Scattering theory in relativistic quantum mechanics

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We construct a relativistic quantum scattering theory in a framework originally suggested by Stueckelberg, where the dynamical evolution of a system in space-time is described by means of an invariant parameter τ. The wave operator for the reduced motion of a two-body system is related to measureable cross sections. The optical theorem is proved, and it is shown that in the nonrelativistic limit the cross section has the same interpretation as in the usual nonrelativistic scattering theory. A perturbation expansion for the S matrix is obtained, and its form is compared with that of the perturbative structure of quantum constraint Hamiltonian dynamics and quantum field theory. The problem of electromagnetic scattering of two charged particles is formulated and it is shown, for a heavy target, that the Rutherford cross section is obtained to lowest order.

I. INTRODUCTION

The problem of merging special relativity and quantum mechanics together into a satisfactory relativistic quantum mechanics (RQM) of interacting particles has not yet been completely worked out. On the other hand, a relativistic quantum field theory (RQFT) has been constructed and gives accurate results. Nevertheless, RQFT requires one to deal with an infinite number of degrees of freedom. One might expect to be able to describe a finite number of particles phenomenologically through a finite number of degrees of freedom. It would be of great value in understanding the foundation of RQFT if there were an underlying quantum mechanics as in the case of nonrelativistic quantum field theory.

Approximate nonrelativistic potential models have been used with good results (e.g., the spectrum of the hydrogen atom or of massive quark bound states). One may conjecture that relativistic scalar and vector potential models will be applicable to a wider range of phenomena.

In this paper, we shall develop a scattering theory in the framework of a manifestly covariant relativistic quantum theory in which there is a canonical evolution[1-4] according to an invariant parameter τ which is a generalization of proper time. This formalism enables us to construct a perturbation theory which facilitates calculations; it also makes possible a comparison with certain aspects of RQFT and with the recently developed quantum relativistic constraint Hamiltonian dynamics (QCHD),[5,6] which appears to form a bridge between these theories in its treatment of direct interactions.

The framework in which we shall study scattering theory should be distinguished from approaches based on a relativistic particle dynamics of the type proposed by Bakamjian and Thomas.[7] Coester,[8] for example, has developed a relativistic scattering theory of this type which is covariant, but not manifestly covariant. In this formalism, the four-dimensional energy-momentum manifold is reduced to R^3 by putting an elementary system in correspondence with a point in the direct integral over mass. The mass parameter is then replaced by an operator h [in $L^2(R^3)$] representing the rest energy of the system in its center of mass, and is supposed to contain the interactions. The coordinate operator is taken to be the Newton-Wigner operator [as it appears in $L^2(R^3)$; see Ref. 4 for a discussion of this point]. The dynamical evolution of the system is considered to be generated by $P^0=H=(\vec{p}^2+h^2)^{1/2}$. The similarity between our approach and Coester's has its source in a theorem cited by Coester,[8] i.e., that the wave operators relating H, H_0 are equal to the wave operators relating h, h_0. The former are defined by limits in t, and the latter are defined by limits in an invariant parameter τ. There is, however, no canonical quantum-theoretical evolution associated with this development in τ. The formulation of the Bakamjian-Thomas theory in the center of mass of a system leads to difficulties in assuring cluster decomposition; this property is quite natural in the framework we shall use.

The perturbation theory which we shall develop, for the case of direct interaction, has a structure in which there is one propagator for each intermediate state, regardless of the number of particles. In this form, the S matrix exactly conserves the sum of the unperturbed single free-particle dynamical generators, corresponding essentially to the sum of squares of the individual particle masses. The conservation of individual particle masses is a dynamical question, and the theory can therefore describe phenomena which appear, in the laboratory, to be inelastic (there are, as we shall show, forms of the direct interaction potential for which the individual particle masses in the two-body problem are precisely, or almost precisely, conserved). If, however, the correlation in τ between different particles is removed in the perturbation expansion, and each particle evolves according to a different parameter, the perturbation expansion decomposes into a form in which there is a product of Feynman propagators, one for each particle in the intermediate states, with a structure, therefore, closer to that of quantum field theory. The differential equations giving rise to this type of perturbation expansion are found to coincide with those obtained in the framework of the many-time formulation of QCHD. In its simplest formulation, when wave operators of normal type exist, the individual particle masses in QCHD are exactly asymptotically conserved in the S matrix.

Classical constraint Hamiltonian dynamics (CCHD)[9] offers an approach to mechanics which is, in some ways, complementary to the single-τ formalism,[4] but eventually finds itself on common ground. In order to describe the motion of an N-particle system, CCHD specifies a set of N first-class constraints (for which the constraint functions, called "constrainors", have mutually vanishing Poisson brackets), usually equations relating $p^{\mu}p_{\mu}$ of each particle to a function corresponding to the interaction of the particle with all of the others (effective mass-shell conditions). This set of N equations specifies a $7N$-dimensional hypersurface for the motion, embedded in the $8N$-dimensional phase space, consisting of the N x^{μ}'s and p^{μ}'s. Additional constraints are, however, required to specify a trajectory in the phase space and, finally, to parametrize the motion along this trajectory with an invariant "time" parameter (to be identified with the τ of Stueckelberg). The motion along this trajectory can be generated by a linear combination of constrainors, a generalized invariant Hamiltonian; the requirement that this linear combination have vanishing Poisson brackets with $N-1$ additional constraints, and Poisson bracket unity with the Nth, for example, provides a set of invertible linear equations for the coefficients and ensures conservation of the constraints.

Since, in the differential geometry of CCHD, the constrainors act as generators of motions in independent directions in the $7N$-dimensional constrained surface,[9] the quantum version of the theory[5] associates a Schrödinger-type equation, each with its own invariant time parameter, with each of the constrainor operators. In the classical limit, this system of equations reduces to the usual description of CCHD. It can be shown[5] in QCHD that the S matrix of scattering theory is independent of the choice of the additional constraints, since the limit of the interaction-picture wave function for all $\tau_i \to \pm\infty$, $i=1,\ldots,N$, is independent of the order of limits, and coincides with the S-matrix of a corresponding single-τ theory. The choice of the additional constraints is analogous to the choice of an interpolating field in Lehmann-Symanzik-Zimmermann (LSZ) quantum field theory.[10] From the point of view of QCHD, the theory that we shall be studying here (at least for the part that concerns direct interaction) is formally equivalent to that of an interpolating theory. From the point of view of the single-τ quantum theory, where one may chose a direct action potential with no restriction other than Poincaré invariance and suitable falloff in spacelike directions for the relative coordinate variables, the QCHD appears as a self-consistent field-type approximation.

A formulation of electromagnetic interactions of charged relativistic particles will also be given in the framework of RQM. We shall show that the instantaneous form of the current density must be integrated over all τ, and that the resulting current density (for any number N of particles) is a conserved four-vector which can serve as the source of electromagnetism. This result was stated and proved (in a slightly different way) by Stueckelberg[11] for the case of one particle. Since the wave function is obtained by integrating a differential equation containing a (τ-independent) vector potential, and the vector potential is obtained from the Maxwell equations in which the source depends bilinearly on the wave functions integrated over all τ, the resulting semiclassical electrodynamics is a rather implicit system of equations. The associated perturbation theory is expected to have at least some of the structure of QCHD, and to resemble RQFT more than the direct-interaction RQM. In this paper, we formulate the problem of the scattering of two charged particles. It is shown, to lowest order, that one obtains the Rutherford cross section, and the corrections due to recoil appear to be quite reasonable. This result verifies, in a heuristic way, the definitions and interpretation we have given for the S matrix and scattering cross section. The problem of electromagnetic self-interaction, and its implication for scattering theory, will also be discussed.

Although RQM has been extended in order to be able to treat particles with spin,[12-14] we shall restrict ourselves in this paper to the case of spinless particles interacting either through direct interaction or the vector potential of electromagnetism.

In Sec. II, the scattering problem is formulated, starting with a brief review of the basic structure of RQM. The cross section is defined in view of the physical nature of a relativistic scattering experiment. In Sec. III, the S matrix is constructed in terms of the limit of the interaction-picture evolution operator. This construction provides a perturbation expansion for the S matrix.

The comparison of the structure of the perturbation expansion in direct interaction scattering in RQM with perturbation expansions in QCHD and RQFT is carried out in Sec. IV. The form of the theory with charged particles in interaction with each other through the electromagnetic field is discussed in Sec. V.

II. SCATTERING THEORY IN RQM

We start with a brief review of relativistic classical and quantum mechanics.[4] In classical mechanics, the state of each particle is described by the eight independent variables $x^\mu = (\vec{x}, t)$ and $p^\mu = (\vec{p}, E)$ [we shall use the metric $(-1,+1,+1,+1)$]. The energy E is independent of $\vec{p}$, and the particles are therefore not restricted to a particular mass shell. To parametrize the evolution of the system, a parameter τ is introduced as an order parameter; it cannot be altered or directly observed, and should not be confused with the geometrical time t which is a physical observable defining the state of a particle (time of occurrence in the laboratory frame). An N-body system is characterized by a scalar function of all $8N$ variables $K(p^\alpha, x^\alpha)$, $\alpha = 1, \ldots, 4N$, which satisfies a generalized Hamilton principle resulting in the canonical equations

$$\frac{dp^\alpha}{d\tau} = -\frac{\partial K}{\partial x_\alpha}, \quad \frac{dx^\alpha}{d\tau} = \frac{\partial K}{\partial p_\alpha} . \tag{2.1}$$

In the nonrelativistic limit,[15] one imposes $dt^i/d\tau = 1$, so that $K = H - \sum E^i$ $(i = 1, \ldots, N)$ and the equations (2.1) reduce to the usual Hamilton equations. In the case of one free particle, one takes

$$K = \frac{1}{2M} p^\mu p_\mu , \tag{2.2}$$

so that

$$\frac{dp^\mu}{d\tau} = 0, \quad \frac{dx^\mu}{d\tau} = p^\mu / M. \tag{2.3}$$

Eliminating τ, these reduce to the familiar equations

$$\frac{dp^\mu}{dt} = 0, \quad \frac{d\vec{x}}{dt} = \frac{\vec{p}}{E} . \tag{2.4}$$

The parameter M is not necessarily the mass of the particle, and for free particles (or particles with purely electromagnetic interaction) it is associated only with the scale of τ. Choosing initial conditions such that

$$K = -M/2 \tag{2.5}$$

so that

$$p^\mu p_\mu \equiv -m^2 = -M^2 , \tag{2.6}$$

the scale of τ is fixed so that

$$-(ds)^2 = dx^\mu dx_\mu = \frac{p^\mu p^\mu}{M^2} (d\tau)^2 = -(d\tau)^2 , \tag{2.7}$$

i.e., τ is essentially equal to the proper time s. An external (direct-interaction) potential can be added to (2.2) to give

$$K = \frac{p^\mu p_\mu}{2M} + V(x^\mu x_\mu) \tag{2.8}$$

and a particle in an external electromagnetic field is described by

$$K = \frac{1}{2M} (p^\mu - eA^\mu)(p_\mu - eA_\mu) . \tag{2.9}$$

The initial condition (2.5) leads to $(d\tau)^2 = (ds)^2$ asymptotically for the case (2.8), and for all τ for (2.9).

For the two-body problem, we may take

$$\begin{aligned} K &= \frac{p_1^\mu p_{1\mu}}{2M_1} + \frac{p_2^\mu p_{2\mu}}{2M_2} + V(x_1 - x_2) \\ &= \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(x) , \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &x^\mu = x_1^\mu - x_2^\mu , \quad p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M_1 + M_2} , \\ &X^\mu = \frac{M_1 x_1^\mu + M_2 x_2^\mu}{M_1 + M_2} , \quad P^\mu = p_1^\mu + p_2^\mu , \\ &M = M_1 + M_2 , \quad m = \frac{M_1 M_2}{M_1 + M_2} . \end{aligned} \tag{2.11}$$

For the general N-body case, the center-of-mass motion can always be extracted as in (2.10), provided that the interaction potential is a function only of the differences between space-time coordinates.

