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Creation of particles by shell-focusing singularities

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The creation of massless scalar particles in asymptotically flat spacetimes containing shell-focusing naked singularities which evolve from nonsingular initial data is studied. In the case where the singularity is marginally naked, i.e., its Cauchy horizon coincides with the event horizon, we are able to compute the spectrum of created particles by Hawking's method. The spectrum of particles is no longer thermal, but can be expressed as a quasithermal spectrum with a frequency-dependent temperature. In the high-frequency limit the effective temperature approaches a constant value greater than the Hawking temperature. In the more general case where the Cauchy horizon and event horizon do not coincide, we calculate the expectation value of the stress-energy tensor of the scalar field in the two-dimensional spacetimes obtained by suppressing the spherical coordinates. In all cases the energy flux along the Cauchy horizon diverges in a positive sense. This strongly suggests that the metric's back-reaction to the flux of created particles will prevent the formation of naked shell-focusing singularities.

I. INTRODUCTION

One of the principal goals of studying quantized matter fields in classical curved-space backgrounds ("semiclassical gravity") is to understand how quantization of matter fields affects the structure and existence of spacetime singularities. Since quantized matter fields need not obey the usual energy conditions,[1] it is conceivable that quantum effects can prevent the occurrence of some classically predicted singularities. One also expects that the effects of curved spacetime on the quantized matter field, e.g., particle creation and vacuum polarization, are strongest (and hence most interesting) in the neighborhood of spacetime curvature singularities.

A fundamental ambiguity in studying quantum field dynamics in a singular spacetime is the choice of quantum state at the spacetime singularity. There is no unique or even obviously preferable choice which resolves the ambiguity.[2] One way to partially circumvent this ambiguity is to study quantized fields in asymptotically flat spacetimes containing naked (either locally or globally) singularities which evolve from regular (nonsingular) initial data. Such spacetimes are, in some sense, counterexamples to the cosmic censorship hypothesis. The fact that only a few such spacetimes are known[3,4] is perhaps the main reason for believing the cosmic censorship hypothesis to be true. Presumably a true and correct form of the cosmic censorship hypothesis will exclude the presently known "counterexamples" as nongeneric, or having unphysical equations of state, etc.

Ford and Parker[5] have calculated the flux of created particles produced by a naked singularity of the "shell-crossing" sort. This sort of naked singularity, which can be made regular by treating it as a distribution (δ-function singularity) does not seem to produce a large flux of particles. Hiscock[6] has studied particle creation by the naked singularity associated with the end point of an evaporating black hole. These singularities are not usually regarded as counterexamples to cosmic censorship, since they owe their existence to quantum effects, while the cosmic censorship hypothesis is usually considered to be a mathematical hypothesis about the classical theory of general relativity with classical sources. Other work has studied the effect of black-hole evaporation on the locally naked singularities in the analytically extended interiors of the Reissner-Nordström[7] and Kerr-Newman[8] black-hole solutions.

In this paper we shall study particle creation in spacetimes containing "shell-focusing" singularities,[4] which are in some ways the strongest presently known counterexamples to the cosmic censorship hypothesis. Unlike the shell-crossing sort of singularity, shell-focusing singularities cannot be handled by defining them as distributions (δ-function singularities). They can, however, be eliminated from consideration by forbidding matter fields (such as dust) which form singularities even in a flat spacetime.

Since such a spacetime is nonsingular in the distant past, the existence of a complete past null infinity ($\mathscr{I}^-$) is guaranteed, and a quantized massless field may be expanded into a complete set of modes on $\mathscr{I}^-$. This is a sufficient condition to allow the computation of the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$ up to the Cauchy horizon, although such calculations are currently only practical in two-dimensional spacetimes[9] or highly symmetric four-dimensional spacetimes.

The Cauchy horizon associated with the formation of the singularity will generally intersect future null infinity ($\mathscr{I}^+$) at some finite value of the retarded time coordinate (asymptotically, $u \approx t - r$). Since the Cauchy horizon is an absolute barrier to prediction, there is no justification for even assuming that the totality of $\mathscr{I}^+$ exists (i.e., the entire range $\infty > u > -\infty$). It is thus generally improper in such spacetimes to assume that the quantized field may be expanded in terms of a complete set of basis functions on $\mathscr{I}^+$. Techniques such as Bogoliubov transformations may be used rigorously only in spacetimes where complete sets of basis functions are known to exist in the asymptotic past and future; there must be no Cauchy horizon intersecting $\mathscr{I}^+$.

To overcome this problem, we collapse a spherically symmetric cloud of matter in such a way that a curvature singularity forms inside and just on, but not outside, the absolute event horizon. Since the Cauchy horizon associated with the singularity then coincides with the event horizon, the existence of a complete future null infinity is guaranteed, and the problem of choosing boundary conditions on the singularity is avoided. The initial vacuum state on $\mathscr{I}^-$ can then be expanded into a complete set of modes on $\mathscr{I}^+$, yielding the created particle spectrum there exactly as in Hawking's original calculation of black-hole radiance.[10] The crucial difference between our work and Hawking's original calculation is that regions of arbitrarily large curvature are visible from $\mathscr{I}^+$ in our model spacetimes; the singularity formed in the collapsing matter is "marginally naked."

