Prolongation analysis of the cylindrical Korteweg-de Vries equation

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The cylindrical Korteweg-de Vries equation is studied systematically within the prolongation scheme of Estabrook and Wahlquist. A non-Abelian Lie algebra associated to the equation is exploited, without using representations, to derive a set of Bäcklund transformations and a nonlinear superposition formula. These are used to provide some examples of explicit solutions.

I. INTRODUCTION

Several studies concerning nonlinear evolution equations make the existence of a correspondence between integrable equations and non-Abelian prolongations likely.[1–4] Since, as far as we know, a general theory of this correspondence has not been set up yet, the accumulation of cases may act as a guide to such a theory.

In this context, this paper is devoted to a systematic analysis, within the prolongation scheme,[1] of the cylindrical Korteweg-de Vries (cKdV) equation

$$V_t+VV_x+V_{xxx}+\frac{1}{2t}V=0\ , \qquad (1.1)$$

which is of interest in plasma physics.[5]

Sec. II deals with the prolongation calculation which gives rise to a non-Abelian Lie algebra used to write explicit Bäcklund transformations (BT) for the cKdV equation (Sec. III). These BT's, which can reduce to a pair of Riccati equations, are derived without using representations of the Lie algebra associated to the cKdV equation.

In Sec. IV we give a nonlinear superposition formula, which is exploited in Sec. V to obtain some explicit solutions of Eq. (1.1), like the Calogero-Degasperis soliton solution.[6] Section V contains also other examples of solutions, such as the Hirota solution[7] and a solution in terms of hyperbolic and rational functions.

In Sec. VI some concluding remarks are presented, while Appendixes A, B, and C contain details of calculation.

II. THE PROLONGATION CALCULATION

For our purposes, it is convenient to write Eq. (1.1) as[8]

$$u_t+t^{-1/2}uu_x+u_{xxx}=0\ , \qquad (2.1)$$

by means of the transformation $V=t^{-1/2}u$.

We now introduce the following prolongation equations[1,2] for Eq. (2.1):

$$y_x^A=F^A(u,x,t;y^B)\ , \qquad (2.2a)$$

$$y_t^A=G^A(u,z,p,x,t;y^B)\ , \qquad (2.2b)$$

where $A,B=1,2\ldots,N$ and $z=u_x$, $p=u_{xx}$, $r=u_{xxx}$.

The integrability conditons[2] for Eqs. (2.2) are

$$r(F_u^A+G_p^A)+(t^{-1/2}uF_u^A+G_u^A)z+G_z^Ap-F_t^A+[F,G]^A+G_x^A=0\ , \qquad (2.3)$$

where

$$[F,G]^A=F^B\frac{\partial G^A}{\partial y^B}-G^B\frac{\partial F^A}{\partial y^B}\ . \qquad (2.4)$$

Dropping the indexes for simplicity and following the procedure of Ref. 2, Eqs. (2.2) and (2.3) yield

$$F_{uuu}=0\ , \qquad (2.5a)$$

$$[F,F_{uu}]+F_{uux}=0\ , \qquad (2.5b)$$

$$L_u+t^{-1/2}uF_u+[F,[F,F_u]]+2[F,F_{ux}]+[F_x,F_u]+F_{uxx}=0\ , \qquad (2.5c)$$

$$[F,L]-F_t+L_x=0\ , \qquad (2.5d)$$

and

$$G=-pF_u+\tfrac{1}{2}F_{uu}z^2+[F,F_u]z+F_{ux}z+L\ , \tag{2.6}$$

where $L\equiv L(u,x,t;y)$ is a function of integration.

Integrating Eqs. (2.5a) and (2.5c), we find

$$F=\tfrac{1}{2}au^2+bu+c\ , \tag{2.7}$$

$$\begin{aligned}L=&-\tfrac{1}{3}(at^{-1/2}+\tfrac{1}{2}[a,f]+\tfrac{1}{2}[b,a_x])u^3\\&-\tfrac{1}{2}(bt^{-1/2}+[b,f]+[c,a_x]+[b,b_x]\\&\quad+[c_x,a]+a_{xx})u^2\\&-([c,f]+2[c,b_x]+[c_x,b]+b_{xx})u-d\ ,\end{aligned} \tag{2.8}$$

where a, b, c, and d are (vector) functions of integration depending on x, t, and the y's only, and $f\equiv[c,b]$.

Substituting for F and L in (2.5b) and (2.5d), and making the coefficient of powers of u equal to zero, we obtain the non-Abelian prolongation algebra:

$$[a,b]=0\ , \tag{2.9a}$$

$$[a,c]=a_x\ , \tag{2.9b}$$

$$[a,[a,f]]=[a,[b,a_x]]=0\ , \tag{2.9c}$$

$$[a,[b,f]]=[b,[a,b_x]]=0\ , \tag{2.9d}$$

$$[b,[b,f]]+[b,[b,b_x]]=0\ , \tag{2.9e}$$

$$\begin{aligned}-a_t=&\,t^{-1/2}(b_x+f)+[a,d]\\&+3([b,[c,f]]+[c,[b,b_x]]\\&\quad+[b,f]_x+[b,b_x]_x)\ ,\end{aligned} \tag{2.9f}$$

$$\begin{aligned}-b_t=&\,b_{xxx}+[b,d]+[c,[c,f]]\\&+2[c,[c,b_x]]+[c,[c_x,b]]\\&+[c,b_{xx}]+[c,f]_x+2[c,b_x]_x\\&+[c_x,b]_x\ ,\end{aligned} \tag{2.9g}$$