In the quantum theory, the states of a system at a given τ are described in the Hilbert space $L^2(R^4; d^4x)$. The operators corresponding to space-time and energy-momentum satisfy the commutation relations

$$[x^\mu, p^\nu] = i g^{\mu\nu} \tag{2.12}$$

and the evolution of the system is described by the equation

$$i\frac{\partial\psi_\tau}{\partial\tau}=K\psi_\tau, \qquad (2.13)$$

where K is the operator corresponding to (2.9) or (2.10) in the cases we shall treat.

It is well known that in relativistic theories based on the Klein-Gordon or Dirac equations, the position operator is not represented by $i\partial/\partial\vec{p}$, but by a somewhat more complicated object, the Newton-Wigner position operator.[16] From Eq. (2.12), one sees that x^μ can be represented by $i\partial/\partial p_\mu$ and it is reasonable to ask whether this operator can correspond to the position of the particle. It has been shown[4] that the operator

$$\vec{x}-\frac{1}{2}\left\{t,\frac{\vec{p}}{E}\right\}, \qquad (2.14)$$

in the direct integral representation obtained by transforming the momentum-space integration (d^4p) to

$$\left[\frac{dm^2d^3p}{2(\vec{p}^2+m^2)^{1/2}}\right],$$

is exactly the Newton-Wigner operator for each value of m^2. There is no corresponding form for t, since this operator is not diagonal in m^2. We conclude from this result that the values of $\vec{x}$ in $L^2(R^4)$ correspond to the physical positions of the particle.

As in nonrelativistic quantum mechanics, the Heisenberg equations are of the same form as the corresponding classical equations of motion (2.1), and describe the motion of the center of the wave packet. For the case of free particles, valid, in a system for which scattering can take place, when the particles are sufficiently far apart, Eq. (2.3) implies the asymptotic relation

$$\langle x^\mu\rangle=\left\langle\frac{p^\mu}{M}\right\rangle\tau. \qquad (2.15)$$

As $\tau\to\pm\infty$, it follows from Eq. (2.15) that the particle moves to remote regions of space and time, corresponding to the usual notion of an asymptotic region for scattering. If the particles move to regions in which the interaction is negligible when $|\tau|$ is sufficiently large, they will then continue to evolve as indicated in the asymptotic relation (2.15). We may then state the asymptotic condition for scattering as

$$\lim_{\tau\to\infty(-\infty)}\|e^{-iK\tau}|\psi\rangle-e^{-iK_0\tau}|\psi_{\text{out(in)}}\rangle\|=0, \qquad (2.16)$$

where K_0 corresponds to the unperturbed evolution operator. This condition is formally identical to the asymptotic condition in nonrelativistic scattering theory, and the formal structure of relativistic scattering theory is therefore quite similar to that of the nonrelativistic theory.[17] It follows from the condition (2.16) that one can define the operators

$$\Omega_\pm=\lim_{\tau\to\mp\infty}e^{iK\tau}e^{-iK_0\tau}, \qquad (2.17)$$

which we shall call wave operators. As in the nonrelativistic theory, one finds that a sufficient condition for the existence of the wave operator Ω_+ is that there exist a dense set of $|\psi_{\text{in}}\rangle$ such that

$$\int_0^\infty d\tau\|Ve^{-iK_0\tau}|\psi_{\text{in}}\rangle\|<\infty. \qquad (2.18)$$

If V were square integrable, the spread of the wave packet (proportional to τ^{-2} in the relativistic case, since there are four Gaussian integrals) would be adequate to ensure the validity of the inequality (2.18). The potential function V is, however, an invariant function of x^μ, and in the absence of other four-vectors, it must be a function of $x^\mu x_\mu$. Such a function cannot be square integrable in $L^2(R^4)$. One must therefore use not only the spread of the wave packet, but also its motion. Choosing a dense set of $|\psi_{\text{in}}\rangle$ for which $\langle p|\psi_{\text{in}}\rangle$ vanishes (along with some of its derivatives) for $p^2\to 0$, Horwitz and Soffer have rigorously shown[18] that the wave operator exists for potentials that are bounded and decrease faster than $|x^2|^{-1/2}$ as $x^2\to\pm\infty$. Asymptotic completeness in the absolutely continuous part of the spectrum of K has recently been proved by Soffer.[19]

With the help of the wave operators, one can define the relativistic S matrix

$$|\psi_{\text{out}}\rangle=\Omega_-^\dagger|\psi\rangle=\Omega_-^\dagger\Omega_+|\psi_{\text{in}}\rangle\equiv S|\psi_{\text{in}}\rangle. \qquad (2.19)$$

From the intertwining relations of the wave operator, one obtains

$$[K_0,S]=0. \qquad (2.20)$$

For the case of a single particle in an external potential, Eq. (2.20) implies that the S matrix conserves $p^\mu p_\mu=-m^2$. For the two-body problem, however, with a dynamical evolution operator of the form (2.10), Eq. (2.20) imples only that $p_1{}^2/2M_1+p_2{}^22M_2$ is automatically conserved. Hence, the conservation of the individual particle masses in a several-particle problem is a dynamical question. It follows from Eqs. (2.1) (considered as Heisenberg equations) that

$$\frac{d}{d\tau}m_i{}^2(\tau)=\left(p_i^\mu\frac{\partial V}{\partial x_i^\mu}+\frac{\partial V}{\partial x_i^\mu}p_i^\mu\right)(\tau), \qquad (2.21)$$

where $\mathcal{O}(\tau)\equiv e^{iK\tau}\mathcal{O}e^{-iK\tau}$, and, therefore, with the help of the asymptotic condition (2.16),[20,21]

$$\langle\psi_{\rm out}|m_i^2|\psi_{\rm out}\rangle-\langle\psi_{\rm in}|m_i^2|\psi_{\rm in}\rangle = \int_{-\infty}^{\infty} d\tau\langle\psi_\tau|\frac{\partial V}{\partial x_i^\mu}p_i^\mu+p_i^\mu\frac{\partial V}{\partial x_i^\mu}|\psi_\tau\rangle\,. \quad (2.22)$$

In the case of a single particle in an external field, the right-hand side of Eq. (2.22) necessarily vanishes. To the extent that each particle in a several-particle problem moves in the effective field of the others, one would expect a similar conservation law to hold (we shall return to this point later). In the classical two-body problem, for example, Pearle[3] has pointed out that [using variables defined in (2.11)]

$$p_1^2=\left(\frac{M_1}{M}P-p\right)^2=\frac{M_1^2}{M^2}P^2-\frac{2M_1}{M}P\cdot p+p^2\,,$$
$$p_2^2=\left(\frac{M_2}{M}P+p\right)^2=\frac{M_2^2}{M^2}P^2+\frac{2M_2}{M}P\cdot p+p^2\,, \quad (2.23)$$

so that, denoting momentum exchange by $q=p'-p=p_2'-p_2=p_1-p_1'$, one obtains

$$p_1'^2=p_1^2-\frac{2M_1}{M}P\cdot q\,,$$
$$p_2'^2=p_2^2+\frac{2M_2}{M}P\cdot q\,. \quad (2.24)$$

The condition for mass conservation is that $P\cdot q=0$. It has been suggested,[22] in a slightly different context, that a potential of the form $V(x^\mu-P^\mu(x\cdot P)/P^2)$ will assure this condition, since the Fourier transform is then proportional to $\delta(P\cdot q)$. A potential of this form, however, would contribute to the center-of-mass motion in the Heisenberg equations as a term depending on the relative motion, and the center-of-mass motion would not decouple from the relative-motion problem.[23] A potential of the form $V(x^\mu+n^\mu(x\cdot n))$, where $n^2=-1$ (see Refs. 12 and 13 for a discussion of the possible significance of such a timelike vector) could be considered. If the center-of-mass momentum has a direction not very different from n^μ, $P_\mu q^\mu$ will be small.

The conservation of individual particle masses can be guaranteed if K is such that other constants of the motion are admissible. The constraint Hamiltonian formalism developed by Rohrlich and others[9] for classical mechanics (CCHD) and recently extended to quantum mechanics (QCHD),[5,6] described in Sec. I, effectively conserves a set of functions (constrainors) of the form $p_a^2+m_a^2+\Phi_a$, $a=1,2,\ldots,N$, where the Φ_a play the role of potentials which vanish asymptotically, assuring the conservation of individual particle masses.

On the other hand, there are physical processes for which mass shifts do occur, in inelastic scattering or decay processes (such as β decay) with amplitudes crossing equivalent to an inelastic scattering process. This formalism offers the possibility of constructing models for such processes. In the following, we shall not assume the existence of additional constants of the motion specifying asymptotic particle masses, and return to this point in our analysis of the perturbation expansion.

We now turn to study the definition of the differential cross section. Let us consider the two-body problem in terms of the variables defined in Eqs. (2.11), and separate out the motion of the center of mass. The problem then formally resembles that of a particle scattering in the potential of an external source. One should remember, however, that the "particle" momentum p^μ may be spacelike, and its coordinate is relative, e.g., $t=t_1-t_2$ is the relative time, at a given τ, of the occurrence of the two particles (particles of the beam and target).[24]

The wave packets of the beam leaving the accelerator are assumed to be associated with wave packets with relative coordinates in the asymptotic region. By the analog of Ehrenfest's theorem, the centers satisfy the free-particle equations of motion

$$x_{i\mu}^*(\tau)=x_{i\mu}^*(0)+\frac{p_\mu^*}{M}\tau\,,$$
$$p_\mu^*(\tau)=p_\mu^*=\text{const.} \quad (2.25)$$

Working in the laboratory frame, where the accelerator, the source of the potential, and the detector are at rest, we choose the direction of $\vec{p}^*$ to be the z axis. The wave packets start at $\tau=0$ at a point $z_i=-L$, where L is the distance of the accelerator from the origin, but the other coordinates x_i, y_i, and t_i are random and have to be averaged over. The wave packets displaced in t will sample the t-dependent potential, a function of $\vec{x}^2-t^2$, at different points. The different wave packets to be averaged over can be defined in terms of a representative incoming wave function $\psi_{\rm in}(p)$ as

$$\psi_{\rm in}^i(p)=\psi_{\rm in}^{\vec{\rho},x^0}(p)=e^{-i\vec{\rho}\cdot\vec{p}}e^{ix^0p^0}\psi_{\rm in}(p)\,. \quad (2.26)$$

The set of wave packets obtained in this way is to effectively "cover" the potential, with random spatial and temporal displacements. The corresponding procedure in the nonrelativistic case is straightforward to carry out, since the potential is bounded, or falls off, in each direction. A relativistically invariant potential cannot be bounded in this way because of the diminishingly small regions near the light cone where it is nonzero; these prevent the possibility of "covering" it by a beam of finite dimensions [if $V(x^2)$ is nonzero for any value of $x^2=s$, then it will have this value for all $\vec{x}$, t such that $\vec{x}^2=s+t^2$; for $|\vec{x}|,t\to\infty$, this region becomes Euclidean close to the light cone]. Since, however, wave packets with no zero-mass components move away from the light cone,[18] there will be decreasingly small overlap with incident wave packets on these tails of the potential for larger relative time displacements. The asymptotic fringes of the potential can therefore be neglected and a wide enough beam can cover it effectively.