The quantum particle creation in this model is computable and finite as seen by distant observers, except in one special case. The spectrum is no longer thermal, due to two separate effects: (1) Wave packets scatter in the region of intense curvature very near the singularity. (2) Even neglecting such scattering, the geometrical optics approximation to wave propagation near the singularity differs from that in weakly curved spacetime. The resultant spectrum can be expressed as quasithermal, with a frequency-dependent temperature $T(\omega)$. If scattering near the singularity is neglected, the spectrum becomes purely thermal, but at a temperature greater than Hawking's value $T_H = 1/8\pi M$.

In the more general case where the shell-focusing singularity can also be locally naked (Cauchy horizon inside the event horizon) or globally naked (Cauchy horizon outside the event horizon), two-dimensional stress-energy tensor calculations show that the created energy flux always diverges (with positive energy density) on the Cauchy horizon. It thus seems likely that the gravitational back-reaction to this diverging flux will prevent the shell-focusing type of naked singuarlity from forming.

Our conclusions depend, of course, on our assumptions. Spherical symmetry is probably not a key assumption here. The most important assumption we make is that the gravitational collapse is self-similar, i.e., spacetime admits a homothetic Killing vector in some region around the point of formation of the event horizon, and that the Cauchy horizon of the singularity is also a surface where the homothetic Killing vector becomes null. Furthermore, in the case for which we compute particle creation at late times, namely, the case of a marginally naked singularity, we are also assuming that this same null surface is the absolute event horizon. We expect all of our results to be substantially different in non-self-similar collapse. Finally, we work entirely within the semiclassical approximation, and the results may be greatly different in some respects, e.g., for late-time particle creation, in a better approximation to full quantum gravity.

In Sec. II of this paper the properties of the model spacetimes containing shell-focusing singularities are detailed and discussed. In Sec. III the spectrum of created particles in the marginally naked case is calculated. Section IV derives the two-dimensional quantum stress-energy tensors in the more general case.

II. SPACETIMES WITH SHELL-FOCUSING SINGULARITIES

The model spacetimes we will consider in this paper are described by spherically symmetric, imploding null fluid Vaidya metrics[11]:

$$ds^2 = -\left[1 - \frac{2m(v)}{r}\right] dv^2 + 2dv\,dr + r^2 d\Omega^2 , \tag{1}$$

where $-\infty < v < \infty$, $0 \leq r < \infty$, and $d\Omega^2$ is the metric of the unit two-sphere. The mass is chosen as a function of advanced time to be

$$m(v) = \begin{cases} 0, & v < 0 \\ \mu v, & 0 < v < v_0 \\ M, & v > v_0 , \end{cases} \tag{2}$$

where M, v_0, and μ are constants. The mass of the final, Schwarzschild black hole is M. The causal structure of the spacetime depends on the relative values of the three constants M, v_0, and μ.

The causal structure of the model spacetimes is easily studied because of the linearity of the mass function $m(v)$ in the region $0 < v < v_0$. In this region the vector field ξ, defined by

$$\xi = v\frac{\partial}{\partial v} + r\frac{\partial}{\partial r} , \tag{3}$$

is a homothetic Killing vector, i.e., satisfies

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2g_{\mu\nu} . \tag{4}$$

This symmetry makes it possible to explicitly construct double-null coordinate systems in all three portions of the model spacetime. In the initial Minkowskian region ($v < 0$) we simply use the usual advanced and retarded time coordinates:

$$ds^2 = -dU\,dv + r^2 d\Omega^2 , \tag{5}$$

where $U = v - 2r$. In the null fluid region $v_0 > v > 0$, we first adopt new coordinates ζ and z, defined by

$$z = \frac{v}{r} , \tag{6}$$

$$\zeta = \ln v . \tag{7}$$

The metric in this region then takes the form

$$ds^2 = e^{2\zeta}\left[-\left[1 - \frac{2}{z} - 2\mu z\right] d\zeta^2 - \frac{2}{z^2} d\zeta\,dz + \frac{1}{z^2} d\Omega^2 \right] . \tag{8}$$

A double-null coordinate system is obtained by defining

$$\eta = \zeta - 2z^* , \tag{9}$$

$$dz^* = \frac{dz}{z(2\mu z^2 - z + 2)} , \tag{10}$$

yielding finally

$$ds^2 = e^{2\zeta}\left[-\left[1 - \frac{2}{z} - 2\mu z\right] d\zeta\,d\eta + \frac{1}{z^2} d\Omega^2 \right] , \tag{11}$$

where z is now to be considered an implicit function of ζ and η. The maximum analytic extension of the metric given by Eq. (11) (except $v < 0$) is illustrated in Fig. 1 for $0 < \mu < \frac{1}{16}$. The null surfaces defined by the zeros of $g_{\eta\zeta}$, at coordinate values

$$z = z_\pm = \frac{1}{4\mu}[1 \pm (1 - 16\mu)^{1/2}] , \tag{12}$$

are homothetic Killing horizons[12]; the homothetic Killing vector [Eq. (3)] becomes null there. The surfaces defined by $z = z_+$, $\zeta = -\infty$, and $z = \infty$ (ζ arbitrary) are curvature singularities. This is easily seen by examining the quantity

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48m^2(v)}{r^6} = 48\mu^2 z^6 e^{-4\zeta} . \tag{13}$$