$$-c_t=d_x+[c,d]\ . \tag{2.9h}$$

Although we expect that the algebra defined by the commutation relations (2.9) is an infinite-dimensional one,[1,3] at present there is no rigorous proof that this occurs. However, in order to find *explicit* Bäcklund transformations for Eq. (2.1), we shall try to *close* algebra (2.9) assuming that[1,2] (i) the set of variables $\{y^A\}$ has one element only, say y (pseudopotential of the first kind[4]), (ii) b, c, and $f\equiv[c,b]$ are linearly independent, and (iii) b_x, c_x, $[c,f]$, and d are given by

$$b_x=A_1b+A_2c+A_3f\ , \tag{2.10}$$

$$c_x=B_1b+B_2c+B_3f\ , \tag{2.11}$$

$$[c,f]=C_1b+C_2c+C_3f\ , \tag{2.12}$$

$$d=D_1b+D_2c+D_3f\ , \tag{2.13}$$

where A_i, B_i, C_i, and D_i are (scalar) functions of x and t only.

From the requirement (i) and from (2.9a)–(2.9d) one can see that a must be zero to avoid inconsistency.

Inserting now (2.10)–(2.13) in the commutation relations (2.9), and exploiting the Jacobi identity, we obtain the (non-Abelian) finite-dimensional Lie algebra:

$$[b,f]=-C_2b\ , \tag{2.14a}$$

$$[c,f]=C_1b+C_2c\ , \tag{2.14b}$$

$$[c,b]=f\ , \tag{2.14c}$$

which can be related to the Lie algebra SL(2,R) (see Appendix C), and

$$b_x=A_1b+A_3f\ , \tag{2.15}$$

$$c_x=B_1b+B_2c+B_3f\ , \tag{2.16}$$

$$b_t=\gamma_1b+\gamma_3f\ , \tag{2.17}$$

$$c_t=\delta_1b+\delta_2c+\delta_3f\ . \tag{2.18}$$

Using the Jacobi identity and the integrability conditions $b_{xt}=b_{tx}$, $c_{tx}=c_{xt}$, and the relations (2.14), we obtain

$$C_2=\tfrac{1}{3}t^{-1/2}(1+A_3)^{-1}\ , \tag{2.19}$$

$$A_{3x}=-(B_2+A_1)(1+A_3)\ , \tag{2.20}$$

$$C_{1x}=2(B_2C_1-C_2B_1)\ , \tag{2.21}$$

$$C_{2x}=C_2(B_2+A_1)\ , \tag{2.22}$$

$$A_{1t}=\gamma_{1x}+C_2(\gamma_3B_3-A_3\delta_3)\ , \tag{2.23}$$

$$A_{3t}=\gamma_{3x}-A_3\delta_2+\gamma_3B_2\ , \tag{2.24}$$

$$\begin{aligned}B_{1t}=&\,\delta_{1x}+\delta_1(A_1-B_2)+B_1(\delta_2-\gamma_1)\\&+C_1(A_3\delta_3-B_3\gamma_3)\ ,\end{aligned} \tag{2.25}$$

$$B_{2t}=\delta_{2x}+C_2(A_3\delta_3-B_3\gamma_3)\ , \tag{2.26}$$

$$B_{3t}=\delta_{3x}-B_1\gamma_3-B_3\gamma_1+\delta_1A_3+\delta_3A_1\ , \tag{2.27}$$

$$C_{1t}=2C_1(\delta_2+D_3C_2)\ , \tag{2.28}$$

$$C_{2t}=C_2(\gamma_1+\delta_2)\ . \tag{2.29}$$

The quantities γ_1, γ_3, δ_1, δ_2, and δ_3 are defined in terms of A_i, B_i, C_i, and D_i as follows:

$$\gamma_1=-(A_{1xx}+3A_1A_{1x}+A_1^3)+C_2[B_1(1+A_3)^2-B_{3x}(1+A_3)-2B_3A_1(1+A_3)+D_3] , \quad (2.30)$$

$$\gamma_3=D_2-[2A_{1x}+A_1^2+2(1+A_3)B_3C_2+(1+A_3)^2C_1](1+A_3) , \quad (2.31)$$

$$\delta_1+D_3C_1=-[D_{1x}+D_1A_1+D_2B_1+D_3(B_3C_2+A_3C_1)]=0 , \quad (2.32)$$

$$\delta_2=-[D_{2x}+D_2B_2+D_3C_2(1+A_3)] , \quad (2.33)$$

$$\delta_3=-[D_1(1+A_3)+D_2B_3+D_{3x}+D_3(B_2+A_1)] . \quad (2.34)$$

III. BÄCKLUND TRANSFORMATIONS

Let us look for a solution $\psi(u,x,t;y)$ of Eq. (2.1) such that u satisfies the equation itself, and the first-kind pseudopotential y satisfies the prolongation equations (2.2).

Substituting the derivatives of ψ with respect to u, x, t, and y, and requiring that the resulting equation be satisfied in u, z, p, and r identically, we obtain

$$\psi=-u+\rho(x,t;y) , \quad (3.1)$$

where

$$\rho=-2\left[\frac{c}{b}-3t^{1/2}\left[\frac{f+b_x}{b}\right]_x\right] , \quad (3.2)$$

and b, c, and f are ordinary functions.

From (3.1) and (3.2) we have the Bäcklund transformations

$$(\psi+u)_x=\rho_yF+\rho_x , \quad (3.3a)$$

$$(\psi+u)_t=\rho_yG+\rho_t , \quad (3.3b)$$

where F and G as given by (2.7) and (2.6) can be expressed in terms of the elements b and c of the Lie algebra (2.14).