The probability that a particle associated asymptotically with wave function $\psi^i_{\rm in}(p)$ will be found after the interaction scattered into the region d^4p around p^μ is given by

$$w(d^4p \leftarrow \psi^i_{\rm in}) = |\psi^{i,\,sc}_{\rm out}(p)|^2 d^4p\,, \tag{2.27}$$

where we include in $\psi^{i,\,\rm sc}_{\rm out}$ only the part of $\psi^i_{\rm out}$ actually scattered. We shall be interested only in the direction of the outgoing momentum and in the energy, and therefore integrate over the magnitude of $\vec{p}$ to get the probability of the particle emerging with energy in dp^0 around p^0, and three-momentum in the solid angle $d\Omega$ around $\vec{p}$,

$$w(d\Omega\, dp^0 \leftarrow \psi^i_{\rm in}) = d\Omega\, dp^0 \int_0^\infty d|\vec{p}|\,|\vec{p}|^2 |\psi^{i,\,\rm sc}_{\rm out}(\vec{p}, p^0)|^2 . \tag{2.28}$$

The total number of observed scatterings into $d\Omega\, dp^0$ is the sum over probabilities for each packet,

$$N_{sc}(d\Omega\, dp^0) = \sum_i w(d\Omega\, dp^0 \leftarrow \psi^i_{\rm in}) = \int d^2\rho \int dx^0\, w(d\Omega\, dp^0 \leftarrow \psi^{\vec{\rho},\,x^0}_{\rm in})\, n_{\rm inc}\,, \tag{2.29}$$

where $n_{\rm inc}$ is the number of packets per unit area and unit time perpendicular to the motion of the wave packet. Since we have assumed the beam to effectively "cover" the potential, the limits can be extended to infinity without changing the value of the integral. For $n_{\rm inc}$ constant, the cross section can then be defined as

$$\sigma(d\Omega\, dp^0 \leftarrow \psi_{\rm in}) = \frac{N_{\rm sc}(d\Omega\, dp^0)}{n_{\rm inc}} = \int d^2\rho \int dx^0\, w(d\Omega\, dp^0 \leftarrow \psi^{\vec{\rho},\,x^0}_{\rm in})\,. \tag{2.30}$$

Note that this definition is given in terms of a number divided by a density; it is equivalent to a rate divided by a flux, if we define the rate as $N_{\rm sc}/\Delta\tau$, and the flux as $n_{\rm inc}/\Delta\tau$, where $\Delta\tau$ is the pulse length in τ. As in the nonrelativistic theory, the beam defines a direction; working in four dimensions, one therefore obtains a three-dimensional cross section.[25] We shall show later that under certain conditions this cross section factors into a spatial part with the usual interpretation of a two-dimensional cross section, and a factor T which describes the extension of the potential in relative time (of the order of the spatial range divided by c).

In the treatment of antiparticles, what happens in the physical world in time is not always identical to the accelerator and detector picture described here in τ. To formulate the problem of antiparticle scattering (on a particle target), we consider the initial state to be an "uncontrolled"[17] state $|\psi_{\rm in}\rangle$ which goes asymptotically, as $\tau \to -\infty$ to the future in t, i.e., $t \to +\infty$, the region of the detector. For $\tau \to +\infty$, this state develops to a *controlled* $|\psi_{\rm out}\rangle$ which describes the incident beam of antiparticles at $t \to -\infty$. The (relative) momentum is timelike for this process, and the physical scattering state is $\Omega_- |\psi_{\rm out}\rangle$. Taking the scalar produce with $\Omega_+ |\psi'_{\rm in}\rangle$, the scattering state to be detected at $\tau \to -\infty$ (going to the essentially plane-wave state $e^{-iK_0\tau}|\psi'_{\rm in}\rangle$ in this limit) results in $S^\dagger$ as the corresponding scattering matrix. The cross section is then obtained by integrating the square of the scattered part of this amplitude over variables in $|\psi'_{\rm in}\rangle$ which are not measured, and summing over wave packets in the accelerator beam. As can be seen from Fig. 1, a simple

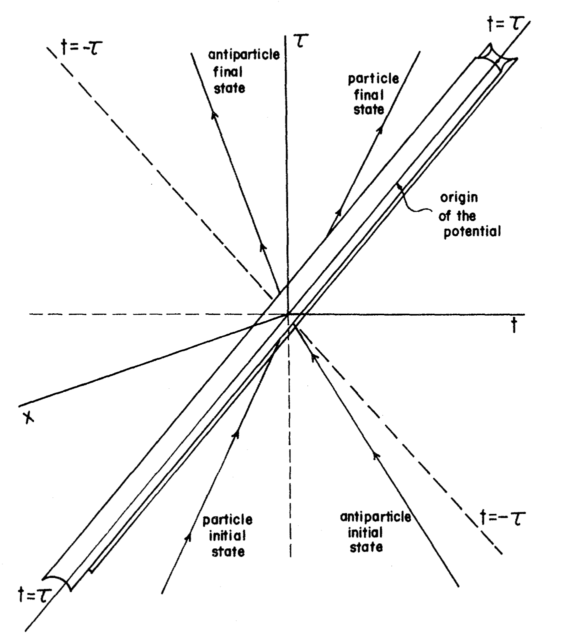

FIG. 1. A t, τ, x diagram of particle and antiparticle scattering on a bounded potential. The difference between the two cases is evident. The form of the invariant potential is starlike (the purely timelike or spacelike part could be suppressed).

transformation of the type $p\to -p$, $t\to -t$ on the particle variables will not put the two cases into physical correspondence.

In the following, we shall describe in detail the kinematical aspects of particle scattering. Analyses of antiparticle scattering, pair creation and annihilation, and consideration of the related question of crossing symmetry, will be given elsewhere.

We shall now complete the computation of (2.30), using Eq. (2.28) for $w(d\Omega\, dp^0 \leftarrow \psi_{\rm in}^i)$ and Eq. (2.26) for $\psi_{\rm in}^i(p)$. To extract the scattered part of the wave function $\psi_{\rm out}^{i,\,\rm sc}$, we note that in

$$\psi_{\rm out}(p)=\int d^4p'\langle p|S|p'\rangle\psi_{\rm in}(p'), \qquad (2.31)$$

the kernel of the S matrix [due to Eq. (2.20)] can always be written as

$$\langle p|S|p'\rangle=\delta^4(p-p') - 2\pi i\delta\left(\frac{p^2}{2M}-\frac{p'^2}{2M}\right)T(p\leftarrow p'), \qquad (2.32)$$

defining the "on-shell" T matrix. The scattered part of $\psi_{\rm out}$ is then

$$\psi_{\rm out}^{\rm sc}(p)=-2\pi i\int d^4p'\delta\left(\frac{p^2}{2M}-\frac{p'^2}{2M}\right)T(p\leftarrow p')\psi_{\rm in}(p'). \qquad (2.33)$$

The expression to be computed is

$$\frac{d\sigma(d\Omega\, dp^0\leftarrow\psi)}{d\Omega\, dp^0}=\int d^2\rho\int dx_0\int_0^\infty d|\vec{p}|\vec{p}^2(2\pi)^2\int d^4p'\delta\left(\frac{p^2}{2M}-\frac{p'^2}{2M}\right)T(p\leftarrow p')\exp(-i\vec{\rho}\cdot\vec{p}'+ix^0p^{0\prime})\psi(p')$$
$$\times\int d^4p''\delta\left(\frac{p^2}{2M}-\frac{p''^2}{2M}\right)T^*(p\leftarrow p'')\exp(i\vec{\rho}\cdot\vec{p}''-ix^0p^{0\prime\prime})\psi^*(p''). \qquad (2.34)$$

Performing the ρ, x^0 integrations, (2.34) becomes

$$\frac{d\sigma}{d\Omega\, dp^0}=(2\pi)^5\int_0^\infty d|\vec{p}|\vec{p}^2\int d^4p'\int d^4p''\delta\left(\frac{p^2}{2M}-\frac{p'^2}{2M}\right)\delta\left(\frac{p'^2}{2M}-\frac{p''^2}{2M}\right)$$
$$\times T(p\leftarrow p')T^*(p\leftarrow p'')\psi(p')\psi^*(p'')\delta^2(p'_\perp-p''_\perp)\delta(p^{0\prime}-p^{0\prime\prime}), \qquad (2.35)$$

where $p'_\perp$ denotes the component of $\vec{p}'$ perpendicular to $\vec{p}^*$. We now perform the integration on d^4p'':

$$\int d^4p''\delta\left(\frac{p'^2}{2M}-\frac{p''^2}{2M}\right)T^*(p\leftarrow p'')\psi^*(p'')\delta^2(p'_\perp-p''_\perp)\delta(p^{0\prime}-p^{0\prime\prime})$$
$$=2M\int d^4p''\delta(p_\perp'^2+p_\parallel'^2-p^{0\prime 2}-p_\perp''^2-p_\parallel''^2+p^{0\prime\prime 2})T^*(p\leftarrow p'')\psi^*(p'')\delta^2(p'_\perp-p''_\perp)\delta(p^{0\prime}-p^{0\prime\prime})$$
$$=\frac{M}{|p'_\parallel|}T^*(p\leftarrow p')\psi^*(p'), \qquad (2.36)$$

where $p'_\parallel$, the component of $\vec{p}'$ parallel to $\vec{p}^*$, appears in the denominator of the last expression because

$$\delta(p_\parallel'^2-p_\parallel''^2)=\frac{1}{2|p'_\parallel|}[\delta(|p'_\parallel|-|p''_\parallel|)+\delta(|p'_\parallel|+|p''_\parallel|)].$$

The second term does not contribute if the wave packets are narrow enough around p^*. Substituting Eq. (2.36) into (2.35), we obtain

$$\frac{d\sigma}{d\Omega\, dp^0}=(2\pi)^5 2M^2\times\int_0^\infty d|\vec{p}|\vec{p}^2\times\int d^4p'\frac{1}{|p'_\parallel|}\delta(p^2-p'^2)|T(p\leftarrow p')|^2|\psi(p')|^2. \qquad (2.37)$$

The integral over $|\vec{p}|$ can now be performed to obtain

$$\frac{d\sigma}{d\Omega\, dp^0}(d\Omega\, dp^0 \leftarrow \psi)$$

$$= (2\pi)^5 M^2 \int d^4p' \frac{q'}{|p'_{\parallel}|}\,|T((q',\Omega,p^0) \leftarrow p')|^2\,|\psi(p')|^2\,, \tag{2.38}$$

where $q' = (p^{02} + p'^2)^{1/2}$, and we denote the value of p selected by the δ function in (2.37) as $p = (q', \Omega, p^0)$. If $\psi(p)$ is different from zero only in a small region around p^*, and in this region the T-matrix element does not change significantly in magnitude, the integral can be well approximated by

$$\frac{d\sigma(d\Omega\, dp^0 \leftarrow p^*)}{d\Omega\, dp^0} = (2\pi)^5 M^2 \frac{q^*}{|\vec{p}^*|}\,|T((q^*,\Omega,p^0) \leftarrow p^*)|^2\,, \tag{2.39}$$

where we have replaced the designation of the wave packet by its average momentum p^*, since the result does not depend on its shape, and $q^* = (p^{02} + p^{*2})^{1/2}$. Equivalently, one may write

$$\frac{d\sigma(d\Omega\, dp^0 \leftarrow p^*)}{d\Omega\, dp^0} = (2\pi)^5 M^2 \frac{|\vec{p}|}{|\vec{p}^*|}\,|T(p \leftarrow p^*)|^2 \bigg|_{p^2 = p^{*2}}\,. \tag{2.40}$$

As in nonrelativistic scattering theory, an optical theorem can be proved as a direct consequence of the unitarity of S. Starting from the relation $S^\dagger S = 1$ and expressing S in terms of T by (2.32), one obtains

$$i\delta\left(\frac{p^2}{2M} - \frac{p'^2}{2M}\right)[T(p' \leftarrow p) - T^*(p \leftarrow p')]$$

$$= 2\pi \int d^4p''\,\delta\left(\frac{p'^2}{2M} - \frac{p''^2}{2M}\right)\delta\left(\frac{p^2}{2M} - \frac{p''^2}{2M}\right) \times T^*(p'' \leftarrow p')T(p'' \leftarrow p)\,.$$

Factoring out a common δ function and equating p to p' results in

$$\mathrm{Im}\,T(p \leftarrow p) = -\pi \int d^4p''\,\delta\left(\frac{p''^2}{2M} - \frac{p^2}{2M}\right)|T(p'' \leftarrow p)|^2\,. \tag{2.41}$$