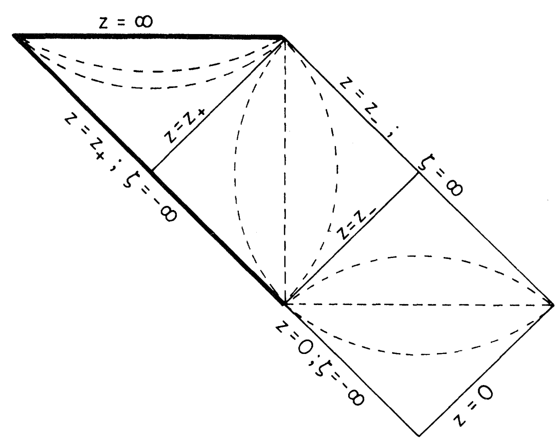

FIG. 1. Maximum analytic extension of the self-similar (homothetic) Vaidya metric ($0 < \mu < \frac{1}{16}$), except for region $m < 0$. Dashed lines indicate surfaces of constant z. Bold lines are curvature singularities.

The outer homothetic Killing horizon, at $z = z_-$ (ζ varies), is also the Cauchy horizon. When $\mu = \frac{1}{16}$, the two homothetic Killing horizons coalesce. For $\mu > \frac{1}{16}$, spacetime is globally hyperbolic; i.e., the singularity is not even locally naked.

In order to obtain a model spacetime which is asymptotically flat and vacuum in the distant future, we cut the homothetic Vaidya solution along a $v = v_0 =$constant (hence, ζ=constant) null surface and attach it to a portion of a Schwarzschild spacetime [see Fig. (2)]. The mass of the final Schwarzschild black hole (M) is chosen such that the Cauchy horizon coincides with the event horizon of the black hole. Quantitatively, this means that at $v = v_0$ and $r = 2M$ we have

$$z = z_- = \frac{v_0}{2M} \tag{14}$$

or

$$\frac{M}{v_0} = \frac{1+(1-16\mu)^{1/2}}{8} . \tag{15}$$

Note that for all values of μ ($0 < \mu \leq \frac{1}{16}$), Eq. (15) implies that

$$M > \mu v_0 . \tag{16}$$

This simply means that there is a δ-functional shell of null fluid at $v = v_0$, whose mass ($M - \mu v_0$) is chosen such that Eq. (15) is satisfied. The δ-functional nature of this shell does not affect our results. We could equally well have used a matching region in which the density smoothly rolls off to zero. If the mass of the shell is chosen to be less than the value determined by Eq. (15), then the Cauchy horizon lies outside the event horizon of

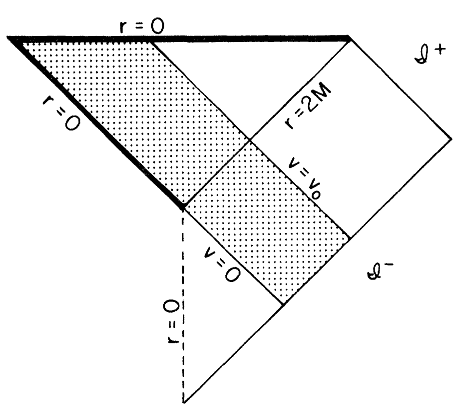

FIG. 2. Vaidya model spacetime of collapse to a marginally naked shell-focusing singularity. Spacetime to the past of $v=0$ is described by the Minkowski metric; to the future of $v=v_0$, by the Schwarzschild metric with mass M. The central region ($0 < v < v_0$) is the homothetic Vaidya null fluid.

the black hole, and a globally naked singularity results. If the mass is chosen to be greater than the critical mass, the Cauchy horizon lies inside the event horizon and the singularity is visible only to observers who fall into the black hole.

In the final Schwarzschild region we can of course adopt the usual double-null coordinates u and v, where

$$u = v - 2r^* , \tag{17}$$

$$r^* = r + 2M \ln \left[\frac{r}{2M} - 1 \right] \tag{18}$$

and the metric has its usual form

$$ds^2 = - \left[1 - \frac{2M}{r} \right] du\, dv + r^2 d\Omega^2 , \tag{19}$$

with r to be considered an implicit function of u and v.

It is also possible to create shell-focusing singularities in the collapse of inhomogeneous spherically symmetric dust clouds (the Tolman-Bondi solutions).[4,13,14] We have chosen to work with the Vaidya metric models in this paper for reasons of calculational simplicity. We do not believe our principal conclusion (divergence of the created flux) would be changed by using some other type of matter to form the shell-focusing singularity.

III. PARTICLE CREATION BY A MARGINALLY NAKED SHELL-FOCUSING SINGULARITY

There are two features of our model spacetime which seem to be important for the success of the particle creation calculation. First, the marginally naked character of the shell-focusing singularity guarantees that a complete future null infinity ($\mathscr{I}^+$) exists on which we can Fourier analyze the quantum field. If the singularity were globally naked, the existence of a complete $\mathscr{I}^+$ could not be assumed. If the singularity were only locally naked (Cauchy horizon inside the event horizon), then Hawking's original calculation[10] is sufficient and there is no new physics. Second, the existence of a homothetic Killing vector in the null fluid region ($0 < v < v_0$) makes it possible to separate the massless scalar wave equation,

$$\Phi_{,\alpha}{}^{;\alpha} = 0 \tag{20}$$

into ordinary differential equations in this region. The scalar wave equation of course also separates in the Schwarzschild ($v > v_0$) and Minkowski ($v < 0$) regions. It is then possible to match solutions of Eq. (20) across the boundaries at $v = 0$ and $v = v_0$, allowing one to calculate the spectrum of created particles at $\mathscr{I}^+$ by Hawking's method utilizing Bogoliubov transformations.

