4] for small separation or numerically[8] for large separation. Numerical study of the two-monopole system has also been done in Ref. 9 using a completely different approach.

Recently, Nahm[10] has adapted the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction for instantons[11] to the monopole problem, and we will refer to it as the ADHMN construction. In the ADHMN construction, the regularity of the solution is automatic. Two other advantages of the ADHMN approach are as follows: (a) it is easily generalized to gauge groups beyond SU(2), and (b) it allows the exact construction of Green's functions for particles propagating in the background of the multimonopole solutions.

The purpose of this paper is to study two arbitrarily separated monopoles in the ADHMN construction and, in particular, to locate the two zeros of the Higgs field (which are defined to be the location of the monopoles). The main result of this paper is an exact analytical expression for the Higgs field on the axis connecting the two zeros. From this expression we compute the zeros of the Higgs field for small and large separations.

The results of this paper should help clarify the nature of the parameter space for two monopoles which in turn is relevant to understanding the dynamics of two monopoles[12] (e.g., scattering of two-far-apart monopoles approaching each other).

II. STATEMENT OF PROBLEM

Let us define in four-dimensional Euclidean space (x_1,x_2,x_3,x_4) the matrix-valued fields $(\partial_\mu \equiv \partial/\partial x_\mu)$:

$$A_\mu \text{ and } F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (\mu,\nu=1,2,3,4)\ . \tag{2.1}$$

For SU(2) Yang-Mills gauge theory, A_μ (the gauge potentials) and $F_{\mu\nu}$ (the gauge field strengths) are 2×2 anti-Hermitian traceless matrices.

The problem, simply stated, is to solve the self-duality equations

$$F_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma} \tag{2.2a}$$

for A_μ subject to the following requirements:

(1) In all gauges A_μ are static (independent of x_4):

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$$\partial_4 A_\mu = 0 \ . \tag{2.2b}$$

In this case A_4 is referred to as the Higgs field.

(2) In some gauge A_μ are nonsingular functions of

$$(x_1, x_2, x_3) \ . \tag{2.2c}$$

(3) The gauge-invariant scalar function

$$H \equiv (-\tfrac{1}{2}\,\mathrm{Tr} A_4{}^2)^{1/2} \tag{2.2d}$$

has the following asymptotic form:

$$H \to c - \frac{n}{2r} + O(r^{-2}) \quad \text{as } r \equiv (x_1{}^2 + x_2{}^2 + x_3{}^2)^{1/2} \to \infty \ , \tag{2.2e}$$

where $c>0$ is an arbitrary constant and n is a positive integer called the topological charge which, in appropriate units, is also the magnetic charge of the solution.

Motivated by considerations from differential topology,[13] one can define the location of the monopoles as the zeros of the function H. In particular, for $n=2$, the two-separated-monopole solution will be defined to be the one where H has exactly two distinct (simple) zeros. The limit where the two distinct zeros of H degenerate into one (double) zero corresponds to the axisymmetric configuration of two superimposed monopoles.[1]

III. ADHMN CONSTRUCTION FOR $n=2$

Following Ref. 10, the ADHMN construction for $n=2$ begins by defining three real functions f_1, f_2, and f_3 of a real variable z satisfying the equations

$$\frac{df_1}{dz} = f_2 f_3, \quad \frac{df_2}{dz} = f_1 f_3, \quad \frac{df_3}{dz} = f_1 f_2 \ . \tag{3.1}$$

We then define the 4×4 matrices

$$\tilde{\mathscr{X}} \equiv \begin{bmatrix} x_3 - ix_4 & x_1 - ix_2 & 0 & 0 \\ x_1 + ix_2 & -x_3 - ix_4 & 0 & 0 \\ 0 & 0 & x_3 - ix_4 & x_1 - ix_2 \\ 0 & 0 & x_1 + ix_2 & -x_3 - ix_4 \end{bmatrix}, \quad \tilde{\mathscr{F}} = \frac{1}{2}\begin{bmatrix} f_3 & 0 & 0 & f_1 - f_2 \\ 0 & -f_3 & f_1 + f_2 & 0 \\ 0 & f_1 + f_2 & -f_3 & 0 \\ f_1 - f_2 & 0 & 0 & f_3 \end{bmatrix} \tag{3.2}$$

and consider the following linear matrix differential equation ($I \equiv 4\times4$ identity matrix):

$$\left[\frac{d}{dz} I + \tilde{\mathscr{X}} + \tilde{\mathscr{F}}\right]\tilde{\mathscr{V}} = 0 \tag{3.3}$$

over a symmetric interval $-z_s \le z \le z_s$. Equation (3.3) will have four linearly independent (4×1 column vector) solutions $\tilde{\mathscr{V}}$, and we require that only two of them be orthonormalizable in the sense that

$$\int_{-z_s}^{+z_s} \tilde{\mathscr{V}}_\alpha^\dagger \tilde{\mathscr{V}}_\beta \, dz = \delta_{\alpha\beta}, \quad \alpha,\beta = 1,2 \ . \tag{3.4}$$