Carrying out the integral over $|\vec{p}''|$ and comparing the result with the expression for the total cross section

$$\sigma_{\mathrm{tot}}(p^*) = \int d\Omega\, dp^0 \frac{d\sigma(d\Omega\, dp^0 \leftarrow p^*)}{d\Omega\, dp^0}$$

$$= (2\pi)^5 M^2 \frac{1}{|\vec{p}^*|} \int d\Omega\, dp^0 q^*\,|T((q^*,\Omega,p^*) \leftarrow p^*)|^2\,, \tag{2.42}$$

we obtain the optical theorem

$$\mathrm{Im}\,T(p \leftarrow p) = -\frac{|\vec{p}|}{32\pi^4 M}\,\sigma_{\mathrm{tot}}(p)\,. \tag{2.43}$$

We conclude this section with a remark about the nonrelativistic limit. In the nonrelativistic limit[15] (see also Pearle[3]), the potential $V(x)$ becomes effectively independent of t, and hence $[p^0, S] = 0$. The T matrix is therefore proportional to an energy δ function. We therefore write

$$T(p \leftarrow p^*) = \delta(p^0 - p^{0*})T_{\mathrm{NR}}(\vec{p} \leftarrow \vec{p}^*)\,. \tag{2.44}$$

Furthermore, $p^0 = p^{0*}$ implies that $q^* = |\vec{p}^*|$, so that

$$\frac{d\sigma(d\Omega \leftarrow \vec{p}^*)}{d\Omega} = \int dp^0 \frac{d\sigma(d\Omega\, dp^0 \leftarrow p^*)}{d\Omega\, dp^0}$$

$$\sim [2\pi\delta(p^{0*} - p^{0*})](2\pi)^4 M^2\,|T_{\mathrm{NR}}(\vec{p} \leftarrow \vec{p}^*)|^2\,. \tag{2.45}$$

The infinite factor in front of the nonrelativistic cross section corresponds to the integration over x^0 required to sample the potential during its spread in t. Since the potential becomes constant in t in the nonrelativistic limit, the series of experiments for different values of t yield the same result, and the three-dimensional cross section obtained in the asymptotic result (2.45) is just proportional to the time width of this sampling pulse of wave packets. The relevant physical quantity is this cross section divided by the pulse width, a two-dimensional cross section

$$\frac{d\sigma_{\mathrm{NR}}(d\Omega \leftarrow \vec{p}^*)}{d\Omega} \sim (2\pi)^4 M^2\,|T_{\mathrm{NR}}(\vec{p} \leftarrow \vec{p}^*)|^2\,, \tag{2.46}$$

coinciding with the usual nonrelativistic cross section. Using the expression (2.45) in the optical theorem (2.43), one finds that both σ_{tot} and $\mathrm{Im}\,T(p \leftarrow p)$ contain the same factor of time. The optical theorem therefore also goes over to the usual form for nonrelativistic scattering theory in this limit.

III. PERTURBATION EXPANSION AND FEYNMAN RULES

The most powerful and general technique for the calculation of S-matrix elements can be found in a perturbation expansion in successive powers of the potential. This allows for the calculation of approximate results, the use of the pictorial Feynman diagram method, and a way of comparing the structure of the theory with that of RQFT. We shall develop this technique for both direct action type of interactions and for scattering on an external electromagnetic field by interaction through a vector potential.

For a dynamical evolution operator of the form

$K = K_0 + V$, we define the interaction picture

$$|\psi_\tau\rangle_I = e^{iK_0\tau}|\psi_\tau\rangle\,, \tag{3.1}$$

so that Eq. (2.13) becomes

$$i\frac{\partial}{\partial\tau}|\psi_\tau\rangle_I = V_I(\tau)|\psi_\tau\rangle_I\,, \tag{3.2}$$

where

$$V_I(\tau) = e^{iK_0\tau}Ve^{-iK_0\tau}\,. \tag{3.3}$$

According to the asymptotic condition (2.16),

$$\lim_{\tau\to+\infty(-\infty)}|\psi_\tau\rangle_I = |\psi_{\text{out (in)}}\rangle\,, \tag{3.4}$$

and the operator $U_I(\tau,\tau_0)$ governing the evolution of $|\psi_\tau\rangle_I$ satisfies the differential equation

$$i\frac{d}{d\tau}U_I(\tau,\tau_0) = V_I(\tau)U_I(\tau,\tau_0)\,, \tag{3.5}$$

which, with the initial condition $U_I(\tau_0,\tau_0)=1$, is equivalent to the integral equation

$$U_I(\tau,\tau_0) = 1 - i\int_{\tau_0}^{\tau}d\tau' V_I(\tau')U_I(\tau',\tau_0)\,. \tag{3.6}$$

For small V, one may reasonably hope that the iterative expansion

$$U_I(\tau,\tau_0) = 1 - i\int_{\tau_0}^{\tau}d\tau' V_I(\tau') + (-i)^2\int_{\tau_0}^{\tau}d\tau'\int_{\tau_0}^{\tau'}d\tau'' V_I(\tau')V_I(\tau'') + \cdots \tag{3.7}$$

will converge. The S operator is defined by

$$S = \lim_{\substack{\tau\to+\infty\\ \tau_0\to-\infty}} U_I(\tau,\tau_0)\,. \tag{3.8}$$

The nth-order contribution to the matrix element of S is

$$\langle\phi|S^{(n)}|\psi\rangle = (-i)^n\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\cdots\int_{-\infty}^{\tau_{n-1}}d\tau_n\langle\phi|e^{iK_0\tau_1}Ve^{-iK_0(\tau_1-\tau_2)}Ve^{-iK_0(\tau_2-\tau_3)}V\cdots e^{-iK_0(\tau_{n-1}-\tau_n)}Ve^{-iK_0\tau_n}|\psi\rangle\,. \tag{3.9}$$

Taking for $|\psi\rangle$ an element of the dense set of states for which the inequality (2.18) is valid assures that the last integration, on τ_n, is absolutely convergent. Hence, we may insert a factor $e^{\epsilon\tau_n}$, with the limit $\epsilon\to 0$ implied, and replace $|\psi\rangle$ by an improper momentum eigenvector. Inserting complete sets of such intermediate states, we obtain

$$\begin{aligned}\langle p'|S^{(n)}|p\rangle = (-i)^n\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\cdots\int_{-\infty}^{\tau_{n-1}}d\tau_n\int d^4q_1\cdots d^4q_{n-1}\\ \times e^{ik(p')\tau_1}\langle p'|V|q_1\rangle e^{-ik(q_1)\tau_1}e^{ik(q_1)\tau_2}\langle q_1|V|q_2\rangle\\ \times e^{-ik(q_2)\tau_2}\cdots e^{-ik(q_{n-1})\tau_{n-1}}e^{ik(q_{n-1})\tau_n}e^{\epsilon\tau_n}\langle q_{n-1}|V|p\rangle e^{-ik(p)\tau_n}\,,\end{aligned} \tag{3.10}$$

where $k(q)\equiv q^2/2M$. Carrying out the $d\tau_n$ integral, one obtains

$$\int_{-\infty}^{\tau_{n-1}}d\tau_n e^{ik(q_{n-1})\tau_n}e^{\epsilon\tau_n}e^{-ik(p)\tau_n} = \frac{ie^{\{i[k(q_{n-1})-k(p)-i\epsilon]\tau_{n-1}\}}}{k(p)-k(q_{n-1})+i\epsilon}\,. \tag{3.11}$$

The factor $e^{\epsilon\tau_{n-1}}$ is therefore available for the next integration. Note also that $k(q_{n-1})$ cancels and is replaced by $k(q_{n-2})$. This process continues to the last integration, which results in a $k(p)$-conserving δ function. The matrix element of S is therefore

$$\begin{aligned}\langle p'|S^{(n)}|p\rangle = -2\pi i\delta(k(p)-k(p'))\int d^4q_1\cdots d^4q_{n-1}\langle p'|V|q_1\rangle\,.\\ \times\langle q_1|V|q_2\rangle\cdots\langle q_{n-1}|V|p\rangle\frac{1}{k(p)-k(q_1)+i\epsilon}\frac{1}{k(p)-k(q_2)+i\epsilon}\cdots\frac{1}{k(p)-k(q_{n-1})+i\epsilon}\,,\end{aligned} \tag{3.12}$$

or, in a more compact form,

$$\langle p'|S^{(n)}|p\rangle = -2\pi i\delta(k(p)-k(p'))\langle p'|V(G_0V)^{n-1}|p\rangle\,, \tag{3.13}$$

where

$$G_0 \equiv G_0(k(p)) = \frac{1}{k(p)-K_0+i\epsilon} \tag{3.14}$$

has the form of a Feynman propagator for a spin-0 particle of mass squared equal to $-2Mk(p)$. The propagator (3.14) was also obtained by Feynman[3] in a proper time formalism. The series

$$\langle p|S|p'\rangle = \delta^4(p-p') + \sum_{n=1}^{\infty} \langle p|S^{(n)}|p'\rangle \qquad (3.15)$$

can be represented as a series of Feynman diagrams with the following rules (see Fig. 2).

(a) Draw a diagram of n vertices in a column, with momentum p entering from below and p' leaving above.

(b) For each vertex with momentum q_i entering and q_{i-1} leaving, write down a factor $\tilde{V}(q_i - q_{i-1}) = \langle q_i|V|q_{i-1}\rangle$ [assuming V a local potential of the form $V(x)$]. There is four-momentum conservation at each vertex for the two particles involved, since $x = x_1 - x_2$ and V is translation invariant.

(c) For each internal line carrying momentum q_i there is a propagator $2M(p^2 - q_i^2 + i\epsilon)^{-1}$ and integration on d^4q_i.

(d) There is an overall factor $-2\pi i\delta(p^2/2M - p'^2/2M)$.

Note that the explicit appearance of M in Eq. (3.12) can be removed by redefining the potential as $\overline{V}(q) = 2M\tilde{V}(q)$; it then becomes part of the coupling constant. This is, of course, a result of factoring $(2M)^{-1}$ from the definition of K, resulting only in a change of scale of τ which changes nothing in the physics of the potential problem.

We have so far been considering the scattering

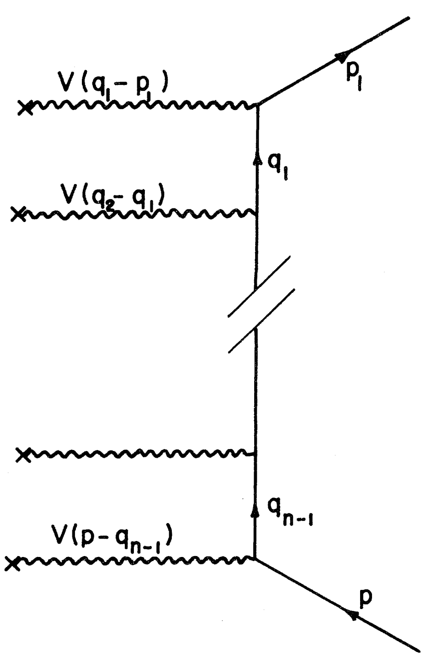

FIG. 2. The nth-order Feynman diagram for a particle in an external potential.

of two particles with direct interaction, but the S-matrix expansion applies equally well to the scattering of a single particle in an external potential. In this case, the δ function in (3.12) corresponds to the conservation of m^2 of the particle, and one can see in an explicit way the possibility of pair annihilation and creation in the scattering process. An antiparticle is characterized by a negative sign for the 4th component of its four-momentum, so that the corresponding particle line moves backwards in t as τ goes forward, as in the usual Feynman diagram. Consider, for example, the first-order term in the S matrix:

$$\langle p|S^{(1)}|p'\rangle = -2\pi i\delta(m^2 - m'^2)\overline{V}(p-p') . \qquad (3.16)$$

This expression contains the following four processes differing only by the signs of p^0, $p^{0\prime}$ or, equivalently, according to our convention, m, m': (i) particle scattering $m = m' > 0$, (ii) antiparticle scattering $m = m' < 0$, (iii) pair creation $-m = m' > 0$, (iv) pair annihilation $m = -m' > 0$.