Equations (3.3) can be used to obtain some explicit solutions of Eq. (2.1). To this end, we introduce the function

$$v=\frac{bu-c}{bu+c} , \quad (3.4)$$

and observe that for one-dimensional prolongations, the expression

$$\frac{f^2}{b^2}+2\frac{[b,f]c}{b^2}=\frac{f^2}{b^2}-2C_2\frac{c}{b} \quad (3.5)$$

is constant with respect to y. One easily sees that (3.5) coincides with C_1.

Starting now from

$$v_x=v_yF+v_uz+\frac{\partial v}{\partial x} , \quad (3.6a)$$

$$v_t=v_yG+v_uu_t+\frac{\partial v}{\partial t} , \quad (3.6b)$$

and using (2.14)–(2.29), (3.4), and (3.5), we are led to the following relations:

$$-W_x=\left[1-\frac{B_3}{u}+(1+A_3)W\right](C_1+2C_2uW]^{1/2}+\left[\frac{z}{u}+A_1-B_2\right]W-\frac{B_1}{u} , \quad (3.7a)$$

$$-W_t=\left[-\frac{p}{u}+\frac{A_1z}{u}-\tfrac{1}{3}ut^{-1/2}-\alpha-\frac{(D_1+\delta_3)}{u}\right](C_1+2C_2uW)^{1/2}+\left[\frac{z}{u}-A_1\right](1+A_3)(C_1+2C_2uW)$$
$$+\left\{\left[-\frac{(1+A_3)}{3}t^{-1/2}u+(\gamma_3-D_2)\right](C_1+2C_2uW)^{1/2}+\frac{u_t}{u}+\left[\ln\frac{C_2}{C_1}\right]_t\right\}W , \quad (3.7b)$$

where $W=(1-v)/(1+v)$, and

$$\alpha=A_1^2+A_{1x}+C_1(1+A_3)^2+\tfrac{1}{3}B_3t^{-1/2} . \quad (3.8)$$

Using now the expressions (2.15) and (2.16) for b_x and c_x, $c/b=uW$, and $f/b=(C_1+2C_2uW)^{1/2}$, from (3.1) and (3.2) we have

$$\psi+u=-2\left[(1+A_3)uW-B_3-3t^{1/2}A_{1x}+\frac{A_1}{C_2}(C_1+2C_2uW)^{1/2}\right] . \quad (3.9)$$

We have proved that the system (3.7) is integrable, provided that the relations from (2.15) to (2.29) are satisfied. Through the relation (3.9) one can produce a new solution ψ of Eq. (2.1) in terms of a known solution of the same equation.

We point out that in deriving Eqs. (3.7) and (3.9), we have not made use of representations of the Lie algebra (2.14). Our procedure is essentially based on the fact that the quantity (3.5) does not depend on the pseudopotential y.

In order to put Eqs. (3.7) in a form suitable for practical use, let us introduce the functions $R(x,t)$ and $\beta(x,t)$ defined, respectively, by

$$A_1-B_2=-\frac{R_x}{R} \tag{3.10}$$

and

$$\beta=\int^x \frac{B_1(\xi,t)}{R(\xi,t)}\,d\xi\,. \tag{3.11}$$

From (2.23) and (3.10) one has

$$A_1=\tfrac{1}{2}\left[\ln\frac{C_2}{R}\right]_x . \tag{3.12}$$

On the other hand, with the help of (3.11), Eq. (2.21) yields

$$C_1=C_2R(\varphi-2\beta)\,, \tag{3.13}$$

where φ denotes an integration function of t only.

Now, in virtue of (2.19), (3.10), (3.11), (3.13), and using the expression (A8) quoted in Appendix A, Eq. (3.7a) can be written as

$$T_x=-\left[\frac{C_2}{R}\right]^{1/2}\left[u-\tfrac{1}{2}xt^{-1/2}-\mu+t^{1/2}(A_{1xx}+\tfrac{3}{2}A_1{}^2+2A_{1x})+\tfrac{1}{6}t^{-1/2}\frac{R}{C_2}T^2\right], \tag{3.14}$$

where

$$T=\left[\varphi-2\left[\beta-\frac{uW}{R}\right]\right]^{1/2} . \tag{3.15}$$

In a similar but less simple manner (see Appendix B) we can express (3.7b) in the form

$$\begin{aligned}T_t=\left[\frac{C_2}{R}\right]^{1/2}\Bigg[&p-A_1z+\tfrac{1}{3}u^2t^{-1/2}+u[\tfrac{1}{3}(A_{1x}-A_{1xx})+\tfrac{1}{6}xt^{-1}+\tfrac{1}{3}\mu t^{-1/2}]\\&-E(x,t)+\tfrac{1}{2}\frac{R}{C_2}T^2\left[\frac{ut^{-1}}{9}+\frac{t^{-1/2}}{3}[\tfrac{2}{3}(A_{1x}-A_{1xx})+\tfrac{1}{3}xt^{-1}+\tfrac{2}{3}\mu t^{-1/2}]\right]\Bigg]\\&+\left\{\tfrac{1}{3}t^{-1/2}(A_1u-z)+\tfrac{1}{2}\left[X(x,t)+\tfrac{1}{3}t^{-1}-\left[\ln\frac{R}{C_2}\right]_t\right]\right\}T\,,\end{aligned} \tag{3.16}$$

where $E(x,t)$ and $X(x,t)$ are respectively given by (B3) and (B7) of Appendix B.