Thus, Eq. (3.3) must have two non-normalizable solutions, and this can only happen if the matrix $\tilde{\mathscr{F}}$ diverges at $z = \pm z_s$. The ADHMN construction then states that the solution of our problem [Eq. (2.2)] for $n=2$ is given by

$$[A_\mu]_{\alpha\beta} \equiv \alpha\text{th row and } \beta\text{th column of } A_\mu = \int_{-z_s}^{+z_s} \tilde{\mathscr{V}}_\alpha^\dagger \partial_\mu \tilde{\mathscr{V}}_\beta \, dz \ . \tag{3.5}$$

The real symmetric matrix $\mathscr{F}$ can be diagonalized by the constant orthogonal matrix

$$\tilde{\mathscr{D}} \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \tag{3.6}$$

so that we can replace Eq. (3.3) with

$$\left[\frac{d}{dz} I + \mathscr{X} + \mathscr{F}\right]\mathscr{V} = 0 \ , \tag{3.7a}$$

where

$$\mathscr{X} \equiv \tilde{\mathscr{D}}^{-1}(\tilde{\mathscr{X}} + Ix_4 i)\tilde{\mathscr{D}} = \begin{bmatrix} 0 & x_3 & x_1 & -ix_2 \\ x_3 & 0 & -ix_2 & x_1 \\ x_1 & ix_2 & 0 & -x_3 \\ ix_2 & x_1 & -x_3 & 0 \end{bmatrix}, \tag{3.7b}$$

$$\mathscr{F}\equiv\widetilde{\mathscr{D}}^{-1}\widetilde{\mathscr{F}}\widetilde{\mathscr{D}}=\frac{1}{2}\begin{bmatrix} f_1-f_2+f_3 & 0 & 0 & 0 \\ 0 & f_3-f_1+f_2 & 0 & 0 \\ 0 & 0 & f_1+f_2-f_3 & 0 \\ 0 & 0 & 0 & -f_1-f_2-f_3 \end{bmatrix}, \tag{3.7c}$$

$$\mathscr{V}\equiv\mathscr{V}(x_1,x_2,x_3,z)\equiv\widetilde{\mathscr{D}}^{-1}\widetilde{\mathscr{V}}e^{-ix_4z} . \tag{3.7d}$$

Note that the x_4 variable has disappeared in Eq. (3.7a). In terms of $\mathscr{V}$ the gauge potentials, Eq. (2.5), become ($l=1,2,3$)

$$[A_l]_{\alpha\beta}=\int_{-z_s}^{z_s}\mathscr{V}_\alpha^\dagger\partial_l\mathscr{V}_\beta dz ,$$
$$[A_4]_{\alpha\beta}=i\int_{-z_s}^{z_s}z\mathscr{V}_\alpha^\dagger\mathscr{V}_\beta dz . \tag{3.8}$$

Let $\mathscr{Q}$ be an arbitrary constant ($\equiv$independent of x_1,x_2,x_3,x_4, and z) 4×4 orthogonal matrix and $U(x)$ an arbitrary 2×2 unitary matrix function of x_1,x_2, and x_3. Under the transformation

$$\mathscr{V}_\alpha\rightarrow\mathscr{V}'_\alpha=(\mathscr{Q}\mathscr{V})_{\alpha'}U_{\alpha'\alpha}\quad(\alpha,\alpha'=1,2) , \tag{3.9}$$

the gauge potentials, Eq. (3.8), transform as

$$A_l\rightarrow A'_l=U^{-1}A_lU+U^{-1}\partial_lU,\quad A_4\rightarrow A'_4=U^{-1}A_4U , \tag{3.10}$$

which are precisely gauge transformations. In particular, the scalar function H defined by (2.2d) is gauge invariant.

IV. EXPLICIT SOLUTIONS FOR f_1,f_2,f_3

Integration of Eq. (3.1) implies $f_i^2-f_j^2$ $=$constant$=c_{ij}$. Since $c_{13}=c_{12}+c_{23}$, only two of these constants of integration are independent. By appropriate rescaling of z and the f's we can further fix one of these constants, c_{13}, to be 1. We also require that the f's diverge at the symmetrical end points $z=\pm z_s$ and are thus led to the following solution of Eq. (3.1):

$$\frac{df_3}{dz}=(1+f_3^2)^{1/2}(1+\delta^2+f_3^2)^{1/2},$$
$$f_3(z=0)=0 , \tag{4.1}$$
$$f_1=(1+f_3^2)^{1/2},\quad f_2=(1+\delta^2+f_3^2)^{1/2} ,$$

δ is an arbitrary real number.