From (3.16), one may also see that not all potentials can be the cause of pair creation or annihilation in first order. The condition is that $V(q)$ have nonvanishing value for q such that $-q^2 \geq 4m^2$ [choosing $p = (0,0,0,m)$, $p' = (\vec{p}', E')$ with $E' < 0$, one has $(p-p')^2 = -2m^2 + 2mE' = -2m^2 - 2m(m^2 + \vec{p}'^2)^{1/2} \leq 4m^2$]. Higher-order diagrams have similar characteristics. Each can be decomposed into four terms representing higher-order corrections to the first-order approximations to the four processes listed above. A given potential may create pairs in the nth order term if $\overline{V}(q) \neq 0$ for q such that

$$-\Big(\sum_n q_n\Big)^2 \geq 4m^2 .$$

We now turn to a study of perturbation theory for a particle in an external electromagnetic field.

The K operator, in accordance with Eq. (2.9), is taken to be

$$K = \frac{p^2}{2M} - \frac{e}{M}(p_\mu A^\mu(x) + A_\mu(x)p^\mu) + \frac{e^2}{2M}A_\mu(x)A^\mu(x) , \qquad (3.17)$$

and hence the potential operator V is given by

$$V = -\frac{e}{2M}(p\cdot A + A\cdot p) + \frac{e^2}{2M}A^2 . \qquad (3.18)$$

Its matrix element is

$$2M\langle p|V|q\rangle = \langle p|\overline{V}|q\rangle = -e(p_\mu + q_\mu)\tilde{A}^\mu(p-q) + e^2\tilde{A}^2(p-q) , \qquad (3.19)$$

where

$$\tilde{A}_\mu(p-q)=(2\pi)^{-4}\int d^4x\, e^{i(p-q)x}A_\mu(x),$$
$$\tilde{A}^2(p-q)=\int d^4k\tilde{A}_\mu(p-k)\tilde{A}^\mu(k-q). \quad (3.20)$$

The expression for $\langle p'|S^{(n)}|p\rangle$ is then a sum of terms of orders in e ranging from e^n to e^{2n}. Each term is represented by a diagram with s single vertices and d double vertices such that $s+d=n$, while $s+2d=r$ gives the order of the diagram in e. All vertices are arranged in a column of increasing τ, and the Feynman rules are then as follows.

(a) A factor $-e(q_\mu+q'_\mu)\tilde{A}_\mu(k)\delta^4(q'-q-k)$ for each single vertex where q_μ, q'_μ, and k_μ are the momenta of the incoming particle line, outgoing particle line, and external potential line respectively.

(b) A factor $e^2\tilde{A}_\mu(k)\tilde{A}_\mu(k')\delta^4(q'-q-k-k')$ for each double vertex where $q_\mu, q'_\mu, k_\mu, k'_\mu$ are the momenta of the incoming particle line, outgoing particle line, and the two potential lines respectively.

(c) A factor $-(q^2+m^2-i\epsilon)^{-1}$ for each internal particle line carrying momentum q, where m^2 is the mass squared of the particle line entering the diagrams from below ($m^2=-p^2$).

(d) An overall factor of $-2\pi i\delta(p^2-p'^2)$, where p and p' are the momenta entering the diagram from below and leaving it from above, respectively.

These rules closely resemble those for a spinless particle in an external vector potential in QED, as given explicitly by Rohrlich.[26]

IV. COMPARISON WITH PERTURBATION EXPANSIONS IN RQFT AND QCHD

The group of change of scale of τ is a symmetry group for the evolution in all cases of a single particle in an external potential, as originally pointed out by Broyles and Pearle.[3] This symmetry is broken only when the system is in nontrivial interaction with another system. Then, there are at least two mutually exclusive choices of initial conditions,[4] one of which makes the proper time of both systems equal to τ, while the other makes the proper time of the composite system equal to τ. The first choice seems to be natural in the case of a scattering system,[4] and the second in the case of a composite system with internal motion.[21] This has also been noticed by Takabayasi,[27] who uses the term "gauge fixing" for the choice of scale of τ. We shall choose the initial conditions (2.5) for each particle, which imply that asymptotically, for $\tau\to-\infty$, $-p_1^2\equiv m_1^2=M_1^2$, and $-p_2^2\equiv m_2^2=M_2^2$. In this case $P^2\neq M^2$, and the center-of-mass proper time does not coincide with τ.

When one considers systems of two or more particles, it is possible to choose a single τ to describe the evolution, as we have done,[4] or to use a multiple τ formalism. Feynman[3] proposed the use of a separate τ for each particle, and Droz-Vincent has recently formulated the problem of several particles in interaction in this framework. Horwitz and Rohrlich[5,6] have found this approach appropriate for the development of quantum constraint Hamiltonian dynamics (QCHD). We shall maintain the single-τ approach in our development of a perturbation expansion for the two-body problem which makes explicit the two-body nature of the system. In this form, the extension to N-body systems is straightforward. We shall then study a modification of this perturbation expansion which leads to a structure more closely analogous to RQFT, and obtain from this the equations of QCHD. One finds that, from the point of view of the single-τ theory, QCHD emerges in the approximation that each particle can be considered as moving in a potential created by the others.

A direct application of the method of Sec. III, using the form (2.10) for K in place of the (reduced motion) form (2.8) results in a perturbation expansion for the S matrix for which the nth order term is

$$\langle p_1p_2'|S^{(n)}|p_1p_2\rangle=\langle P'p'|S^{(n)}|Pp\rangle=\delta^4(P-P')(-2\pi i)\,\delta\left(\frac{p^2}{2m}-\frac{p'^2}{2m}\right)$$
$$\times\int d^4q_1\cdots d^4q_{n-1}\langle p'|V|q_1\rangle\cdots\langle q_{n-1}|V|p\rangle\frac{1}{(p^2-q_1^2)/2m+i\epsilon}\cdots\frac{1}{(p^2-q_{n-1}^2)/2m+i\epsilon}\ . \quad (4.1)$$

There are $n-1$ particle propagators in the nth order term, as compared with $2(n-1)$ in the usual field theory expressions. To see the difference in structure clearly, we study explicitly the second-order term. From Eq. (3.10), we have

$$\langle p_1'p_2'|S^{(2)}|p_1p_2\rangle=-\delta^4(P-P')\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\int d^4q\,e^{i(p'^2/2m)\tau_1}\langle p'|V|q\rangle e^{-i(q^2/2m)\tau_1}e^{i(q^2/2m)\tau_2}\langle q|V|p\rangle$$
$$\times e^{-i(p^2/2m+i\epsilon)\tau_2} \quad (4.2)$$

which, using the identities

$$\frac{q_1^{\,2}}{2M_1}+\frac{q_2^{\,2}}{2M_2}=\frac{q^2}{2m}+\frac{Q^2}{2M},$$
$$\int d^4q_1\int d^4q_2=\int d^4Q\int d^4q\,, \tag{4.3}$$
$$\langle p_1p_2|V|q_1q_2\rangle=\delta^4(P-Q)\langle p|V|q\rangle$$

is equivalent to

$$\langle p_1'p_2'|S^{(2)}|p_1p_2\rangle=-\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\int d^4q_1\int d^4q_2\exp\left[i\left(\frac{p_1'^2}{2M_1}+\frac{p_2'^2}{2M_2}\right)\tau_1\right]$$
$$\times\langle p_1'p_2'|V|q_1q_2\rangle\exp\left[-i\left(\frac{q_1^{\,2}}{2M_1}+\frac{q_2^{\,2}}{2M_2}\right)(\tau_1-\tau_2)\right]\langle q_1q_2|V|p_1p_2\rangle$$
$$\times\exp\left[-i\left(\frac{p_1^{\,2}}{2M_1}+\frac{p_2^{\,2}}{2M_2}+i\epsilon\right)\tau_2\right]. \tag{4.4}$$

Carrying out the τ integrations in (4.4) one obtains

$$\langle p_1'p_2'|S^{(2)}|p_1p_2\rangle=-2\pi i\delta\left(\frac{p_1^{\,2}-p_1'^2}{2M_1}+\frac{p_2^{\,2}-p_2'^2}{2M_2}\right)\int d^4q_1d^4q_2\langle p_1'p_2'|V|q_1q_2\rangle$$
$$\times\langle q_1q_2|V|p_1p_2\rangle\frac{1}{(p_1^{\,2}-q_1^{\,2})/2M_1+(p_2^{\,2}-q_2^{\,2})/2M_2+i\epsilon}. \tag{4.5}$$

This result corresponds to the diagram of Fig. 3(a), in terms of the relative motion coordinates [i.e., the formula for $S^{(2)}$ obtained from Eq. (4.1)]; the corresponding diagram in quantum field theory would have a structure of the type shown in Fig. 3(b). We have assumed in these figures that the (phenomenological) two-body amplitudes $\langle p_1'p_2'|V|p_1p_2\rangle$ occur in field theory as two vertices connected by a propagator, and represented this structure by a wavy line.

The generalization of the result (4.5) to the case of N particles is evident. For any number of particles in the intermediate state, $S^{(2)}$ will contain only a single propagator with a sum of quadratic terms in the denominator.

In the derivation of the expansion containing the term (4.4), the development of the entire system was assumed to be governed by a single parameter τ. A generalization of Eq. (4.4) to

$$-\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\int_{-\infty}^{\infty}d\sigma_1\int_{-\infty}^{\sigma_1}d\sigma_2\int d^4q_1d^4q_2\exp\left[i\left(\frac{p_1^{\,2}}{2M_1}\tau_1+\frac{p_2^{\,2}}{2M_2}\sigma_1\right)\right]\langle p_1'p_2'|V|q_1q_2\rangle$$
$$\times\exp\left\{-i\left[\frac{q_1^{\,2}}{2M_1}(\tau_1-\tau_2)+\frac{q_2^{\,2}}{2M_2}(\sigma_1-\sigma_2)\right]\right\}\langle q_1q_2|V|p_1p_2\rangle$$
$$\times\exp\left[-i\left(\frac{p_1^{\,2}}{2M_1}\tau_2+\frac{p_2^{\,2}}{2M_2}\sigma_2\right)\right]e^{\epsilon_1\tau_1}e^{\epsilon_2\sigma_2}, \tag{4.6}$$

in which a different τ is associated with each particle, will yield, upon integration over τ_1, τ_2, σ_1, σ_2

$$4\pi^2\delta\left(\frac{p_1^{\,2}-p_1'^2}{2M_1}\right)\delta\left(\frac{p_2^{\,2}-p_2'^2}{2M_2}\right)\int d^4q_1d^4q_2\langle p_1'p_2'|V|q_1q_2\rangle$$
$$\times\langle q_1q_2|V|p_1p_2\rangle\frac{1}{(p_1^{\,2}-q_1^{\,2})/2M_1+i\epsilon_1}\frac{1}{(p_2^{\,2}-q_2^{\,2})/2M_2+i\epsilon_2}. \tag{4.7}$$

This expression coincides with the second-order Feynman diagram of Fig. 3(b) with momentum conservation assured at each vertex by the form of V, and conservation of individual masses by the two δ functions.

We remark that the expression (4.7) can also be obtained by "gluing" together two Feynman diagrams of the type shown in Fig. 2 for scattering in an external field, with appropriate δ functions equating their momenta. For the second-order term we would obtain the diagram of Fig. 4(a) and its crossed counterpart, Fig. 4(b). The second-order composite matrix element for Fig. 4(a) is

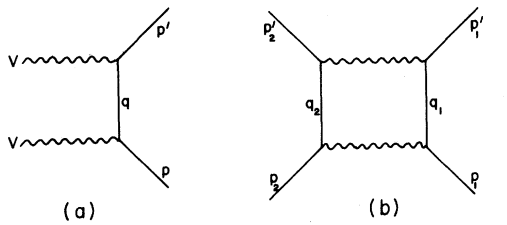

FIG. 3. A second-order diagram in (a) RQM and in (b) RQFT.