The decomposition of the scalar field near $\mathscr{I}^+$ and $\mathscr{I}^-$ is standard and follows the usual treatments.[10,15,16] Outgoing modes in continuum normalization near $\mathscr{I}^+$ have the form

$$p_{\omega lm} = N\omega^{-1/2} r^{-1} P_{\omega l}(r^*) e^{-i\omega u} Y_{lm}(\theta,\phi) , \quad (21)$$

where N is a normalization constant. As is well known, the radial functions $P_{\omega l}(r^*)$ approach constant values as $r^* \to \pm\infty$ (near infinity and the event horizon).

In the null fluid region the scalar wave equation separates in the (ζ, z) coordinate system defined by Eqs. (6)–(8). The solutions have the form

$$q_{\sigma lm} = \exp(\sigma\zeta/2) Q_{\sigma l}(z) Y_{lm}(\theta,\phi) \quad (22)$$

with the functions $Q_{\sigma l}(z)$ satisfying

$$zh(z) Q_{,zz} + (2\mu z^2 + \sigma) Q_{,z} + [l(l+1) - (\sigma/z)] Q = 0 \quad (23)$$

with

$$h(z) = (2\mu z^2 - z + 2)$$
$$= 2\mu(z - z_+)(z - z_-) . \quad (24)$$

Fortunately we shall only need to find solutions of Eq. (23) near the horizon ($z = z_-$) and near $z = 0$ (Minkowski space boundary and $\mathscr{I}^-$). Near $z = z_-$, the two independent solutions of Eq. (23) are approximately given by

$$Q(z) \simeq (z_- - z)^s \quad (25)$$

with

$$s = 0 \quad (26)$$

or

$$s = (2 + \sigma)/(4 - z_-) . \quad (27)$$

Near $z = 0$ the two independent solutions of Eq. (23) are approximately given by

$$Q(z) \simeq z^s , \quad (28)$$

with

$$s = 1 \quad (29)$$

or

$$s = -\sigma/2 . \quad (30)$$

Finally, near $\mathscr{I}^-$ (for all v) incoming modes have the form

$$f_{\omega' lm} = N(\omega')^{-1/2} r^{-1} F_{\omega' l}(r) e^{-i\omega' v} Y_{lm}(\theta,\phi) , \quad (31)$$

where, as before, the functions $F_{\omega' l}(r)$ tend to a constant value as $r \to \infty$ and N is a normalization constant.

The final outgoing wave can be expressed as an integral over the incoming modes (l,m indices are hereafter suppressed for notational simplicity)

$$p_\omega = \int_0^\infty (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} \bar{f}_{\omega'}) d\omega' . \quad (32)$$

Following the usual procedure,[10,15,16] one can set up annihilation and creation operators for ingoing particles at past null infinity and for outgoing particles at future null infinity, and find that the expectation value of the number operator for the outgoing mode of frequency ω is

$$\langle 0_- | N_\omega | 0_- \rangle = \int_0^\infty | \beta_{\omega\omega'} |^2 d\omega' , \quad (33)$$

where the initial state ($| 0_- \rangle$) is the usual vacuum state on $\mathscr{I}^-$. Thus, to obtain the created particle spectrum we must calculate the values of $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$.

There are two sorts of scattering contributions to $\alpha_{\omega\omega'}$. First, there will be some scattering in the static Schwarzschild region (just as in Hawking's calculation). Second, the waves will be scattered within the null fluid, especially near the curvature singularity. This scattering by the singularity is a new effect computable within our model spacetimes.

Let us now find the form of $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ by tracing a unit probability wave packet back from near $\mathscr{I}^+$ at late times and large values of r^*. Near $\mathscr{I}^+$ at late times the scalar field will have the approximate form

$$p_\omega \simeq N_\omega^{-1/2} r^{-1} e^{-i\omega u} , \quad (34)$$

where we have chosen N and $P_\omega(r^*)$ such that as $r^* \to +\infty$, $P_\omega(r^*) \to 1$. Tracing this wave back along the event horizon to retarded time $v = v_0$, we find that the scalar field on $v = v_0$ near the horizon is

$$p_\omega \simeq K_0 \omega^{-1/2} e^{2i\omega r^*} , \quad (35)$$

where K_0 is the product of the normalization constant, a scattering amplitude for the static Schwarzschild geometry and the asymptotic limit of the factor $r^{-1} P_\omega(r^*)$ as $r^* \to -\infty$ ($r \to 2M$). This segregation of factors is justified by the fact that $r^{-1}P_\omega(r^*)$ varies much more slowly near the horizon than does the exponential factor. Utilizing the definition of r^* [Eq. (16)] and again separating out those factors which vary slowly near the horizon (into a new constant, K_1), the scalar field at $v=v_0$ takes the form

$$p_\omega \simeq K_1\omega^{-1/2}\left[\frac{r}{2M}-1\right]^{4i\omega M} . \quad (36)$$

This must now be matched onto the solutions of the scalar wave equation in the null fluid region near the horizon, Eqs. (22) and (25)–(27). Transforming Eq. (36) into the (ζ,z) coordinate system, we obtain

$$p_\omega \simeq K_2\omega^{-1/2}(z_- - z)^{4i\omega M} \quad (37)$$

at $v=v_0$ near the horizon, and additional constant factors have been included in K_2. Comparing Eq. (37) with Eqs. (22) and (25)– (27), it is clear that the match at $v=v_0$ is onto a nearly pure solution of Eq. (23), with σ given by

$$(2+\sigma)/(4-z_-)=4i\omega M \quad (38)$$

or

$$\sigma=-2+4i\omega M(4-z_-) , \quad (39)$$

and with boundary conditions fixed by Eq. (26).