Equation (4.1) can be explicitly solved in terms of Jacobian elliptic functions:

$$f_3=\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)},\quad f_1=\frac{1}{\mathrm{cn}(u,k)} ,$$
$$f_2=\frac{1}{k'}\frac{\mathrm{dn}(u,k)}{\mathrm{cn}(u,k)} ,$$
$$u\equiv\frac{1}{k'}z,\quad k\equiv\frac{\delta}{(1+\delta^2)^{1/2}} , \tag{4.2}$$
$$k'\equiv(1-k^2)^{1/2}=\frac{1}{(1+\delta^2)^{1/2}} ,$$

and we note that f_1,f_2,f_3 diverge at $z=\pm k'K$ where

$$K\equiv\int_0^{\pi/2}\frac{dy}{(1-k^2\sin^2 y)^{1/2}} . \tag{4.3}$$

For $\delta=0$, $f_3=\tan z$, $f_1=f_2=\sec z$, and we regain the axially symmetric $n=2$ monopole solution.[10] As noted in Ref. 10, Eq. (3.7) implies that as $r\rightarrow\infty$, $\mathscr{V}\rightarrow e^{rz}$ and Eq. (2.8) gives $H\rightarrow z_s$ as $r\rightarrow\infty$. Thus the constant c in Eq. (2.2e) will be $z_s=k'K$:

$$c=z_s=k'K \Longrightarrow H\rightarrow k'K \quad\text{as } r\rightarrow\infty . \tag{4.4}$$

From now on, by f_1, f_2, and f_3 we will mean the expressions defined by Eq. (4.2).

V. DISCRETE SYMMETRIES OF H

We begin by proving covariance of Eq. (3.7) under space inversion (parity operation P):

$$P:\ x'_1=-x_1,\quad x'_2=-x_2,\quad x'_3=-x_3 . \tag{5.1}$$

In the space-inverted system Eq. (3.7) will appear as

$$P:\ \left[\frac{d}{d(-z)}+\mathscr{X}-\mathscr{F}\right]\mathscr{V}'(-x_1,-x_2,-x_3,z)=0 . \tag{5.2}$$

From Eq. (4.2) we have

$$f_3(-z)=-f_3(z),\quad f_1(-z)=f_1(z) ,$$
$$f_2(-z)=f_2(z) , \tag{5.3}$$

and if we define the constant orthogonal matrix

$$\mathscr{D}\equiv\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}, \tag{5.4}$$

then it is easy to show that

$$\mathscr{D}^{-1}\mathscr{F}(z)\mathscr{D}=-\mathscr{F}(-z),\quad \mathscr{D}^{-1}\mathscr{X}\mathscr{D}=\mathscr{X}. \tag{5.5}$$

From Eqs. (3.7), (5.2), and (5.5) it follows that

$$P:\ \mathscr{V}'(-x_1,-x_2,-x_3,z)=\mathscr{D}\mathscr{V}(x_1,x_2,x_3,-z), \tag{5.6}$$

and Eq. (3.8) implies

$$P:\ A_4'(-x_1,-x_2,-x_3)=-A_4(x_1,x_2,x_3), \tag{5.7}$$

where A_4' is some gauge transform of A_4 [see Eq. (3.10)]. Squaring both sides of Eq. (5.7) and taking the trace, we obtain the gauge-invariant statement

$$H(-x_1,-x_2,-x_3)=H(x_1,x_2,x_3), \tag{5.8}$$

which is our first discrete symmetry of H.

Let us now consider rotations R_1,R_2,R_3 of 180° about the x_1, x_2, and x_3 axes, respectively:

$$\begin{aligned}R_1&:\ x_1'=x_1,\ x_2'=-x_2,\ x_3'=-x_3,\\ R_2&:\ x_2'=x_2,\ x_1'=-x_1,\ x_3'=-x_3,\\ R_3&:\ x_3'=x_3,\ x_1'=-x_1,\ x_2'=-x_2.\end{aligned} \tag{5.9}$$

If we define the constant orthogonal matrices

$$\mathscr{D}_1\equiv\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix},\quad \mathscr{D}_2\equiv\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix},\quad \mathscr{D}_3\equiv\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \tag{5.10}$$

then it is easy to show that

$$\begin{aligned}R_1&:\ \mathscr{V}'(x_1,-x_2,-x_3,z)=\mathscr{D}_1\mathscr{V}(x_1,x_2,x_3,z)\Longrightarrow A_4'(x_1,-x_2,-x_3)=A_4(x_1,x_2,x_3),\\ R_2&:\ \mathscr{V}'(-x_1,x_2,-x_3,z)=\mathscr{D}_2\mathscr{V}(x_1,x_2,x_3,z)\Longrightarrow A_4'(-x_1,x_2,-x_3)=A_4(x_1,x_2,x_3),\\ R_3&:\ \mathscr{V}'(-x_1,-x_2,x_3,z)=\mathscr{D}_3\mathscr{V}(x_1,x_2,x_3,z)\Longrightarrow A_4'(-x_1,-x_2,x_3)=A_4(x_1,x_2,x_3).\end{aligned} \tag{5.11}$$