Performing now the change of variable

$$\phi=\left[\frac{R}{C_2}\right]^{1/2}T+3A_1t^{1/2}\,, \tag{3.17}$$

Eqs. (3.14) and (3.15) read

$$\phi_x=-u+h-\tfrac{1}{6}t^{-1/2}\phi^2\,, \tag{3.18a}$$

$$\begin{aligned}\phi_t=&\tfrac{1}{3}t^{-1/2}u(u+h)+p-h_{xx}-\tfrac{2}{3}t^{-1/2}h^2\\&+\tfrac{2}{3}t^{-1/2}(-h_x-\tfrac{1}{2}z+\tfrac{3}{4}t^{-1/2})\phi\\&+\tfrac{1}{18}t^{-1}(u+2h)\phi^2\,,\end{aligned} \tag{3.18b}$$

where

$$h=t^{1/2}(A_{1x}-A_{1xx})+\tfrac{1}{2}xt^{-1/2}+\mu\,. \tag{3.19}$$

The integrability condition for the system (3.18) implies that the function (3.19) has to satisfy the differential equations

$$h_x=\tfrac{1}{2}t^{-1/2} \tag{3.20}$$

and

$$h_t+h_{xxx}+2t^{-1/2}hh_x-\tfrac{1}{2}t^{-1}h=0 \tag{3.21}$$

simultaneously. The solution of these is

$$h=\tfrac{1}{2}xt^{-1/2}+kt^{-1/2}\ , \tag{3.22}$$

where k is a constant of integration.

From the substitution of (3.22), (3.18) become the pair of Riccati equations,

$$\phi_x=-u+\tfrac{1}{2}xt^{-1/2}+kt^{-1/2}-\tfrac{1}{6}t^{-1/2}\phi^2\ , \tag{3.23a}$$

$$\begin{aligned}\phi_t=&p+\tfrac{1}{3}t^{-1/2}u\,(u+\tfrac{1}{2}xt^{-1/2}+kt^{-1/2})\\&-\tfrac{2}{3}t^{-1/2}(\tfrac{1}{2}xt^{-1/2}+kt^{-1/2})^2\\&+\tfrac{1}{3}t^{-1/2}(\tfrac{1}{2}t^{-1/2}-z)\phi\\&+\tfrac{1}{18}t^{-1}(u+xt^{-1/2}+2kt^{-1/2})\phi^2\ .\end{aligned} \tag{3.23b}$$

Going back to Eq. (3.9) and following the same procedure used to find (3.23), we are led to the remarkable formula

$$\psi=u+2\phi_x\ , \tag{3.24}$$

which can yield a new solution $\psi(x,t)$ of the cKdV equation (2.1) from a given solution $u(x,t)$.

IV. THE THEOREM OF PERMUTABILITY

The Bäcklund transformations (3.23) can be exploited to construct solutions of the cKdV equation more elaborate than, say, those obtained in Sec. V [see (5.8) and (5.12)]. Of course, this generally could be achieved through the integration of more complicated differential equations (subsection V A). However, for the cKdV equation a nonlinear superposition formula, i.e., a theorem of permutability,[9] holds, which gets the same goal without using quadratures.

In order to derive the superposition formula, let us deal with two solutions of Eq. (2.1), ψ_1 and ψ_2, corresponding to the same starting solution u. From (3.24) and (3.23a) we have

$$\psi_1=-u+xt^{-1/2}+2k_1t^{-1/2}-\tfrac{1}{3}t^{-1/2}\phi_1{}^2 \tag{4.1}$$

and

$$\psi_2=-u+xt^{-1/2}+2k_2t^{-1/2}-\tfrac{1}{3}t^{-1/2}\phi_2{}^2\ . \tag{4.2}$$

Then, let us introduce the potential functions Ω_0, Ω_1, and Ω_2 defined by

$$u=-\Omega_{0x}\ , \tag{4.3}$$

$$\psi_1=-\Omega_{1x}\ , \tag{4.4}$$

$$\psi_2=-\Omega_{2x}\ . \tag{4.5}$$

From (3.24), (4.3), (4.4), and (4.5) we obtain

$$\phi_1=\tfrac{1}{2}(\Omega_0-\Omega_1) \tag{4.6}$$

and

$$\phi_2=\tfrac{1}{2}(\Omega_0-\Omega_2)\ . \tag{4.7}$$

The substitution of (4.6) and (4.7) in (4.1) and (4.2) gives, respectively,

$$\psi_1=-u+xt^{-1/2}+2k_1t^{-1/2}-\tfrac{1}{12}t^{-1/2}(\Omega_0-\Omega_1)^2\ , \tag{4.8}$$

$$\psi_2=-u+xt^{-1/2}+2k_2t^{-1/2}-\tfrac{1}{12}t^{-1/2}(\Omega_0-\Omega_2)^2\ . \tag{4.9}$$

Now our aim is to look for a function Ω_3 such that

$$\Omega_{3x}=-\psi_3\ , \tag{4.10a}$$

$$\psi_3=-\psi_1+xt^{-1/2}+2k_2t^{-1/2}-\tfrac{1}{12}t^{-1/2}(\Omega_1-\Omega_2)^2\ , \tag{4.10b}$$

$$\psi_3=-\psi_2+xt^{-1/2}+2k_1t^{-1/2}-\tfrac{1}{12}t^{-1/2}(\Omega_2-\Omega_3)^2\ , \tag{4.10c}$$

where ψ_3 is required to fulfill the cKdV equation (2.1). In doing so, let us subtract (4.9) from (4.8) and compare the result with the expression of $\psi_1-\psi_2$ as obtained from (4.10). Thus we are led to the relation

$$\Omega_3=\Omega_0+24\frac{k_1-k_2}{\Omega_1-\Omega_2}\ , \tag{4.11}$$

which is consistent with Eqs. (4.10).