Near $v=0$, the initial boundary of the null fluid, a general solution of the scalar wave equation is given by Eqs. (22) and (28)–(30):

$$p_\omega \simeq Ae^{\sigma\zeta/2}z^{-\sigma/2}+Be^{\sigma\zeta/2}z , \quad (40)$$

or in (v,r) coordinates,

$$p_\omega \simeq Ar^{\sigma/2}+Bv^{1+\sigma/2}r^{-1} . \quad (41)$$

Tracing the wave described by Eq. (37) back to the neighborhood of $v=0$, and putting it in the form of Eq. (41), we find

$$p_\omega \simeq A_\omega K_2\omega^{-1/2}r^{-1}r^{i\omega\chi}+B_\omega K_2\omega^{-1/2}r^{-1}v^{i\omega\chi} , \quad (42)$$

where

$$\chi=2M(4-z_-) . \quad (43)$$

The first term in Eq. (42) represents an incoming wave at $v=0$ (the transmitted wave), while the second term is the portion of the wave scattered to $\mathscr{I}^-$ from near the singularity. A_ω and B_ω are transmission and scattering coefficients for the homothetic Vaidya metric, related by the conservation of probability,

$$|A_\omega|^2-|B_\omega|^2=1 . \quad (44)$$

The scattering amplitude B_ω decreases exponentially as a function of ω for sufficiently large ω.

Tracing the transmitted wave through $v=0$, then through $r=0$ and hence back to $\mathscr{I}^-$, the scalar field at $\mathscr{I}^-$ is found to be (for small positive and negative values of v)

$$p_\omega \simeq \begin{cases} A'_\omega K_2\omega^{-1/2}r^{-1}\exp(i\omega\chi\ln|v|), & v<0 \\ B_\omega K_2\omega^{-1/2}r^{-1}\exp(i\omega\chi\ln v), & v>0 \end{cases} \quad (45)$$

where

$$A'_\omega=A_\omega 2^{-i\omega\chi}(-1)^{l+1} , \quad (46)$$

so that $|A'_\omega|=|A_\omega|$.

The expression for p_ω for $v<0$ is very similar to Hawking's original result, especially if χ is rewritten as

$$\chi=4M\frac{2(1-16\mu)^{1/2}}{1+(1-16\mu)^{1/2}} . \quad (47)$$

The wave on $\mathscr{I}^-$ for $v>0$, which was scattered by the singularity, is a mirror image of the $v<0$ wave, except for a rescaling factor of B_ω/A'_ω.

The values of $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ can now be found by Fourier transforming Eq. (32) with respect to v and evaluating the resulting integrals using Eq. (45) for p_ω. The Fourier transform yields the equations

$$\alpha_{\omega\omega'}=r(\omega')^{1/2}\int_{-\infty}^{\infty}e^{i\omega' v}p_\omega dv , \quad (48)$$

$$\beta_{\omega\omega'}=r(\omega')^{1/2}\int_{-\infty}^{\infty}e^{-i\omega' v}p_\omega dv . \quad (49)$$

Replacing p_ω with the expressions of Eq. (45) and performing the integrations, we find

$$\alpha_{\omega\omega'}=K_2\left[\frac{\omega'}{\omega}\right]^{1/2}(A'_\omega I_1+B_\omega I_2) , \quad (50)$$

$$\beta_{\omega\omega'}=K_2\left[\frac{\omega'}{\omega}\right]^{1/2}(A'_\omega I_3+B_\omega I_4) , \quad (51)$$

where

$$I_1=\int_{-\infty}^{0}e^{i\omega' v}\exp(i\omega\chi\ln|v|)dv , \quad (52)$$

$$I_2=\int_{0}^{\infty}e^{i\omega' v}\exp(i\omega\chi\ln v)dv , \quad (53)$$

$$I_3=\int_{-\infty}^{0}e^{-i\omega' v}\exp(i\omega\chi\ln|v|)dv , \quad (54)$$

$$I_4=\int_{0}^{\infty}e^{-i\omega' v}\exp(i\omega\chi\ln v)dv . \quad (55)$$

Since, by Eq. (45), $p_\omega(v)=p_\omega(-v)$, it follows that [as a simple change of variable to $w=-v$ in Eqs.