We thus arrive at the following discrete symmetries of H:

$$\begin{aligned}H(x_1,x_2,x_3)&=H(-x_1,-x_2,x_3)\\ &=H(x_1,-x_2,-x_3)\\ &=H(-x_1,x_2,-x_3).\end{aligned} \tag{5.12}$$

Equations (5.8) and (5.12) prove that if H has exactly two zeros, then those zeros must necessarily be on one of the coordinate axes (x_1,x_2, or x_3) and located symmetrically about $x_1=x_2=x_3=0$. If H has exactly one zero, then it must necessarily be at $x_1=x_2=x_3=0$. These results were originally found in the AW approach by O'Raifeartaigh and Rouhani.[14]

VI. EXACT EXPRESSION FOR $H(x_1=0,x_2,x_3=0)$

We will now compute the function H on the x_2 axis: $x_1=x_3=0$, for, as it turns out, the zeros of H are precisely on this axis. In this section H and A_4 will stand for $H(x_1=0,x_2,x_3=0)$ and $A_4(x_1=0,x_2,x_3=0)$, respectively. It will be useful to define a variable $\bar{u}$ by

$$\bar{u}\equiv\tfrac{1}{2}(u-K)=\tfrac{1}{2}\left[\frac{z}{k'}-K\right],\quad -K\le\bar{u}\le 0 \tag{6.1}$$

so that $\int_{-z_s}^{z_s}dz\to 2k'\int_{-K}^{0}d\bar{u}$, etc.

Equations (3.8) and (3.7a) show that in order to compute A_4 on the x_2 axis, one can immediately set $x_1=x_3=0$ in Eq. (3.7a) and solve for $\mathscr{V}$ on the x_2 axis. (Note that this would not be true if we had to compute A_l for $l=1,2,3$.) Thus, we must solve the following equation:

$$\left[\frac{d}{dz}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}f_1-f_2+f_3&0&0&-2ix_2\\0&f_3-f_1+f_2&-2ix_2&0\\0&2ix_2&f_1+f_2-f_3&0\\2ix_2&0&0&-f_1-f_2-f_3\end{bmatrix}\right]\mathscr{V}=0\ . \tag{6.2}$$

As shown in Appendix A, Eq. (6.2) must have exactly two normalizable solutions of the form

$$\mathscr{V}_1=\begin{bmatrix}w_1\\0\\0\\w_4\end{bmatrix},\quad \mathscr{V}_2=\begin{bmatrix}0\\w_2\\w_3\\0\end{bmatrix}, \tag{6.3}$$

where w_1 and w_2 can be taken to be purely real with w_3 and w_4 purely imaginary. Since $\mathscr{V}_1^\dagger\mathscr{V}_1=0$, it follows that

$$[A_4]_{12}=[A_4]_{21}=0\ , \tag{6.4a}$$

$$-i[A_4]_{11}=\int_{-z_s}^{z_s}(w_1{}^2+|w_4|^2)z\,dz\Big/\int_{-z_s}^{z_s}(w_1{}^2+|w_4|^2)dz\ , \tag{6.4b}$$

$$-i[A_4]_{22}=\int_{-z_s}^{z_s}(w_2{}^2+|w_3|^2)z\,dz\Big/\int_{-z_s}^{z_s}(w_2{}^2+|w_3|^2)dz\ , \tag{6.4c}$$

$$H=|[A_4]_{11}|=|[A_4]_{22}|\ . \tag{6.4d}$$

In order to compute H we need only find $[A_4]_{11}$. From Eq. (6.3) it follows that w_1 and w_4 satisfy the following equations:

$$\left[\frac{d}{dz}+\tfrac{1}{2}(f_1-f_2+f_3)\right]w_1-ix_2w_4=0\ , \tag{6.5a}$$

$$\left[\frac{d}{dz}-\tfrac{1}{2}(f_1+f_2+f_3)\right]w_4+ix_2w_1=0\ . \tag{6.5b}$$