Differentiating Eq. (4.11) with respect to x and using (4.6), (4.7), (4.8), and (4.9) we obtain the superposition formula

$$\psi_3=u+\frac{2t^{-1/2}(k_2-k_1)(\psi_1-\psi_2)}{[(-\psi_1-u+xt^{-1/2}+2k_1t^{-1/2})^{1/2}-(-\psi_2-u+xt^{-1/2}+2k_2t^{-1/2})^{1/2}]^2} \tag{4.12}$$

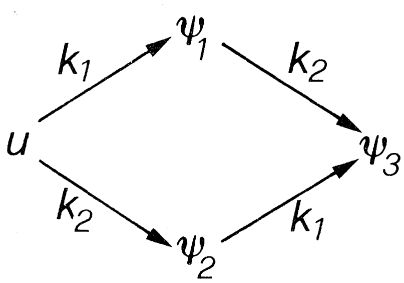

FIG. 1. Schematic representation of the permutability theorem.

represented schematically in Fig. 1.

V. EXAMPLES OF EXPLICIT SOLUTIONS

Equation (3.23a) can be linearized by putting

$$\phi(x,t)=6t^{1/2}\frac{\omega_x(x,t)}{\omega(x,t)}\ , \tag{5.1}$$

to give

$$\omega_{xx}-\tfrac{1}{6}t^{-1/2}(kt^{-1/2}+\tfrac{1}{2}xt^{-1/2}-u)\omega=0\ . \tag{5.2}$$

This equation is the Schrödinger equation corresponding to the eigenvalue problem associated to the cKdV equation.[6]

Furthermore, performing the change of variables[7]

$$\xi=\tfrac{1}{2}xt^{-1/2}\ ,$$
$$\tau=-\tfrac{1}{8}t^{-1/2}\ , \tag{5.3}$$
$$v(\xi,\tau)=-\tfrac{2}{3}t^{1/2}(u-\xi)\ ,$$

Eq. (5.2) reads

$$\omega_{\xi\xi}+(\lambda^2-v)\omega=0\ , \tag{5.4}$$

where $\lambda^2=-\frac{2}{3}k$ and $v(\xi,\tau)$ satisfies the Korteweg-de Vries (KdV) equation

$$v_\tau-6vv_\xi+v_{\xi\xi\xi}=0\ . \tag{5.5}$$

A. The Hirota solution

Let us put $u=\frac{1}{2}xt^{-1/2}$ [a special solution of Eq. (2.1)] in Eq. (5.2). Then, after integrating the resulting equation, from (5.1) one obtains

$$\phi=(6k)^{1/2}\tanh\{(6k)^{-1/2}[kxt^{-1/2}+q(t)]\}\ , \tag{5.6}$$

where $k>0$.

Inserting (5.6) in (3.23) one has

$$q(t)=\tfrac{4}{3}k^2t^{-1/2}+m\ , \tag{5.7}$$

m being an integration constant.

Then Eq. (3.24) provides the following solution of the cKdV equation:

$$\psi(x,t;k)=\tfrac{1}{2}xt^{-1/2}+2kt^{-1/2}\text{sech}^2[(6k)^{-1/2}(kxt^{-1/2}+\tfrac{4}{3}k^2t^{-1/2}+m)]\ . \tag{5.8}$$

The expression (5.8) has been found also by Hirota through a different procedure.[7]

Since Eqs. (3.23) (for $u=\frac{1}{2}xt^{-1/2}$) are invariant under the replacement $\phi\to k/(6\phi)$, the cKdV equation affords also the solution

$$\tilde{\psi}(x,t;k)=\tfrac{1}{2}xt^{-1/2}-2kt^{-1/2}\text{csch}^2[(6k)^{-1/2}(kxt^{-1/2}+\tfrac{4}{3}k^2t^{-1/2}+m)]\ . \tag{5.9}$$

Now let us go back to the superposition formula (4.12). Choosing $\psi_1\equiv\psi(x,t;k_1)$ and $\psi_2\equiv\tilde{\psi}(x,t;k_2)$ as given, respectively, by (5.8) and (5.9), where $k_1<k_2$, we find a solution for the cKdV equation which corresponds, through the transformation (5.3), to the two-soliton solution of the KdV equation.[7,10] This procedure can be extended to build up solutions of the cKdV equation corresponding to multisoliton solutions of the KdV equation.

To conclude, we observe that starting from $u=\frac{1}{2}xt^{-1/2}$ and $k=0$, Eqs. (5.1), (5.2), (3.24), and (3.23b) provide the rational solution

$$\psi=\tfrac{1}{2}xt^{-1/2}-\frac{3t^{-1/2}}{(\frac{1}{2}xt^{-1/2}+c_0)^2}\ , \tag{5.10}$$

where c_0 is an arbitrary constant.

B. Solutions in terms of Airy functions

Setting $u=\bar{u}=$const (the trivial solution of the cKdV equation), Eq. (5.2) becomes the Airy equation.[11] Then, from (5.1) one has

$$\phi=(18)^{1/3}t^{1/6}\frac{\dot{\mathrm{Ai}}(s_k)+C\,\dot{\mathrm{Bi}}(s_k)}{\mathrm{Ai}(s_k)+C\,\mathrm{Bi}(s_k)}\,, \tag{5.11}$$

where $\mathrm{Ai}(s_k)$ and $\mathrm{Bi}(s_k)$ are two linearly independent Airy functions,[11] and $\dot{\mathrm{Ai}}$ and $\dot{\mathrm{Bi}}$ denote their derivatives carried out with respect to the variable $s_k=(12t)^{-1/3}(x+2k-2\bar{u}t^{1/2})$.

Substitution from (5.11) in (3.23b) (with $u=\bar{u}$) implies that C must be constant.