(53) and (55) will show]

$$I_2=I_3 \tag{56}$$

and

$$I_4=I_1 \ . \tag{57}$$

The two integrals we are left to evaluate, I_1 and I_3, are precisely the Fourier transforms evaluated by Hawking originally.[10] We thus find that $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are given by

$$\alpha_{\omega\omega'}=f(-A'_\omega e^{\pi\omega\chi/2}+B_\omega e^{-\pi\omega\chi/2}) \ , \tag{58}$$

$$\beta_{\omega\omega'}=f(A'_\omega e^{-\pi\omega\chi/2}-B_\omega e^{\pi\omega\chi/2}) \ , \tag{59}$$

where

$$f=i\left[\frac{\omega'}{\omega}\right]^{1/2}K_2\Gamma(1+i\omega\chi) \times\exp[(-1-i\omega\chi)\ln(\omega')] \ . \tag{60}$$

It is now possible to obtain the number spectrum of created particles by relating $\alpha_{\omega\omega'}$ to $\beta_{\omega\omega'}$. Note that in the limit of no scattering by the singularity ($B_\omega \rightarrow 0$),

$$|\alpha_{\omega\omega'}|^2=e^{2\pi\omega\chi}|\beta_{\omega\omega'}|^2 \ , \tag{61}$$

which is characteristic of a thermal spectrum with positive temperature $kT=(2\pi\chi)^{-1}$. In the opposite limit of zero transmittance ($A'_\omega \rightarrow 0$), where the wave is perfectly reflected by the singularity, we have

$$|\alpha_{\omega\omega'}|^2=e^{-2\pi\omega\chi}|\beta_{\omega\omega'}|^2 \ , \tag{62}$$

which is characteristic of a thermal spectrum with *negative* temperature $kT=-(2\pi\chi)^{-1}$. However this limit is never approached, due to Eq. (44), and the positive temperature always dominates.

In the general case, Eqs. (58) and (59) imply that

$$|\alpha_{\omega\omega'}|^2=\frac{|A'_\omega e^{\pi\omega\chi/2}-B_\omega e^{-\pi\omega\chi/2}|^2}{|A'_\omega e^{-\pi\omega\chi/2}-B_\omega e^{\pi\omega\chi/2}|^2}|\beta_{\omega\omega'}|^2 \ . \tag{63}$$

Conservation of probability may be expressed as

$$\int_0^\infty (|\alpha_{\omega\omega'}|^2-|\beta_{\omega\omega'}|^2)d\omega'=\Gamma_\omega \ , \tag{64}$$

where Γ_ω is the transmission coefficient for a wave of frequency ω propagating backward from $\mathscr{I}^+$ through the static Schwarschild geometry and the low-curvature portion of the Vaidya null fluid region. If we now combine Eqs. (33), (44), (46), (63), and (64) we can compute the spectrum of created particles:

$$N_\omega=\frac{\Gamma_\omega}{e^{2\pi\omega\chi}-1}|1-C_\omega e^{\pi\omega\chi}|^2 \ , \tag{65}$$

where $C_\omega=B_\omega/A'_\omega$, with $|C_\omega|<1$ by Eq. (44). We can further define a frequency-dependent effective temperature $T(\omega)$ which gives rise to the same spectrum,

$$T(\omega)=\omega/\ln[1+(e^{2\pi\omega\chi}-1)|1-C_\omega e^{\pi\omega\chi}|^{-2}] \ , \tag{66}$$

so that

$$N_\omega \equiv \Gamma_\omega/(e^{\omega/T(\omega)}-1) \ . \tag{67}$$

Equations (65)–(67) are the main results of this section. In the semiclassical approximation, the emission of particles from the black hole at late times is substantially altered by the marginally naked singularity that is present at the formation of the event horizon. Even when scattering near the singularity is neglected ($C_\omega=0$), the temperature involved in our calculated spectrum is shifted from the Hawking temperature by a constant factor:

$$kT=(2\pi\chi)^{-1} =kT_H\left[\frac{1+(1-16\mu)^{1/2}}{2(1-16\mu)^{1/2}}\right] \ . \tag{68}$$

There are several interesting features in this constant factor. Note that T diverges in the limit $\mu\rightarrow\frac{1}{16}$. This limit corresponds to the two homothesy horizons coalescing (see Sec. II). The temperature of our model spacetime is always greater than the Hawking temperature for all acceptable values of μ, if scattering is neglected ($C_\omega=0$). However, when scattering is included, $T(\omega)$ may be either greater or less than T_H. This preservation of the thermal character of the spectrum (up to scaling the temperature) is connected with the homothesy of our model. A similar scaling, and the lack of a thermal spectrum for nonhomothetic Vaidya metrics, was found in constructing models of evaporating black holes.[6]

It seems quite likely that the shift in temperature is associated with a divergent flux of created particles along the horizon. In two dimensions, it is known that only the Hawking temperature gives a finite stress-energy tensor on the event horizon.[9] In four dimensions the situation is complicated by scattering, but it seems reasonable to expect a divergent flux on the horizon as seen by an observer freely falling into the black hole. As we shall see, this view is supported by the two-dimensional stress-energy tensor calculations presented in the next section.

We also remark that N_ω cuts off exponentially at large ω, so that the total, frequency-integrated emission per unit time from the black hole is finite. The scattering amplitude B_ω can be estimated in the limit $\omega \rightarrow \infty$ in the well-known Born approximation as

$$B_\omega \cong C_\omega \sim e^{-2\pi\omega X/(4-z_-)} = e^{-4\pi\omega M} \tag{69}$$

up to possible power-law corrections. Since $0 < X < 4M$ [cf. Eq. (47)] for $\frac{1}{16} > \mu > 0$, Eq. (66) becomes

$$T(\omega) \cong (2\pi X)^{-1}$$

as $\omega \rightarrow \infty$. In this limit, scattering near the singularity becomes negligible and the spectrum becomes thermal. However the temperature still differs from the Hawking temperature T_H [cf. Eq. (68)].