Let us define the real function Λ by

$$w_1\equiv\Lambda(f_1+f_3)^{1/2}\ . \tag{6.6}$$

If we substitute Eq. (6.6) into (6.5a), solve for w_4, and then substitute in Eq. (6.5b), we find that Λ must satisfy the following equation:

$$\frac{d^2\Lambda}{d\bar{u}^2}=(B+2k^2\mathrm{sn}^2\bar{u})\Lambda\ , \tag{6.7}$$

where $B\equiv 4k'^2x_2{}^2-(1+k^2)$. Equation (6.7) is Lamé's equation (of order 1). Note that since the coefficients of Eq. (6.7) are regular for all finite real values of $\bar{u}$, so will all its solutions. If we now define a parameter t through the relation

$$B\equiv 4k'^2x_2{}^2-(1+k^2)$$
$$\equiv -1-k^2\mathrm{cn}^2t\ , \tag{6.8}$$

then it is shown in Ref. 15 that an exact solution of Eq. (6.7) is given by

$$\mathscr{L}(\bar{u})\equiv\frac{\theta_1[(\pi/2K)(\bar{u}+t)]}{\theta_4[(\pi/2K)\bar{u}]}e^{-\bar{u}Z(t)}\ , \tag{6.9}$$

where θ_1 and θ_4 are theta functions and Z is Jacobi's zeta function.

In order that $\mathscr{V}_1$ be normalizable, we must have

$$\int_{-K}^{0}w_1{}^2d\bar{u}=\int_{-K}^{0}\Lambda^2(f_1+f_3)d\bar{u}<\infty\ .$$

However, (f_1+f_3) has a simple pole at $\bar{u}=0$, and therefore it is essential that $\Lambda(\bar{u}=0)=0$. [Because of the regularity of Eq. (6.7), Λ can have a simple zero at $\bar{u}=0$.] Now since $\mathrm{sn}^2(-\bar{u})=\mathrm{sn}^2(\bar{u})$, it follows from Eqs. (6.7) and (6.9) that we can obtain a normalizable solution by choosing

$$\Lambda(\bar{u})=\mathscr{L}(\bar{u})-\mathscr{L}(-\bar{u})\ . \tag{6.10}$$

From now on, Λ will be defined by Eq. (6.10).

It is fortunate that, with Λ defined by Eqs. (6.7) and (6.10), all of the integrals in Eq. (6.4b) can be evaluated exactly—some of the details can be found in Appendix B. In order to present the result we now define the function $S(t)$ by

$$S(t)\equiv\left[\frac{d\ln\Lambda}{d\bar{u}}\right]_{\bar{u}=-K}$$
$$=\frac{-(\mathrm{sn}t\,\mathrm{dn}t)}{\mathrm{cn}t}\tanh[KZ(t)]\ . \tag{6.11}$$

The exact expression for $[A_4]_{11}$ is found to be

$$-i[A_4]_{11}=-k'K+\frac{2k'}{B+1+k^2-S^2}\times\left[S-2(B+1+k^2)\frac{dS}{dB}\right]. \tag{6.12}$$

Taking the absolute value of Eq. (6.12) we obtain $H(x_1=0,x_2,x_3=0)$.

In terms of the parametrization (6.8), Eq. (6.12) becomes

$$-i[A_4]_{11}=-k'K+\frac{2k'}{(k^2\mathrm{sn}^2t-S^2)}\times\left[S-\frac{\mathrm{sn}t}{\mathrm{cn}t\,\mathrm{dn}t}\frac{dS}{dt}\right]. \tag{6.13}$$

As we will show, it is sometimes useful to make the following change of parametrization:

$$t\equiv t'+iK' \quad \text{where } K'\equiv\int_0^{\pi/2}\frac{dy}{(1-k'^2\sin^2y)^{1/2}}. \tag{6.14}$$

Equations (6.8) and (6.11) in terms of t' become

$$B=4k'^2x_2{}^2-(1+k^2)=-1+\frac{\mathrm{dn}^2t'}{\mathrm{sn}^2t'}, \tag{6.15a}$$

$$S(t)=-\frac{\mathrm{cn}t'}{(\mathrm{sn}t')(\mathrm{dn}t')}\coth\left\{K\left[Z(t')+\frac{\mathrm{cn}t'\mathrm{dn}t'}{\mathrm{sn}t'}\right]\right\}, \tag{6.15b}$$

and Eq. (6.12) becomes

$$-i[A_4]_{11}=-k'K+\frac{2k'\mathrm{sn}^2t'}{1-S^2\mathrm{sn}^2t'}\left[S+\frac{\mathrm{sn}t'}{\mathrm{cn}t'\mathrm{dn}t'}\frac{dS}{dt'}\right]. \tag{6.16}$$