In virtue of (3.24), Eq. (5.11) yields

$$\psi=\bar{u}+(18)^{1/3}t^{1/6}s_k-(\tfrac{3}{2})^{1/3}t^{-1/6}\left[\frac{\dot{\mathrm{Ai}}(s_k)+C\,\dot{\mathrm{Bi}}(s_k)}{\mathrm{Ai}(s_k)+C\,\mathrm{Bi}(s_k)}\right]^2 . \tag{5.12}$$

For $\bar{u}=0$, (5.12) reproduces the solution reported in Ref. 12. Furthermore, using the permutability theorem of Sec. IV we are able to find a soliton solution of the Calogero-Degasperis (CD) type[6] for any value of the constant $\bar{u}$. Specifically, from (4.6) and (5.11) we can write

$$\Omega_k^{(k_1)}(s_k)=\Omega_0-2(18^{1/3}t^{1/6})\frac{\dot{\mathrm{Ai}}(s_k)+C(k-k_1)\dot{\mathrm{Bi}}(s_k)}{\mathrm{Ai}(s_k)+C(k-k_1)\mathrm{Bi}(s_k)}\,, \tag{5.13}$$

where $\Omega_{0x}=-\bar{u}$, and the replacement $\phi_1\to\phi$ has been performed.

Then, exploiting (4.11) where $\Omega_2\equiv\Omega_k^{(k_1)}(s_k)$ and $\Omega_1\equiv\Omega_{k_1}^{(k_1)}(s_{k_1})$, and taking the limit $k\to k_1$, we get

$$\Omega_1^{\mathrm{CD}}(s_{k_1})=\Omega_0+24\left[\left.\frac{\partial\Omega_k^{(k_1)}(s_k)}{\partial k}\right|_{k=k_1}\right]^{-1} \tag{5.14}$$

which yields the soliton solution of the CD type:

$$\psi^{\mathrm{CD}}=\bar{u}-2(12)^{1/3}t^{-1/6}\frac{\mathrm{Ai}(s_k)\dot{\mathrm{Ai}}(s_k)}{\dot{\mathrm{Ai}}^2(s_k)-s_k\mathrm{Ai}^2(s_k)+(C/\pi)(\tfrac{3}{2}t)^{1/3}}-(12)^{1/3}t^{-1/6}\frac{\mathrm{Ai}^4(s_k)}{\left[\dot{\mathrm{Ai}}^2(s_k)-s_k\mathrm{Ai}^2(s_k)+(C/\pi)(\tfrac{3}{2}t)^{1/3}\right]^2}\,. \tag{5.15}$$

This procedure is also followed in order to determine multisoliton solutions. This can be achieved applying the superposition formula (5.6) to the potentials $\Omega_{k_1}^{(k_1)}(s_{k_1})$, $\Omega_{k_{N+1}}^{(k_1)}(s_{k_{N+1}}),\ldots,$ $\Omega_{k_N}^{(k_N)}(s_{k_N})$, $\Omega_{k_{2N}}^{(k_N)}(s_{k_{2N}})$, and making the limits $k_{N+1}\to k_1,\ldots,k_{2N}\to k_N$. For example, the potential for the double-soliton solution reads

$$\Omega_{12}^{\mathrm{CD}}=\Omega_{12}-\frac{(k_1-k_2)^2(\Omega_{12}-\Omega_0)[(\Omega_{12}-\Omega_0)^2-(\Omega_{12}-\Omega_0)(\Omega_1^{\mathrm{CD}}+\Omega_2^{\mathrm{CD}})+(\Omega_1^{\mathrm{CD}}-\Omega_0)(\Omega_2^{\mathrm{CD}}-\Omega_0)]}{(k_1-k_2)^2(\Omega_1^{\mathrm{CD}}-\Omega_0)(\Omega_2^{\mathrm{CD}}-\Omega_0)+k_1k_2(\Omega_{12}-\Omega_0)^2}\,, \tag{5.16}$$

where

$$\Omega_{12}=\Omega_0+24\frac{k_2-k_1}{\Omega_{k_2}^{(k_2)}(s_{k_2})-\Omega_{k_1}^{(k_1)}(s_{k_1})}\,. \tag{5.17}$$

C. Solutions in terms of combinations of hyperbolic and rational functions

Let us put in (5.2) the expression (5.10). After some manipulation, Eq. (5.2) becomes

$$\omega_{zz}-\left[\frac{2}{z^2}+\frac{1}{4}\right]\omega=0\,, \tag{5.18}$$

where

$$z=(\tfrac{8}{3}k)^{1/2}(\tfrac{1}{2}xt^{-1/2}+c_0)\ . \tag{5.19}$$

Equation (5.18) is a special case of the Whittaker equation[11] whose general solution can be written as

$$\omega(x,t)=C_1(t)\left[\cosh\tfrac{1}{2}z-\frac{2}{z}\sinh\tfrac{1}{2}z\right]+C_2(t)\left[\frac{2}{z}\cosh\tfrac{1}{2}z-\sinh\tfrac{1}{2}z\right], \tag{5.20}$$

where C_1 and C_2 are functions of integration.

Substituting (5.20) in (5.1), we find

$$\phi=(6k)^{1/2}\left[-\frac{2}{z}+\frac{C\sinh\frac{1}{2}z-\cosh\frac{1}{2}z}{C[\cosh\frac{1}{2}z-(2/z)\sinh\frac{1}{2}z]+(2/z)\cosh\frac{1}{2}z-\sinh\frac{1}{2}z}\right], \tag{5.21}$$

where $C\equiv C_1/C_2$ can be determined from (3.23b).