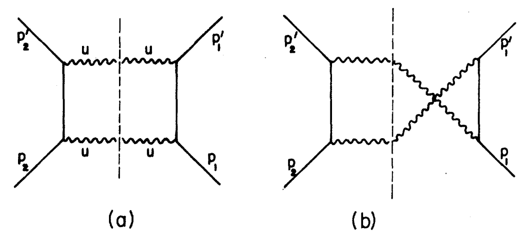

FIG. 4. A diagram and its crossed counterpart.

$$\langle p_1'p_2'|S^{(2)}|p_1p_2\rangle = (2\pi)^2\delta\left(\frac{p_1^2-p_1'^2}{2M_1}\right)\delta\left(\frac{p_2^2-p_2'^2}{2M_2}\right)$$
$$\times\int d^4q_1\langle p_1'|u_1|q_1\rangle\langle q_1|u_1|p_1\rangle\frac{1}{(p_1^2-q_1^2)/2M_1+i\epsilon_1}$$
$$\times\int d^4q_2\langle p_2'|u_2|q_2\rangle$$
$$\times\langle q_2|u_2|p_2\rangle\frac{1}{(p_2^2-q_2^2/2M_2)+i\epsilon_2}\delta^4(p_1'-q_1+p_2'-q_2)\delta^4(q_1-p_1+q_2-p_2)\,. \tag{4.8}$$

With the definition

$$\langle p_1'p_2'|V|p_1p_2\rangle = \delta^4(p_1'+p_2'-p_1-p_2)\langle p_1'|u_1|p_1\rangle\langle p_2'|u_2|p_2\rangle\,, \tag{4.9}$$

the expression (4.8) becomes identical to (4.7). Note that the potential V is given in terms of the single vertex potentials u by

$$V(q)=u_1(q)u_2(-q)\,. \tag{4.10}$$

In the crossed diagram of Fig. 4(b), the δ functions also combine in the proper way to ensure agreement with the corresponding field theory expression [crossed diagrams arise in solutions of Eq. (4.12) containing advanced limits in the integrals[6]]. We infer from this discussion of "gluing" that expressions of the form (4.7) can be interpreted, from the point of view of the single-τ formalism, in terms of each particle moving in the field of the other as if it were an external field. The examination of the differential equations giving rise to a perturbation expansion of the form (4.6) leads to a similar conclusion.

The interaction-picture wave function, in the generalization corresponding to Eq. (4.6), is of the form

$$|\psi_{\tau\sigma}\rangle_I = U_I(\tau\sigma,\tau_0\sigma_0)|\psi_{\tau_0\sigma_0}\rangle_I\,, \tag{4.11}$$

where the second-order term in the perturbation expansion for U_I is of the form (4.6) with finite upper (σ,τ) and lower (σ_0,τ_0) limits. Differentiating with respect to both variables, we find the equation

$$-\frac{\partial^2}{\partial\sigma\,\partial\tau}|\psi_{\tau\sigma}\rangle_I = V_I(\tau,\sigma)|\psi_{\tau,\sigma}\rangle_I\,, \tag{4.12}$$

where

$$V_I(\tau,\sigma)=e^{iK_1^0\tau}e^{iK_2^0\sigma}Ve^{-iK_2^0\sigma}e^{-iK_1^0\tau}\,. \tag{4.13}$$

For the Schrödinger picture wave function, defined by

$$|\psi_{\tau\sigma}\rangle_S = e^{-iK_1^0\tau}e^{-iK_2^0\sigma}|\psi_{\tau\sigma}\rangle_I\,, \tag{4.14}$$

Eq. (4.12) becomes

$$\left(i\frac{\partial}{\partial\tau}-K_1^0\right)\left(i\frac{\partial}{\partial\sigma}-K_2^0\right)|\psi_{\tau\sigma}\rangle_S = V|\psi_{\tau\sigma}\rangle_S\,, \tag{4.15}$$

which is clearly of noncanonical form.

Equation (4.15) has the appearance of the composition of two separate differential equations, one for the motion of each particle in the effective field of the other, of the form

$$\left(i\frac{\partial}{\partial\tau}-K_1^0\right)|\psi_{\tau\sigma}\rangle_S = \Phi_1|\psi_{\tau\sigma}\rangle_S\,,$$
$$\left(i\frac{\partial}{\partial\sigma}-K_2^0\right)|\psi_{\tau\sigma}\rangle_S = \Phi_2|\psi_{\tau\sigma}\rangle_S\,. \tag{4.16}$$

The condition that these equations be integrable, and that there can exist a V independent of the order of application of the operators on the left side of Eq. (4.15), is that

$$[K_1,K_2]|\psi_{\tau\sigma}\rangle_S = 0\,, \tag{4.17}$$

where

$$K_1 = K_1^0 + \Phi_1, \quad K_2 = K_2^0 + \Phi_2 . \tag{4.18}$$

If these conditions are satisfied, then

$$\begin{aligned} V|\psi_{\tau\sigma}\rangle_S &= ([\Phi_1, K_2] + \Phi_1\Phi_2)|\psi_{\tau\sigma}\rangle_S \\ &= ([\Phi_2, K_1] + \Phi_2\Phi_1)|\psi_{\tau\sigma}\rangle_S . \end{aligned} \tag{4.19}$$

These are precisely the equations of QCHD,[5,6] where K_1, K_2 are the constrainors discussed in Sec. I.[28]

In the course of our modification of the perturbation expansion for which the second-order term is given in Eq. (4.4), it was necessary to destroy the correlation in τ between different particles. Such a situation might be imagined to occur when the forces between particles are mediated by dynamical emission and absorption processes, so that the exact correlation between x_1 and x_2 in $V(x_1, x_2)$ may be altered by dynamical effects[29]. A total loss of correlation is perhaps, however, an idealization at the opposite extreme of a single-τ formalization. It has, in fact, been argued by many authors,[9] that the parametric description of a world line requires the selection of some curve in the space of many τ's (the imposition of a correlation), and this selection is often called a "gauge" condition. It was shown by Horwitz and Rohrlich[5] that with a wide choice of such conditions (satisfying a positivity requirement), the S matrix for scattering processes is independent of this choice. They used an asymptotic condition of the form

$$|\psi_{\text{out}}\rangle = \lim_{[\tau]\to\infty} |\psi[\tau]\rangle_I , \tag{4.20}$$

where $[\tau]$ is the set of all of the τ's. The detailed description of the development of a state, however, would require some choice of gauge which could replace $|\psi[\tau]\rangle_S$ by a function of a single τ, and thus describe, for example, the Ehrenfest motion of a wave packet along some world line in space-time.

Considerations from quantum field theory indeed offer some understanding for the difference in the structure of Eqs. (4.4) and (4.6), and hence of the difference between RQM and QCHD from the point of view of the canonical single τ formalism. Denote by Δ the τ difference of the points connected by the potential; this quantity can be considered as conjugate to the mass parameter κ^2 appearing in the corresponding field theory propagator. The $\Delta=0$ potential in Eq. (2.10), for example, would be represented by a propagator with all values of κ^2; this is to be contrasted with the "naive" field theory description of the interaction as mediated by a particle of a distinct mass. An interaction of the latter type would connect points of all possible Δ.

In the Källén-Lehmann spectral representation,[30] the dressed propagators are not of a distinct κ^2 value. For example, if the interaction is mediated by a meson of mass μ, the dressed propagator for such a meson line carrying momentum q is

$$G(q^2) = \int_0^\infty d\kappa^2 \rho(\kappa^2) \frac{1}{q^2 - \kappa^2 + i\epsilon} , \tag{4.21}$$

where $\int_0^\infty \rho(\kappa^2) d\kappa^2 = 1$. The weight function $\rho(\kappa^2)$ contains a δ function at the mass of the single particle and a continuum from $(2\mu)^2$ on. Thus G may be written as

$$G(q^2) = \frac{Z_3}{q^2 - \mu^2 + i\epsilon} + \int_{(2\mu)^2}^\infty d\kappa^2 \frac{\sigma(\kappa^2)}{q^2 - \kappa^2 + i\epsilon} , \tag{4.22}$$

where $0 \leq Z_3 \leq 1$. This expression is intermediate between the single-κ^2 field theory and the single-τ potential theory, and would correspond to an effective potential acting over a range of Δ's.

V. THE ELECTROMAGNETIC INTERACTION

A particularly interesting system in which particles are linked by a interaction which is not described by an equal-τ potential is that of a system of charged particles with electromagnetic interaction. The basic dynamical equations for charged particles in electromagnetic interaction with each other are Eq. (2.13), where

$$K = \sum_{i=1}^N \frac{[p_i^\mu - e_i A^\mu(x_i)][p_{i\mu} - e_i A_\mu(x_i)]}{2M_i} \tag{5.1}$$

and the Maxwell equations

$$\partial_\nu F^{\mu\nu}(x) = J^\mu(x) , \tag{5.2}$$

which determine, up to gauge transformations, the field $A_\mu(x)$. In the absence of external sources, the charged particles themselves form the current $J^\mu(x)$, and it will be the first task of this section to determine how this is done.

The wave function $\psi_\tau(x)$ corresponds to the probability distribution for finding a particle at a point in space-time, at a particular value of τ. If the particle is charged, its motion will be associated with a current. One can easily show, however, that this current cannot be determined from a knowledge of ψ_τ at just one τ. Consider, for example, the following classical argument. Suppose that, corresponding to the information contained in $\psi_\tau(x)$, a charged particle occurs at a point $(\vec{x}_0, t_0)$ in space-time. The charge density for this event is $e\delta^3(\vec{x} - \vec{x}_0)\delta(t - t_0)$. Since the Maxwell equations (5.2) require a divergenceless current, the space part of the current associated with this event is determined by

$$\nabla \cdot \vec{J} + \frac{\partial}{\partial t}[e\delta^3(\vec{x} - \vec{x}_0)\delta(t - t_0)] = 0. \tag{5.3}$$

Up to an undetermined curl, the solution of Eq. (5.3) is

$$\vec{J}(x) = -\frac{e}{4\pi}\frac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|^2}\,\delta'(t-t_0)\,. \qquad (5.4)$$

Hence the space part of the current at a point $\vec{x}$ is zero except for a short time before t_0, when a surge of current passes inward from infinity, and a short time after t_0 when it goes out again. Placing two charged events infinitesimally close to each other does not require a transport of charge in between, but only before the first and after the second (the charge can just be transferred from one event to the other). An infinite sequence of charged events, placed along some world line with no finite end points is therefore required for the construction of a conserved current which can be a source for the Maxwell equations, if we do not wish to accept the transport of charges to and from infinity which is unrelated to the motions of the particles that we can describe with our dynamical equations. Let us consider the construction of such a current for a classical system of N-charged particles. The charge density is

$$\rho_\tau(x) = \sum_{i=1}^N e_i\,\delta^4(x - x_i(\tau))\,, \qquad (5.5)$$

where $x_i(\tau)$ is the function describing the world line of the ith particle. Then,

$$\frac{\partial\rho_\tau(x)}{\partial\tau} = \sum_{i=1}^N\left[-e_i\frac{\partial}{\partial x^\mu}\delta^4(x-x_i(\tau))\frac{dx_i^\mu}{d\tau}(\tau)\right]. \qquad (5.6)$$

For an evolution function of the form (5.1), it follows that

$$\frac{dx_i^\mu}{d\tau} = \frac{\partial K}{\partial p_{i\mu}} = \frac{p_i^\mu - e_iA^\mu(x_i)}{M_i}\,, \qquad (5.7)$$

so that Eq. (5.6) becomes

$$\frac{\partial\rho_\tau}{\partial\tau} = -\partial_\mu j_\tau^\mu(x)\,, \qquad (5.8)$$

where

$$j_\tau^\mu(x) = \sum_{i=1}^N e_i\frac{p_i^\mu - e_iA^\mu(x_i)}{M_i}\,\delta^4(x-x_i(\tau))\,. \qquad (5.9)$$

Equation (5.8) shows that the "instantaneous" current (5.9) is not conserved. Integrating both sides of Eq. (5.8) with respect to τ, however, to obtain the current associated with the whole world line, we obtain

$$-\partial_\mu J^\mu(x) = \rho_\infty(x) - \rho_{-\infty}(x)\,, \qquad (5.10)$$

where

$$J_\mu(x) = \int_{-\infty}^{\infty} d\tau\, j_\tau^\mu(x)\,. \qquad (5.11)$$

Assuming that, as $\tau\to\pm\infty$ the $x_i(\tau)$ leave the region of space-time where we shall be studying the effects of the current (the range of x), $\lim_{\tau\to\pm\infty}\rho_\tau(x)$ is effectively zero (this argument was given by Stueckelberg[11]), and hence

$$\partial_\mu J^\mu(x) = 0\,. \qquad (5.12)$$

In calculating the vector potential from Eq. (5.2), in the neighborhood of a space-time point x, the retarded solution, for example, will depend on the function J^μ only near the past light cone of the point x. Termination of the integral (5.11) at values of τ such that the $x_i(\tau)$ are exterior to this region will suffice for this result. Each region in space time will receive contributions from corresponding segments of the world lines of the source particles.