The particle emission at late times can further be described in terms of correlations or lack of correlations among emission of various numbers of particles in various different modes. For black holes, Wald,[17] Parker,[18] and Hawking[19] showed that such correlations were completely absent, just as for blackbody radiation. In our model spacetime with marginally naked singularity, correlations can be determined using the methods of Refs. 17 and 19. We shall not give the argument, because the result is exactly the same: There is no correlation of any sort. Therefore, for measurements restricted to a single frequency ω, the late-time emission has all the statistical properties of thermal radiation at temperature $T(\omega)$. For measurements involving different frequencies, the emission can in general be distinguished from thermal radiation by the form of N_ω, but there are still no correlations present; the density matrix is diagonal in Fock space.

IV. THE GENERAL CASE IN TWO DIMENSIONS

In the more general case where the Cauchy horizon and event horizon do not coincide, the shell-focusing singularity is either globally naked or locally naked. In both these cases, it would still be desirable to obtain information concerning created particle fluxes along the Cauchy horizon. This can be achieved if we study the two-dimensional model spacetimes obtained by setting $d\Omega^2=0$ in Eq. (1). We can then compute the expectation value of the stress-energy tensor of a massless scalar field in the two-dimensional spacetime by standard techniques.[9,20]

Since all two-dimensional spacetimes are conformally flat, any two-dimensional spacetime metric can be put into the double-null form

$$ds^2 = C^2(\bar{u},\bar{v}) d\bar{u}\, d\bar{v} \; . \tag{70}$$

The null coordinates $\bar{u},\bar{v}$ are chosen so that the inital quantum state of the scalar field (which we choose to be the usual vacuum state near $\mathscr{I}^-$) is in fact the vacuum state of the normal modes in $\bar{u},\bar{v}$ coordinates. The in-vacuum state is then the state annhilated by the field operators with $\omega > 0$. The expectation value of the stress-energy tensor of the quantized scalar field is then[9]

$$T_{\bar{u}\bar{u}} = (12\pi)^{-1} C (C^{-1})_{,\bar{u}\bar{u}} \; , \tag{71}$$

$$T_{\bar{v}\bar{v}} = (12\pi)^{-1} C (C^{-1})_{,\bar{v}\bar{v}} \; , \tag{72}$$

$$T_{\bar{u}\bar{v}} = \mathscr{R} C^2 / 96\pi \; , \tag{73}$$

where $\mathscr{R}$ is the two-dimensional scalar curvature.

Since our model spacetimes are past asymptotically flat, scalar field modes will have the form $\exp(-i\omega v)$ near past null infinity. This gives the relation

$$v = \bar{v} \tag{74}$$

valid everywhere in the spacetime. The usual reflection boundary condition at $r=0$ with $v<0$, combined with Eq. (74) yields

$$\bar{u} = U \tag{75}$$

for all $v<0$. Equations (74) and (75) immediately tell us that

$$\langle T_{\mu\nu} \rangle = 0 \quad (v<0) \; , \tag{76}$$

i.e., that the initial vacuum state remains the vacuum state so long as spacetime remains Minkowskian.

We must now relate the null coordinate $\bar{u}$ to the null coordinate η in the null fluid region $0<v<v_0$. At $v=0$,

$$\bar{u} = U - 2r \; . \tag{77}$$

From Eqs. (9) and (10), we find

$$\eta=\zeta-z^*=\ln r+\tfrac{1}{2}\ln(2\mu)+\frac{2\mu z_+}{(1-16\mu)^{1/2}}\ln(z-z_+)-\frac{2\mu z_-}{(1-16\mu)^{1/2}}\ln(z-z_-)\ , \tag{78}$$

which, evaluated at $v=z=0$, becomes

$$\eta=\ln r+\alpha\ , \tag{79}$$

where α is a function only of μ. Equations (77) and (79) then imply that

$$\bar{u}=-2e^{-\alpha}e^{\eta}\ . \tag{80}$$

We can now calculate $\langle T_{\mu\nu}\rangle$ for the null fluid region $0<v<v_0$ by combining Eqs. (7), (11), and (80) to obtain the metric form

$$ds^2=-\left[1-\frac{2}{z}-2\mu z\right]\frac{\bar{v}}{\bar{u}}d\bar{u}\,d\bar{v}\ , \tag{81}$$

where $z=\bar{v}/r$ and $r=r(\bar{u},\bar{v})$. Computing $\langle T_{\mu\nu}\rangle$ by the prescription of Eqs. (71)–(73) and transforming back to ζ,η coordinates we find

$$T_{\eta\eta}=\frac{\mu z^2}{24\pi}(3-z+\tfrac{3}{2}\mu z^2)\ , \tag{82}$$

$$T_{\zeta\zeta}=\frac{\mu z^2}{24\pi}(1-z+\tfrac{3}{2}\mu z^2)\ , \tag{83}$$

$$T_{\zeta\eta}=\frac{\mu z^2}{24\pi}(2-z+2\mu z^2)\ . \tag{84}$$

Since the η,ζ coordinate system is singular on the Cauchy horizon at $z=z_-$, we must construct a new coordinate system which is regular there to determine whether the created stress-energy flux diverges or not. This is accomplished by creating new coordinates