Using now (3.24) and (5.21), we finally obtain the solution

$$\psi=\tfrac{1}{2}xt^{-1/2}-2kt^{-1/2}\frac{1+\frac{1}{4}z^2\text{sech}^2\tilde{z}}{(1-\frac{1}{2}z\tanh\tilde{z})^2}\ , \tag{5.22}$$

where $\tilde{z}=(6k)^{-1/2}(kxt^{-1/2}+\frac{4}{3}k^2t^{-1/2}+\alpha_0)$ and α_0 is an arbitrary constant.

VI. CONCLUDING REMARKS

The prolongation calculation carried out in this paper is a notable example of the correspondence between integrable equations, such as the cKdV equation,[6] and non-Abelian prolongations. Although there is at present no general theory of this correspondence, the accumulation of cases could be important for indicating the way to build up such a theory. Concerning this, we observe that in dealing with a specific prolongation example, we had the opportunity to develop a straightforward procedure for the practical use of the non-Abelian Lie algebra associated to a nonlinear evolution equation, which avoids the use of representations at all (see also Ref. 2).

Many problems remain to be tackled, such as for instance the role (if any) played by the dimension of the non-Abelian Lie algebra in connection with the complete integrability of the nonlinear evolution equation to which this algebra is associated. A more general work around this problem is in progress.

APPENDIX A

In order to derive an expression for $B_3(x,t)$ which does not contain the unknown functions D_i, let us differentiate Eq. (3.13) with respect to t. With the help of (2.28), we obtain

$$[\ln|R(\varphi-2\beta)|]_t=-\gamma_1+\delta_2+2D_3C_2\ . \tag{A1}$$

On the other hand, from the constraint (2.24) one has

$$-\frac{A_{3t}}{1+A_3}=\delta_2+D_3C_2+2A_{1xx}-A_1{}^3-\tfrac{2}{3}t^{-1/2}A_1B_3+\tfrac{2}{3}t^{-1/2}B_{3x}-(2A_1+B_2)\tfrac{1}{3}t^{-1/2}(1+A_3)R(\varphi-2\beta)$$
$$+\tfrac{1}{3}t^{-1/2}(1+A_3)R_x(\varphi-2\beta)-\tfrac{2}{3}t^{-1/2}(1+A_3)R\beta_x\ . \tag{A2}$$

Since [see (2.19)]

$$-\frac{A_{3t}}{1+A_3}=\tfrac{1}{2}t^{-1}+\frac{C_{2t}}{C_2}\ , \tag{A3}$$

comparing (A2) with (A3) and taking into account (2.28) we get

$$\tfrac{1}{2}t^{-1}+\frac{C_{2t}}{C_2}=\frac{C_{1t}}{2C_1}+2A_{1xx}-A_1{}^3-\tfrac{2}{3}t^{-1/2}A_1B_3+\tfrac{2}{3}t^{-1/2}B_{3x}-(2A_1+B_2)\tfrac{1}{3}t^{-1/2}(1+A_3)R(\varphi-2\beta)$$
$$+\tfrac{1}{3}t^{-1/2}(1+A_3)R_x(\varphi-2\beta)-\tfrac{2}{3}t^{-1/2}(1+A_3)R\beta_x\ . \tag{A4}$$

From (3.13) we have

$$\frac{C_{1t}}{C_1}=\frac{C_{2t}}{C_2}+[\ln|R(\varphi-2\beta)|]_t\ . \tag{A5}$$

Having in mind (2.29), substituting (A5) in (A4) we obtain

$$[\ln|R(\varphi-2\beta)|]_t=t^{-1}+\gamma_1+\delta_2-2[2A_{1xx}-A_1{}^3-\tfrac{2}{3}t^{-1/2}A_1B_3+\tfrac{2}{3}t^{-1/2}B_{3x}-(2A_1+B_2)\tfrac{1}{3}t^{-1/2}(1+A_3)$$
$$\times R(\varphi-2\beta)+\tfrac{1}{3}t^{-1/2}(1+A_3)R_x(\varphi-2\beta)-\tfrac{2}{3}t^{-1/2}(1+A_3)R\beta_x]\ . \tag{A6}$$

The comparison between (A6) and (A1) yields

$$B_{3x}=-t^{1/2}(A_{1xx}+\tfrac{3}{2}A_1{}^2+2A_{1x})_x+\tfrac{1}{3}t^{-1/2}\frac{R}{C_2}\beta_x+\tfrac{1}{2}t^{-1/2}+\tfrac{1}{3}t^{-1/2}(\varphi-2\beta)\frac{R}{C_2}A_1\ , \tag{A7}$$

where (2.30) has been exploited.

Integrating Eq. (A7) by parts, we finally obtain

$$B_3=-t^{1/2}(A_{1xx}+\tfrac{3}{2}A_1{}^2+2A_{1x})+\tfrac{1}{2}xt^{-1/2}-\tfrac{1}{6}t^{-1/2}\frac{R}{C_2}(\varphi-2\beta)+\mu\ , \tag{A8}$$

where $\mu\equiv\mu(t)$ is an integration function.

APPENDIX B

Here we shall derive Eq. (3.16). To this end, we need to express the function $D_1+\delta_2$ appearing in (3.7b) first as a quantity which does not contain either the functions D_i or their derivatives explicitly.