For a single particle,

$$-\left(\frac{ds}{d\tau}\right)^2 = \left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx_\mu}{d\tau}\right) = \frac{(p^\mu - eA^\mu)(p_\mu - eA_\mu)}{M^2} = \frac{2K}{M} = -\frac{m^2}{M^2} \qquad (5.13)$$

and hence we may replace the integral over τ in Eq. (5.11) by an integral over proper time to obtain[31]

$$J^\mu(x) = \int_{-\infty}^{\infty} ds\,\frac{p^\mu - eA^\mu(x(s))}{m}\,\delta^4(x - x(s))\,. \qquad (5.14)$$

The Lienard-Wiechert potentials are derived from (5.14) by using

$$A^\mu(x) = 4\pi\int d^4x' D(x-x')J^\mu(x')\,, \qquad (5.15)$$

where

$$D(x-x') = \frac{1}{2\pi}\delta((x-x')^2)\,. \qquad (5.16)$$

Equation (5.15) yields half the retarded plus half the advanced potential.[32]

We now turn to the quantum case. As in Eq. (5.5), we define the Heinsenberg operator

$$\underline{\rho}_\tau(x) = \sum_{j=1}^N e_j\,\delta^4(x - \underline{x}_j(\tau))\,, \qquad (5.17)$$

where $\underline{\mathcal{O}} \equiv e^{iK\tau}\mathcal{O}e^{-iK\tau}$, and K is of the form (5.1). Then,

$$\frac{\partial\underline{\rho}_\tau(x)}{\partial\tau} = \sum_{j=1}^N ie_j[K, \delta^4(x-\underline{x}_j(\tau))] = -\sum_{j=1}^N\frac{e_j}{2M_j}\left\{\frac{\partial}{\partial x_\mu}\delta^4(x-\underline{x}_j(\tau)), \underline{p}_{j\mu}(\tau) - e_jA_\mu(\underline{x}_j(\tau))\right\}, \qquad (5.18)$$

where it has been assumed that $A(\underline{x}_j)$ commutes with $\underline{x}_i(\tau)$ at equal τ (A^μ is a c-number function of

$\underline{x}_j$). Now, let

$$\underline{J}^\mu(x)=\int_{-\infty}^{\infty}d\tau\sum_{j=1}^{N}e_j\left\{\frac{p_j^\mu(\tau)-e_jA^\mu(\underline{x}_j(\tau))}{2M_j},\delta^4(x-\underline{x}_j(\tau))\right\}. \tag{5.19}$$

It then follows from Eq. (5.18) that

$$\frac{\partial}{\partial x^\mu}\underline{J}^\mu(x)=-\int_{-\infty}^{\infty}d\tau\left(\frac{\partial\underline{\rho}_\tau(x)}{\partial\tau}\right). \tag{5.20}$$

The operator $\underline{J}^\mu$ cannot be the source of the electromagnetic field, since it would lead to an operator-valued electromagnetic potential which would not be consistent with Eq. (5.18). Furthermore, the electromagnetic potential should depend on the actual state of the system, i.e., the configuration of charge at each τ. We therefore identify as the source of the electromagnetic field

$$J^\mu(x)=\langle\Psi|\underline{J}^\mu(x)|\Psi\rangle\,, \tag{5.21}$$

where Ψ is the Heinsenberg state of the N-body system. Representing Ψ in terms of configuration-space wave functions,

$$J^\mu(x)=\int_{-\infty}^{\infty}d\tau\sum_{j=1}^{N}\frac{e_j}{2M_ji}\int d^4\xi_1\cdots d^4\xi_{j-1}d^4\xi_{j+1}\cdots d^4\xi_N\psi_\tau^*(\xi_1,\ldots\xi_{j-1},x,\xi_{j+1},\ldots\xi_N)$$
$$\times\left(\frac{\partial}{\partial x_\mu}-ie_jA^\mu(x)\right)\psi_\tau(\xi_1,\ldots\xi_{j-1},x,\xi_{j+1},\ldots\xi_N)$$
$$-\left\{\left[\frac{\partial}{\partial x_\mu}+ie_jA^\mu(x)\right]\psi_\tau^*(\xi_1,\ldots\xi_{j-1},x,\xi_{j+1},\ldots\xi_N)\right\}\psi_\tau(\xi_1,\ldots\xi_{j-1},x,\xi_{j+1},\ldots\xi_N)\,. \tag{5.22}$$

This current is a sum of one-particle currents due to each world line, where the positions of the other particles have been integrated over. For a single particle, it has the form[11]

$$J^\mu(x)=\int_{-\infty}^{\infty}d\tau\frac{e}{2Mi}\left\{\psi_\tau^*(x)\left[\frac{\partial}{\partial x_\mu}-ieA^\mu(x)\right]\psi_\tau(x)-\left[\left(\frac{\partial}{\partial x_\mu}+ieA^\mu(x)\right)\psi_\tau^*(x)\right]\psi_\tau(x)\right\}. \tag{5.23}$$

Note that the fourth component, for the case that $\psi_\tau(x)$ is an approximate eigenfunction of four-momentum (and we neglect the vector potential), is

$$J^0(x)\sim\int_{-\infty}^{\infty}d\tau e\frac{E}{M}\rho_\tau(x)\,, \tag{5.24}$$

proportional to the (τ-integrated) probability density at the point x^μ, but weighted by the factor eE/M inducing a sign change for the antiparticle.

The matrix element of $\underline{\rho}_\tau(x)$ is

$$\rho_\tau(x)=\langle\Psi|\underline{\rho}_\tau(x)|\Psi\rangle$$
$$=\sum_{i=1}^{N}e_i\int d\xi_1{}^4\ldots d\xi_{i-1}{}^4d^4\xi_{i+1}\ldots d^4\xi_N\times|\psi_\tau(\xi_1\ldots\xi_{i-1},x,\xi_{i+1},\ldots,\xi_N)|^2\,. \tag{5.25}$$

Stueckelberg[11] argued, for the one-particle case that he treated, that

$$\lim_{\tau\to\pm\infty}\rho_\tau(x)=0\,, \tag{5.26}$$

in analogy to the classical arguments presented above for the result (5.12), i.e., that as $\tau\to\infty$, the probability that one of the particles remains in a bounded region of space-time goes to zero. A more rigorous argument can be given for the quantum case, assuming the spectrum of K to be absolutely continuous (the total K carries the center-of-mass motion as well). Let k,α label the representation in which K acts as multiplication (α is a degeneracy index). Then,

$$\rho_\tau(x)=\sum_{\alpha,\alpha'}\int dk\,dk'e^{i(k-k')\tau}\psi(k,\alpha)^*\psi(k',\alpha')$$
$$\times\int d^4\xi,\ldots d^4\xi_{i-1}d^4\xi_{i+1}\ldots d^4\xi_N$$
$$\times\langle k\alpha|\xi_1,\ldots\xi_{i-1},x,\xi_{i+1},\ldots\xi_N\rangle$$
$$\times\langle\xi_1,\ldots\xi_{i-1},x,\xi_{i+1},\ldots\xi_N|k'\alpha'\rangle\,. \tag{5.27}$$

Provided that the integral over the ξ_j's is a continuous function of k,k', it follows from the Riemann-Lebesgue lemma that $\rho_\tau(x)$ goes to zero as $\tau\to\infty$. A final integration over x would produce a δ function, leading to the result

$$\int d^4x\rho_\tau(x)=1$$

for all τ. Hence the vanishing of $\rho_\tau(x)$ pointwise can be understood as a spreading of the wave packets. The expectation value of Eq. (5.20) therefore results in

$$\partial_\mu J^\mu(x)=0\,.$$

The equations that must be solved are therefore (5.1) and (5.2), with $J^\mu(x)$ define by Eq. (5.22).

Since $J^\mu(x)$ may depend on $\psi_\tau(x_1\ldots x_N)$ for all τ, one may ask whether this system of equations forms a well-posed problem. This question will be discussed below for the two-body case.

One can understand the notion of the integration over τ from a physical point of view for the construction of the electromagnetic current by noting that τ itself is not an observable, and no observation is performed at a given τ. One has to integrate over all τ's that could have possibly contributed to a given measurement. A simple example[11] is the case of a pair creation at $t=0$: The result of a measurement of the total charge at a time $t_0>0$ is composed of two contributions from the two points on the world line of the particle where $t(\tau)=t_0$. The two contributions are positive and negative, corresponding to positive and negative energy [c.f. Eq. (5.24)] and cancel each other. Thus the total charge is zero for all t.

We now wish to investigate the case of two charged particles interacting electromagnetically, i.e., moving under the influence of the potential created by their current (no external potential). We shall discuss this problem in terms of the τ-integrated current given by

$$J^\mu(x)=\int d\tau\, j_\tau^\mu(x)=j_1^\mu(x)+j_2^\mu(x)\,, \tag{5.28}$$

with $j_\tau^\mu(x)$ given by the integrand of (5.22) applied to the two-body case:

$$\begin{aligned} j_\tau^\mu(x)=&-\frac{i}{2M_1}\int d^4x_2\;\{\psi_\tau^*(x,x_2)[\partial^\mu-ie_1A^\mu(x)]\psi_\tau(x,x_2)-[(\partial^\mu+ie_1A^\mu(x))\psi_\tau^*(x,x_2)]\psi_\tau(x,x_2)\}\\ &-\frac{i}{2M_2}\int d^4x_1\{\psi_\tau^*(x_1,x)[\partial^\mu-ie_2A^\mu(x)]\psi_\tau(x_1,x)\\ &\qquad-[(\partial^\mu+ie_2A^\mu(x))\psi_\tau^*(x_1,x)]\psi_\tau(x_1,x)\}\,. \end{aligned} \tag{5.29}$$

The potential A^μ is given in terms of the current J^μ by

$$A^\mu(x)=\int d^4x' D(x-x')J^\mu(x')=A_1^\mu(x)+A_2^\mu(x)\,, \tag{5.30}$$

and its Fourier transform by

$$\begin{aligned}\tilde{A}^\mu(q)&=\frac{1}{(2\pi)^4}\int d^4x\int d^4x' D(x-x')J^\mu(x')e^{iqx}\\ &=\frac{1}{(2\pi)^4}\tilde{D}(q)\tilde{J}^\mu(q)\,.\end{aligned} \tag{5.31}$$