VII. ZEROS OF H

We begin by evaluating H at the origin $x_1=x_2=x_3=0$. In terms of the parametrization (6.8), we see that $x_2=0$ corresponds to $t=0$. To evaluate Eq. (6.13) in the limit $t\rightarrow 0$, we need the following Taylor series expansions around $t=0$:

$$\mathrm{sn}t=t+\cdots,\quad S(t)=-t^2(K-E)+\cdots,\quad \frac{\mathrm{sn}t}{\mathrm{cn}t\,\mathrm{dn}t}=t+\cdots, \tag{7.1}$$

where

$$E\equiv\int_0^{\pi/2}(1-k^2\sin^2y)^{1/2}dy .$$

Substituting Eq. (7.1) into (6.13) we obtain

$$-i[A_4(x_1=x_2=x_3=0)]_{11}=\frac{k'}{k^2}[(1+k'^2)K-2E]. \tag{7.2}$$

Expanding Eq. (7.2) around $\delta=0$ we find

$$-i[A_4(x_1=x_2=x_3=0)]_{11}=\frac{\pi}{16}\delta^2+O(\delta^4), \tag{7.3}$$

so that $\delta=0$ corresponds to the axisymmetric configuration of two superimposed monopoles with H vanishing at only one point, namely $x_1=x_2=x_3=0$. For $\delta=0$ we can use the parametrization (6.15) to evaluate $H(x_1=0,x_2,x_3=0)$ as follows. For $\delta=k=0$ the elliptic functions become trigonometric functions:

$$\text{For } \delta=k=0:\ \mathrm{sn}t'=\sin t',\ \mathrm{cn}t'=\cos t',\ \mathrm{dn}t'=1,\ Z(t')=0,\ K=\pi/2 . \tag{7.4}$$

Substituting Eq. (7.4) into (6.16) we obtain

$$-i[A_4(x_1=0,x_2,x_3=0,\delta=0)]_{11}=\frac{\pi}{2}+\frac{2\cosh[(\pi/2)\rho]\{\sinh[(\pi/2)\rho]-(\pi/2)\rho\cosh[(\pi/2)\rho]\}}{\rho\{-\rho^2+\sinh^2[(\pi/2)\rho]\}}\,,$$
$$\rho\equiv(4x_2{}^2-1)^{1/2}\,, \qquad (7.5)$$

which agrees completely (after appropriate rescaling) with results obtained using the AW approach.[1] Equation (7.5) has the following Taylor series expansion around $x_2=0$:

$$-i[A_4(x_1=0,x_2,x_3=0,\delta=0)]_{11}$$
$$=-\frac{\pi}{2}\left[3-\frac{\pi^2}{4}\right]x_2{}^2+O(x_2{}^4)\,. \qquad (7.6)$$

Since on the x_2 axis $H=|[A_4]_{11}|$ and $H(-x_2)=H(+x_2)$, it follows from Eqs. (7.3) and (7.6) that to second order in δ and x_2,

$$-i[A_4(x_1=0,x_2,x_3=0)]_{11}$$
$$=\frac{\pi}{16}\delta^2-\frac{\pi}{2}\left[3-\frac{\pi^2}{4}\right]x_2{}^2+\cdots\,, \qquad (7.7)$$

which shows that, to this order, H has two zeros at

$$(\text{zeros of } H \text{ for small } \delta):\ x_2=\pm\delta(24-2\pi^2)^{-1/2}\,. \qquad (7.8)$$

Thus, as asserted, the zeros of H are on the x_2 axis. Note that to obtain the higher-order terms in Eqs. (7.7) and (7.8) we must directly expand Eq. (6.12).

In order to study H for large values of δ and x_2, it is again convenient to use the parametrization (6.15). For large δ we have the following Fourier series expansion for Eq. (6.15a):

$$B+1+k^2=4k'^2x_2{}^2=\frac{1}{\mathrm{sn}^2t'}$$
$$=\frac{\pi^2}{4K'^2}\left[\coth^2\left[\frac{\pi t'}{2K'}\right]+\frac{1}{32\delta^4}\cosh\left[\frac{\pi t'}{K'}\right]+\cdots\right]\,, \qquad (7.9)$$

and since $K\rightarrow\ln\delta$ for large δ, we can consistently make the following approximation for Eq. (6.15b):

$$S\approx\frac{-\mathrm{cn}t'}{\mathrm{sn}t'\mathrm{dn}t'}\,. \qquad (7.10)$$

Inserting Eq. (7.10) into Eq. (6.16) and expanding the resulting expression in a Fourier series valid for large δ we obtain

$$-i[A_4(x_1=0,x_2,x_3=0)]_{11}$$
$$=-k'K+k'\sinh\left[\frac{\pi t'}{K'}\right]+O(k'^2)\,. \qquad (7.11)$$