In doing so, from (2.30) and (2.24) one has

$$-(D_1+\delta_3)=3t^{1/2}(A_{1xx}+3A_1A_{1x}+A_1{}^3)_x-\tfrac{1}{3}t^{-1/2}\left[\frac{R}{C_2}\right]_x\beta_x-\frac{R}{C_2}\beta_{xx}\tfrac{1}{3}t^{-1/2}+B_{3xx}+2B_{3x}A_1+2B_3A_{1_x}$$
$$+3t^{1/2}A_{1_t}+B_3\left[2A_{1x}+A_1{}^2+\tfrac{2}{3}t^{-1/2}B_3+\tfrac{1}{9}t^{-1}\frac{R}{C_2}(\varphi-2\beta)\right]\ . \tag{B1}$$

Using now (A8), (3.11), and (3.12), (B1) becomes

$$-(D_1+\delta_3)=-\tfrac{1}{6}t^{-1/2}(\varphi-2\beta)\frac{R}{C_2}[\tfrac{2}{3}(A_{1x}-A_{1xx})+\tfrac{1}{3}xt^{-1}+\tfrac{2}{3}\mu t^{-1/2}]+E(x,t)\ , \tag{B2}$$

where

$$E(x,t)=t^{1/2}(A_{1x}-A_{1xx})_{xx}+2t^{1/2}A_1(A_{1x}-A_{1xx})_x+A_1t^{-1/2}+3t^{1/2}A_{1_t}$$
$$+\tfrac{2}{3}t^{1/2}(A_{1x}-A_{1xx}+\tfrac{1}{2}xt^{-1}+\mu t^{-1/2})(A_{1x}-A_{1xx}+\tfrac{1}{2}xt^{-1}+\mu t^{-1/2}-\tfrac{3}{2}A_1{}^2)\ . \tag{B3}$$

Furthermore, the quantities α and γ_3-D_2, defined, respectively, by (3.8) and (2.31), read

$$\alpha=\tfrac{1}{2}A_1{}^2+\tfrac{1}{3}(A_{1x}-A_{1xx}+\tfrac{1}{2}xt^{-1}+\mu t^{-1/2})+\tfrac{1}{18}\frac{R}{C_2}t^{-1}(\varphi-2\beta)\ , \tag{B4}$$

and

$$\gamma_3-D_2=-\tfrac{2}{3}(1+A_3)(A_{1x}-A_{1xx}+\tfrac{1}{2}xt^{-1}+\mu t^{-1/2}) . \tag{B5}$$

Now, from (A2) with the help of (A3) we can find an expression for $\delta_2+D_3C_2$ where the functions D_i and their derivatives do not appear. Substituting such an expression in (A1) and taking account of (2.30) and (A8), we are led to the relation

$$[\ln|(R/C_2)(\varphi-2\beta)|]_t=\tfrac{1}{3}t^{-1}+X(x,t) , \tag{B6}$$

where

$$X(x,t)=\tfrac{4}{3}[(A_{1xx}-A_{1x})_x+A_1(A_{1x}-A_{1xx}+\tfrac{1}{2}xt^{-1}+\mu t^{-1/2})] . \tag{B7}$$

Finally, using (B2), (B4), (B5), (B6), (3.13), and (2.19), from (3.7b) we obtain the Eq. (3.16).

APPENDIX C

As we have seen in Sec. II in the case of pseudopotential of the first kind one can associate to the cKdV equation the non-Abelian Lie algebra (2.14). The generators b and c of this algebra depend on both the pseudopotential and the variables x and t through Eqs. (2.10), (2.11), (2.17), and (2.18), which could be tackled only if one were able to find the functions A_i, B_i, C_i, and D_i from the algebraic constraints (2.19)–(2.29). Fortunately, we do not need to know all these functions specifically. For example, we have shown that we can make practical use of the Bäcklund transformations (3.7) without knowing the quantities D_i.

As far as the functions C_i appearing in (2.14) are concerned, we can see that both C_1 and C_2 cannot be constants.

In fact, let us suppose that this is the case. Since we have

$$A_1=\alpha_1(t)+\alpha_2(t)e^x+(k-\mu t^{1/2})t^{-1} \tag{C1}$$

from (3.19) and (3.22), where α_1 and α_2 are functions of integration, with the help of (C1) one can obtain B_2, R, B_1, and β from (2.22), (3.10), (2.21), and (3.11), respectively.

Then, using (3.13) we deduce that $C_1=B_1=\beta=0$, which is not consistent with the condition (B6).

However, this result does not imply that the local structure of the algebra (2.14) is time dependent. For instance, let us put

$$\begin{aligned} b'&=\beta_1 b+\beta_2 c ,\\ c'&=\gamma_1 b+\gamma_2 c , \end{aligned} \tag{C2}$$

where $\beta_i\equiv\beta_i(x,t)$, $\gamma_i\equiv\gamma_i(x,t)$, and $\beta_1\gamma_2-\beta_2\gamma_1\neq 0$.

Inserting (C2) in (2.14) we obtain the SL(2,R) algebra:

$$\begin{aligned} [b',f']&=-b'-\lambda^2c' ,\\ [c',f']&=\lambda^2b'+c' ,\\ [c',b']&=f' , \end{aligned} \tag{C3}$$

where $f'\equiv C_2^{-1}(1-\lambda^2)$, λ is a constant such that $|\lambda|<1$, $\beta_2\neq 0$, and

$$\beta_1=(2C_2)^{-1}C_1^{1/2}(\beta_2^{-1}C_1^{-1/2}\lambda-\beta_2C^{1/2}) , \tag{C4}$$

$$\gamma_1=-\beta_1\left[\frac{1-(1-\lambda^2)^{1/2}}{1+(1+\lambda^2)^{1/2}}\right]^{1/2}-C_1C_2^{-1}\beta_1(1-\lambda^2)^{1/2}\lambda^{-1} , \tag{C5}$$

$$\gamma_2=-\beta_2\left[\frac{1+(1-\lambda^2)^{1/2}}{1-(1-\lambda^2)^{1/2}}\right]^{1/2} . \tag{C6}$$

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