For the function D we shall take the solution of the Maxwell equations that was chosen by Feynman[33]:

$$D(y)=\pi\delta_+(y^2)=\frac{1}{\pi}\int_0^\infty ds\, e^{isy^2} \tag{5.32}$$

with the Fourier transform

$$\tilde{D}(q)=\int d^4x D(x)e^{iqx}=\frac{-1}{q^2+i\epsilon}\,. \tag{5.33}$$

We now turn to the question of whether the problem of solving the system of equations (5.1), (5.2), and (5.29) is well posed, where the wave function of future τ's may influence the wave function at a given τ through $J^\mu(x)$, which depends on the wave function for all τ. Recalling the discussion for the classical case one realizes that $J^\mu(\vec{x}_0,t_0)$ does not, e.g., depend on $\psi_\tau(x,x_2)$ for all τ but only for those τ that satisfy $\psi_\tau(\vec{x}_0,t_0,x_2)\neq 0$ (for example, the vertical intervals in Fig. 5). Moreover, the integration on τ does not take us into the physical future (in t); the causal nature of the theory is fixed by the function $D(x-x')$ in (5.30). In the classical case the problem has been discussed by Wheeler and Feynman.[32]

We now turn to the problem of self energy. Inserting the decomposition of $A^\mu(x)$ in (5.30) into (5.1), one obtains

$$K=\frac{[p_1-e_1A_1(x_1)-e_1A_2(x_1)]^2}{2M_1}+\frac{[p_2-e_2A_2(x_2)-e_2A_1(x_2)]^2}{2M_2}\,, \tag{5.34}$$

where $A_i(x_j)$ is the potential acting on particle j due to the current of particle i; in particular, $A_i(x_j)$ for $i=j$ describes self interaction, i.e., the quantum effect of a particle acting on itself through the electromagnetic interaction. Note that in the classical case this effect can be made to disappear by defining $A_i(x_j)=0$ for $i=j$.[32] In the quantum case, however, the position variable occurs in a distribution. These terms will cause a redefini-

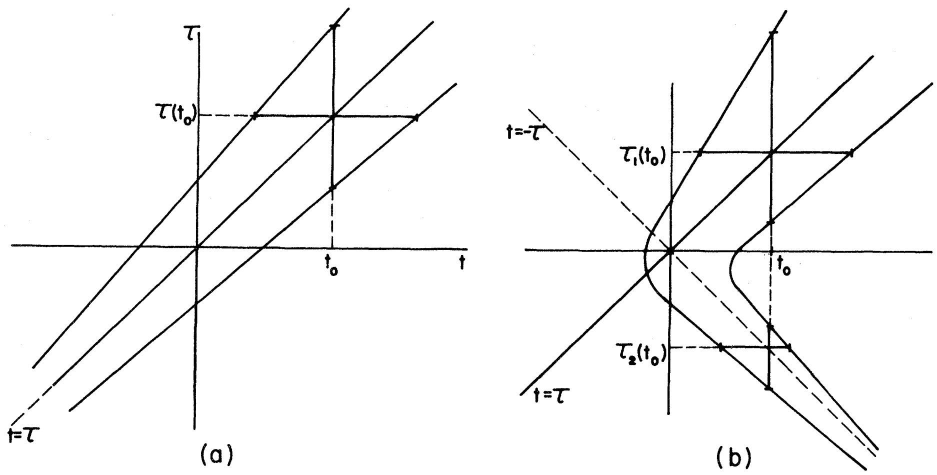

FIG. 5. The τ-t plane projection of the wave packet for a free particle (a) and pair creation (b). The horizontal line is the spread of the wave packet in time; the vertical one is the range of integration in τ.

tion of the free particle's motion in the following way.

The term $A_2^\mu(x_1)$ is a proper interaction term that goes to zero as the particles get infinitely separated when $\tau\to\infty$. The term $A_1^\mu(x_1)$, on the contrary, does not and remains with the particle even when it is very far from the other, and assumed free. Thus the squared mass of the free asymptotic particle is not given by $m_0{}^2=\lim_{\tau\to\infty}\langle -p^2\rangle_{\rm av}$, but rather by $m^2=\lim_{\tau\to\infty}\langle -[p-e\hat{A}(x)]^2\rangle_{\rm av}$, where $\hat{A}(x)$ denotes the action of the particle on itself.

In the perturbation theory we therefore take K_0 to be

$$K_0=\frac{[p_1-e_1A_1(x_1)]^2}{2M_1}+\frac{[p_2-e_2A_2(x_2)]^2}{2M_2}=\frac{\bar{p}_1{}^2}{2M_2}+\frac{\bar{p}_2{}^2}{2M_2} \tag{5.35}$$

and the S-matrix elements are taken between approximate improper eigenstates of the new operators $\bar{p}_1^\mu,\bar{p}_2^\mu$ [whose components commute up to $O(e^2)$], asymptotically the kinetic (noncanonical) momenta. Note that in choosing D as in (5.32) we chose to work with the single κ^2 potential; this also agrees with the fact that in integrating the current over τ, we lose all dependence on Δ, the τ difference between the particles. For a complete, consistent scattering theory, one would have to start again from the beginning. However, we shall apply the scattering theory developed here to one simple case, that of Rutherford scattering, for the sake of illustration.

Assume one of the particles to be very heavy compared with the other. As a first approximation we shall consider the heavy particle as almost classical, having a sharp p_2^μ and suffering no recoil. Denoting the trajectory of particle 2 by (to lowest order) $x_2(\tau)=x_2(0)+(p_2/M_2)\tau$, we have for the current due to particle 2,

$$\begin{aligned}\tilde{J}^\mu(q)&=\int d^4x\,e^{iqx}J^\mu(x)\\&=\int d^4x\,e^{iqx}e_2\int d\tau\frac{p_2^\mu(\tau)}{M_2}\delta^4(x-x_2(\tau))\\&=e_2\int d\tau\frac{p_2^\mu(\tau)}{M_2}e^{iqx_2(\tau)}\,.\end{aligned} \tag{5.36}$$

Using the assumption that $p_2^\mu(\tau)=p_2^\mu=$ const, one may perform the τ integration to obtain

$$\tilde{J}^\mu(q)=2\pi\frac{e_2}{M_2}p_2^\mu e^{iqx_2(0)}\delta\left(\frac{p_2\cdot q}{M_2}\right). \tag{5.37}$$

From (5.31) and (5.33), the potential acting on particle 1 is

$$\begin{aligned}\tilde{A}^\mu(q)&=\frac{1}{(2\pi)^4}\tilde{D}(y)\tilde{J}^\mu(q)\\&=\frac{1}{(2\pi)^3}\frac{e_2}{M_2}\frac{p_2^\mu}{q^2}\,e^{iqx_2(0)}\,\delta\left(\frac{p_2\cdot q}{M_2}\right).\end{aligned} \tag{5.38}$$

We shall consider this case as a problem of a particle in an external potential [the separation of the K operator of (5.34) into a center-of-mass (c.m.) and the relative part may occur only in special cases].

From Eq. (3.13) we have for the first-order T matrix element,

$$\langle p_1'|T|p_1\rangle=\frac{e_1}{2M_1}(p_1+p_1')_\mu\tilde{A}^\mu(p_1'-p_1)\,, \tag{5.39}$$

which upon using (5.38) and working in the rest frame of particle 2, gives

$$|\langle p_1'|T|p_1\rangle| = \frac{e_1 e_2 E_1}{(2\pi)^3 M_1}\frac{1}{(\vec{p}_1' - \vec{p}_1)^2}\delta(E_1 - E_1') \quad (5.40)$$

and the cross section from (2.41) is

$$\frac{d\sigma}{d\Omega\, dp_0} = (2\pi)^{-1} e_1{}^2 e_2{}^2 E_1{}^2 \frac{1}{(\vec{p}_1' - \vec{p}_1)^2}[\delta(E_1 - E_1')]^2 . \quad (5.41)$$

Following the arguments of Sec. II one gets

$$\frac{d\sigma}{d\Omega} = t\, \frac{\alpha_1 \alpha_2 E^2}{4p^4 \sin^4(\theta/2)} , \quad (5.42)$$

where $\alpha = e^2/4\pi$, $p = |\vec{p}_1|$, θ is the scattering angle in the laboratory frame (rest frame of particle 2), and t is an infinite constant denoting the infinite time of exposure to the beam, as discussed in Sec. II. Apart from t, the cross section (5.42) is just the Rutherford cross section for the electromagnetic scattering of a spinless particle off a heavy target (nucleus). Relaxing the assumption of an infinitely heavy target, we now assume its momentum does change so as to satisfy conservation of momentum. For the lowest order we assume that at $\tau = 0$, the momentum p_2 changes into p_2' so as to satisfy $p_1 + p_2 = p_1' + p_2'$. The integral over τ is now of two parts as we have

$$x_2(\tau) = x_2(0) + \frac{p_2}{M}\tau , \quad \tau < 0$$
$$x_2(\tau) = x_2(0) + \frac{p_2'}{M}\tau \quad \tau > 0 . \quad (5.43)$$

The integral in (5.36) gives in this case

$$\tilde{J}^\mu(q) = \frac{e_2}{M_2}\int d\tau\, p_2^\mu(\tau) e^{iqx_2(\tau)} = \frac{e_2}{M_2} e^{iqx_2(0)}\left[p_2^\mu \int_{-\infty}^0 d\tau \exp\left(iq\frac{p_2}{M_2}\tau\right) + p_2'^\mu \int_0^\infty \exp\left(i\frac{q\cdot p_2'}{M_2}\tau\right) d\tau\right]$$
$$= \pi \frac{e_2}{M_2} e^{iqx_2(0)}\left[p_2^\mu\, \delta_+\left(-\frac{q\cdot p_2}{M_2}\right) + p_2'^\mu\, \delta_+\left(\frac{q\cdot p_2'}{M_2}\right)\right]$$
$$= \pi e_2 e^{iqx_2(0)}[p_2^\mu \delta_+(-q\cdot p_2) + p_2'^\mu \delta_+(q\cdot p_2')] . \quad (5.44)$$

The T-matrix element then becomes

$$|\langle p_1'|T|p_1\rangle| = \frac{e_1 e_2}{2(2\pi)^3 M_1}\frac{1}{q^2}[(p_1\cdot p_2')\delta_+(q\cdot p_2') - (p_1\cdot p_2)\delta_+(q\cdot p_2)] \quad (5.45)$$

and may be further simplified by the use of $p_2' = p_2 - q$ and of the laboratory frame where p_2 is pure timelike.

Expression (5.45) constitutes a first correction to (5.40) due to recoil, but it still treats particle 2 as classical (its momentum and position are assumed to have no spread). The treatment presented here, though it gives good results in lowest order, cannot constitute a satisfactory theory for two particles interacting electromagnetically. Such a theory has to be based on a nontrivial modification of the scattering theory given in Secs. II and III, for which a systematic perturbation scheme to all orders remains to be developed.

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$$\langle\psi_{\text{out}}|m_i^2|\psi_{\text{out}}\rangle - \langle\psi_{\text{in}}|m_i^2|\psi_{\text{in}}\rangle = -\left\langle\psi_{\text{in}}\left|\left[K_0, S^\dagger\frac{\partial S}{\partial M_i}\right]\right|\psi_{\text{in}}\right\rangle .$$

One then utilizes the identity (obtained by first calculating $\partial\Omega_+/\partial M_i$ and $\partial\Omega_-^\dagger/\partial M_i$)

$$\frac{\partial S}{\partial M_i} = \lim_{\tau\to\infty}\left[-\Omega_-^\dagger\int_{-\tau}^{\tau} d\tau' e^{iK\tau'}\frac{\partial K_0}{\partial M_i} e^{iK\tau'}\Omega_+ + i\tau\left(\frac{\partial K_0}{\partial M_i}S + S\frac{\partial K_0}{\partial M_i}\right)\right]$$

to obtain (2.22).

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