At this point, we note that the limit $|x_2|\gg|\delta|$ corresponds to $t'\rightarrow 0$, and comparing Eqs. (7.9) and (7.11) we obtain

$$-i[A_4(x_1=0,x_2,x_3=0)]_{11}$$
$$=-k'K+\frac{1}{|x_2|}+O(x_2{}^{-2}),\quad (|x_2|\gg|\delta|) \qquad (7.12)$$

in complete agreement with Eqs. (2.2e) and (4.4). Equation (7.11) shows that for large δ, with an error of $1/\delta$, the zeros of H are at $\pi t'/K'=\sinh^{-1}K\simeq\ln\ln\delta^2$ and inserting this into Eq. (7.9) gives

$$(\text{zeros of } H \text{ for large } \delta):\ x_2=\pm\delta/2\,. \qquad (7.13)$$

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APPENDIX A: NORMALIZATION

From Eq. (3.1) it follows that

$$f_1=\frac{d}{dz}\ln(f_2+f_3),\quad f_2=\frac{d}{dz}\ln(f_1+f_3),\quad f_3=\frac{d}{dz}\ln(f_1+f_2)\,, \qquad (A1)$$

so that any equation of the form

$$\frac{dg}{dz}=(c_1f_1+c_2f_2+c_3f_3)g\ , \tag{A2}$$

where c_1, c_2, and c_3 are constants, can be integrated (within a constant factor) to give

$$g=(f_2+f_3)^{c_1}(f_1+f_3)^{c_2}(f_1+f_2)^{c_3}\ . \tag{A3}$$

Let us now consider Eq. (3.7) at the origin $x_1=x_2=x_3=0$:

$$\left[\frac{d}{dz}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}f_1-f_2+f_3&0&0&0\\0&f_3-f_1+f_2&0&0\\0&0&f_1+f_2-f_3&0\\0&0&0&-f_1-f_2-f_3\end{bmatrix}\right]\overline{\mathscr{V}}=0\ . \tag{A4}$$

Equation (A4) has the following four linearly independent solutions:

$$\overline{\mathscr{V}}_1=\begin{bmatrix}\overline{w}_1\\0\\0\\0\end{bmatrix},\quad \overline{\mathscr{V}}_2=\begin{bmatrix}0\\\overline{w}_2\\0\\0\end{bmatrix},\quad \overline{\mathscr{V}}_3=\begin{bmatrix}0\\0\\\overline{w}_3\\0\end{bmatrix},\quad \overline{\mathscr{V}}_4=\begin{bmatrix}0\\0\\0\\\overline{w}_4\end{bmatrix}, \tag{A5}$$

where

$$\overline{w}_1{}^2=-k'\mathrm{sn}\overline{u}\,\mathrm{cn}\overline{u}\,\mathrm{dn}\overline{u}\ ,\quad \overline{w}_2{}^2=-k'\frac{\mathrm{sn}\overline{u}\,\mathrm{cn}\overline{u}}{\mathrm{dn}^3\overline{u}}\ ,\quad \overline{w}_3{}^2=-k'\frac{\mathrm{sn}\overline{u}\,\mathrm{dn}\overline{u}}{\mathrm{cn}^3(\overline{u})}\ ,\quad \overline{w}_4{}^2=-\frac{1}{k'^3}\frac{\mathrm{cn}\overline{u}\,\mathrm{dn}\overline{u}}{\mathrm{sn}^3\overline{u}}\ . \tag{A6}$$

Note that $(\overline{w}_1{}^2,\overline{w}_2{}^2,\overline{w}_3{}^2,\overline{w}_4{}^2)\geq 0$ for $-K\leq\overline{u}\leq 0$. Since $\mathrm{cn}(-K)=\mathrm{sn}(0)=0$, only $\overline{\mathscr{V}}_1$ and $\overline{\mathscr{V}}_2$ will be normalizable. Now on the x_2 axis the normalizable solutions of Eq. (6.2) must become $\overline{\mathscr{V}}_1$ and $\mathscr{V}_2$ as $x_2\to 0$. Thus, one can search for normalizable solutions of Eq. (6.2) in the form of Eq. (6.3).

APPENDIX B: DERIVATION OF EQ. (6.12)

In this appendix integrals will be indefinite. In order to compute $[A_4]_{11}$ defined by Eq. (6.4b), one must know how to compute the integrals

$$I_1\equiv x_2{}^2\int(w_1{}^2+|w_4|^2)dz\ ,\quad I_2\equiv x_2{}^2\int(w_1{}^2+|w_4|^2)z\,dz\ , \tag{B1}$$

where

$$w_1=\Lambda(f_1+f_3)^{1/2}\ ,\quad w_4=\frac{(f_1+f_3)^{1/2}}{ix_2}\left[\tfrac{1}{2}(f_3+f_1)\Lambda+\frac{d\Lambda}{dz}\right], \tag{B2}$$

where Λ satisfies Eq. (6.7). If we substitute Eq. (B2) into Eq. (B1), repeatedly integrate by parts, and use Eq. (6.7) we find

$$I_1=(f_1+f_3)\left[\tfrac{1}{2}(f_1+f_3-f_2)\Lambda^2+\Lambda\frac{d\Lambda}{dz}\right]+I_3\ , \tag{B3}$$