$$\bar{\eta}=e^{\tau\eta},\quad \bar{\zeta}=e^{-\tau\zeta}\ , \tag{85}$$

where

$$\tau=1-z_-/z_+\ . \tag{86}$$

The metric is regular at $z=z_-$ in the $\bar{\eta},\bar{\zeta}$ coordinate system. Equations (82), (85), and (86) may now be combined to find that

$$T_{\bar{\eta}\bar{\eta}}\simeq D(z-z_-)^{-2}+E(z-z_-)^{-1}+\cdots \tag{87}$$

for $(z-z_-)$ small and positive. D is a positive function of μ, nonzero for all $\frac{1}{16}\geq\mu>0$. Thus the stress energy of the outgoing created particles diverges (in a positive fashion) on the Cauchy horizon for all acceptable values of μ. Note that thus far in our analysis of $\langle T_{\mu\nu}\rangle$ we have not specified the value of M/v_0 [see Eq. (15)]; we have thus far only computed the stress-energy tensor in the null fluid region and shown that it diverges on the Cauchy horizon. This much of the analysis is independent of whether the singularity is locally, marginally, or globally naked.

To find the stress-energy tensor in the final Schwarzschild region $v>v_0$, we must relate $\bar{u}$ to the Schwarzschild null coordinate u defined in Eqs. (15) and (16). In order to encompass the general case (not force the singularity to be marginally naked), we do not require M to have the value given by Eq. (15); any value of M greater than or equal to μv_0 is acceptable.

At the boundary $v=v_0$ we have

$$u=v-2r^*=v_0-2r-4M\ln\left|\frac{r}{2M}-1\right| \tag{88}$$

and

$$\eta=\tfrac{1}{2}\ln(2\mu v_0{}^2-v_0r+2r^2)+\tfrac{1}{2}(1-16\mu)^{-1/2}\ln\left[\frac{v_0-z_-r}{v_0-z_+r}\right]\ . \tag{89}$$

Combining these equations with Eq. (65) we find

$$\frac{d\bar{u}}{du}=-\bar{u}\left[\frac{R-2M}{2\mu v_0{}^2-Rv_0+2R^2}\right]\ , \tag{90}$$

where $R=r(v=v_0,u)$.

It is now possible, using Eqs. (90) and (74), to write the Schwarzschild metric [Eq. (17)] in the canonical $\bar{u},\bar{v}$ coordinate system, and hence to calculate $\langle T_{\mu\nu}\rangle$ for the region $v>v_0$. Two of the three components are simple and unchanged from the ordinary black-hole case[9]:

$$T_{vv}=(24\pi)^{-1}\left[-\frac{M}{r^3}+\frac{3}{2}\frac{M^2}{r^4}\right]\ , \tag{91}$$

$$T_{uv}=-(24\pi)^{-1}\frac{M}{r^3}\left[1-\frac{2M}{r}\right]\ . \tag{92}$$

The third component T_{uu} is a very large and messy expression containing 18 terms. It simplifies considerably if M is chosen to satisfy Eq. (15) (i.e., the marginally naked singularity case); then terms which die off for large values of u may be ignored, and one finds that

$$T_{uu}=(24\pi)^{-1}\left[-\frac{M}{r^3}+\frac{3M^2}{2r^4}+\frac{1}{32M^2}\left[\frac{1+(1-16\mu)^{1/2}}{2(1-16\mu)^{1/2}}\right]^2\right]. \tag{93}$$

This confirms and reproduces our result from Sec. III. The constant term in Eq. (93) appears at large r as an outward-going flux of radiation. The magnitude of the constant term is precisely $(\pi/12)(kT)^2$, where T is the altered Hawking temperature of the shell-focusing singularity derived in Sec. III [Eq. (68)]. The effects of the frequency-dependent temperature in Eq. (66) are not evident here, as there is no scattering in two dimensions ($B_\omega=0$).

In the general case [M not given by Eq. (15)], T_{uu} assumes a fairly simple from only as one approaches the Cauchy horizon; it diverges there, with leading terms of the form

$$T_{uu}\simeq(48\pi)^{-1}\left\{\frac{M^2z_-^{\,2}(z_- -v_0/2M)^2}{4v_0^{\,4}\mu^2(1-16\mu)}[4-(4-z_-)^2]\right\}(z-z_-)^{-2}, \tag{94}$$

where, within this equation, the variable z is restricted to $v=v_0$ and treated as an implicit function of u, i.e., $z=v_0/R$, $R=r(v_0,u)$. Thus, as $z\to z_-$, u approaches its value on the Cauchy horizon,

$$u_c=v_0-\frac{2v_0}{z_-}-4M\ln\left|\frac{v_0}{2Mz_-}-1\right| \tag{95}$$

and T_{uu} diverges along the Cauchy horizon for all values of M, v_0, and μ. The sign of the divergence in T_{uu}, Eq. (94), is determined by the sign of the factor $[4-(4-z_-)^2]$. Since z_- ranges from 2 to 4 as μ ranges from zero to $\frac{1}{16}$, T_{uu} is always positive for all $u<u_c$.

We thus see that the stress-energy tensor of a quantized massless scalar field always diverges along the Cauchy horizon of the shell-focusing singularity in two dimensions. This strongly suggests that the back-reaction of the metric to this stress energy would prevent the formation of the shell-focusing singularity in the real world of four dimensions.

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