$$I_2=(f_1+f_3)\left\{z\left[\tfrac{1}{2}(f_1+f_3-f_2)\Lambda^2+\Lambda\frac{d\Lambda}{dz}\right]-\tfrac{1}{2}\Lambda^2\right\}+I_4\ , \tag{B4}$$

where

$$I_3=-\tfrac{1}{2}\int\Lambda^2(f_3+f_1)[f_2(f_1-f_2+f_3)-f_1f_3]dz=-\frac{k^2}{k'^2}\int\Lambda^2\mathrm{sn}\overline{u}\,\mathrm{cn}\overline{u}\,\mathrm{dn}\overline{u}\,d\overline{u}\ , \tag{B5}$$

$$I_4=-\tfrac{1}{2}\int \Lambda^2(f_3+f_1)\{z[f_2(f_1-f_2+f_3)-f_1f_3]+(f_1+f_3-2f_2)\}dz$$
$$=-\frac{k^2}{k'}\int \Lambda^2(2\bar{u}+K)\mathrm{sn}\bar{u}\,\mathrm{cn}\bar{u}\,\mathrm{dn}\bar{u}\,d\bar{u}-\frac{1}{k'}\int \Lambda^2[-(1+k^2)+2k^2\mathrm{sn}^2\bar{u}]d\bar{u}\ . \tag{B6}$$

In order to evaluate I_3 we multiply both sides of Eq. (6.7) by $d\Lambda/d\bar{u}$ and integrate by parts to obtain

$$\int \Lambda^2\mathrm{sn}\bar{u}\,\mathrm{cn}\bar{u}\,\mathrm{dn}\bar{u}\,d\bar{u}=\frac{1}{4k^2}\left[-\left[\frac{d\Lambda}{d\bar{u}}\right]^2+(B+2k^2\mathrm{sn}^2\bar{u})\Lambda^2\right]\ . \tag{B7}$$

If we multiply both sides of Eq. (6.7) by Λ and integrate by parts, we obtain

$$\int \left[\left[\frac{d\Lambda}{d\bar{u}}\right]^2+(B+2k^2\mathrm{sn}^2\bar{u})\Lambda^2\right]d\bar{u}=\Lambda\frac{d\Lambda}{d\bar{u}}\ . \tag{B8}$$

We can now integrate I_4 by parts and use Eqs. (B7) and (B8) to obtain

$$I_4=-\frac{1}{2k'}\Lambda\frac{d\Lambda}{d\bar{u}}-\frac{1}{4k'}(2\bar{u}+K)\left[-\left[\frac{d\Lambda}{d\bar{u}}\right]^2+(B+2k^2\mathrm{sn}^2\bar{u})\Lambda^2\right]+I_5\ , \tag{B9}$$

where

$$I_5=\frac{1}{k'}(B+1+k^2)\int \Lambda^2 d\bar{u}\ . \tag{B10}$$

In order to evaluate I_5 we regard B as a free parameter in Eq. (6.7) and consider Λ as a function of B. Then $\Lambda(B)$ and $\Lambda(B+\epsilon)$ will satisfy the following equations:

$$\frac{d^2\Lambda(B)}{d\bar{u}^2}=(B+2k^2\mathrm{sn}^2\bar{u})\Lambda(B)\ , \tag{B11a}$$

$$\frac{d^2\Lambda(B+\epsilon)}{d\bar{u}^2}=(B+\epsilon+2k^2\mathrm{sn}^2\bar{u})\Lambda(B+\epsilon)\ . \tag{B11b}$$

Multiplying Eq. (B11a) by $\Lambda(B+\epsilon)$ and Eq. (B11b) by $\Lambda(B)$, subtracting, and then integrating gives

$$\int \Lambda(B+\epsilon)\Lambda(B)d\bar{u}=\frac{1}{\epsilon}\left[\Lambda(B)\frac{d\Lambda(B+\epsilon)}{d\bar{u}}-\Lambda(B+\epsilon)\frac{d\Lambda(B)}{d\bar{u}}\right]\ . \tag{B12}$$

We now take the limit $\epsilon\to 0$ on both sides of Eq. (B12) to obtain the desired integral:

$$\int \Lambda^2 d\bar{u}=\left[\Lambda\frac{d^2\Lambda}{dB\,d\bar{u}}-\left[\frac{d\Lambda}{dB}\right]\frac{d\Lambda}{d\bar{u}}\right]\ . \tag{B13}$$

We have thus exactly computed I_1 and I_2. It only remains to evaluate I_1 and I_2 at the limits $\bar{u}=0$ and $\bar{u}=-K$, which, though tedious, is completely straightforward leading to the result quoted in Eq. (6.12).

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