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Linearized quadrupole waves in general relativity and the motion of test particles

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We write out the explicit form of the metric for a linearized quadrupole gravitational wave in the transverse-traceless gauge. We give a collection of formulas which are useful for testing numerical codes to integrate the full nonlinear Einstein equations. We also show that a test particle initially at rest acquires no energy from a linearized gravitational wave.

I. INTRODUCTION

Progress in general relativity is hampered by the great difficulty in solving the field equations for realistic situations of interest. Insight has often been gained by solving model problems and inferring general results from these idealized cases. Recently, large-scale computer calculations have been used to attack problems not amenable to analytic treatment.[1] Even in such cases, however, one is handicapped by the lack of analytic results to test the computer codes.

In this paper we give the details of a simple analytic solution which has been used to explore several questions in numerical relativity.[2] The solution is simply the well-known quadrupole wave of linearized theory in spherical coordinates. As far as I can determine, no one has ever bothered to write out the metric as an explicit function of the coordinates t, r, θ, and ϕ in the transverse-traceless (TT) gauge.[3]

Besides testing numerical relativity, the metric can also be used in a model calculation of an unsettled question in general relativity theory: the interaction of a strong gravitational wave with matter. It has sometimes been speculated that a strong outgoing burst of gravitational waves, for example produced by gravitational collapse, might interact with surrounding matter and accelerate it. To answer this question properly requires a reliable calculation in full nonlinear general relativity theory. We show in this paper that a free particle initially at rest remains at rest when interacting with a linearized wave (whether quadrupole or not). We discuss why previous results using plane waves are ambiguous. This result means that a particle can absorb energy from the wave only if (i) it is interacting with other particles, or (ii) it has nonzero energy initially, or (iii) the wave is strong so that nonlinear effects are important. Whether any of these three possibilities could be important remains an open question.

II. THE QUADRUPOLE WAVE

In linearized theory, one writes the metric $g_{\alpha\beta}$ as

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}\ , \tag{1}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and $h_{\alpha\beta}$ is a small perturbation. To linear order, indices on $h_{\alpha\beta}$ can be raised and lowered with $\eta_{\alpha\beta}$.

A very convenient gauge for studying linearized gravitational waves is the TT (transverse-traceless) gauge.[4] In this gauge, the perturbation is a purely spatial symmetric tensor $\underline{h}$, with components h_{ij}, satisfying

$$\nabla\cdot\underline{h}=0,\ \text{ i.e., } h^{ij}{}_{|j}=0\ , \tag{2}$$

$$\mathrm{Tr}\underline{h}=0,\ \text{ i.e., } h^{i}{}_{i}=0\ . \tag{3}$$

Here the vertical bar denotes a covariant derivative in ordinary Euclidean space, where we will be using spherical coordinates. The TT gauge is analogous to the Lorentz gauge condition of electromagnetism with the scalar potential set to zero in empty space.

The linearized Einstein equations in vacuum become in the TT gauge

$$\Box\underline{h}=0,\ \text{ i.e., } h_{ij\,|k}{}^{k}=0\ . \tag{4}$$

The symmetric tensor $\underline{h}$ would in general have six independent components. However, the four conditions (2) and (3) reduce this number to two. We can identify these as two independent polarization states, or equivalently as even- and odd-parity waves.

General solutions of Eqs. (2)–(4) are easy to write down, e.g., as a superposition of plane waves in Cartesian coordinates. In spherical coordinates solutions can be written down as multipole expansions using tensor spherical harmonics.[5] The solution of most interest is the quadrupole wave, since quadrupole radiation is likely to be the strongest mode from realistic sources.[4]

Since we want to present the metric as an explicit function of time, we cannot Fourier transform with respect to t. Accordingly, the methods in the Appendix of Ref. 6 are the most useful for us. Quadrupole solutions of Eqs. (3) and (4) have total angular momentum $J=2$ and "orbital" angular momentum $L=0,1,2,3$, and 4. Following Mathews,[7] one can construct a linear combination of $L=0,2$, and 4 ("even parity") to also satisfy Eq. (2). Similarly, there is an odd-parity solution made up of $L=1$ and 3. For each parity, there are five independent modes corresponding to azimuthal quantum number $M=\pm 2,\pm 1,0$. The explicit form of the even-parity metric is

$$\begin{aligned} ds^2 = &-dt^2+(1+Af_{rr})dr^2+(2Bf_{r\theta})r\,dr\,d\theta+(2Bf_{r\phi})r\sin\theta\,dr\,d\phi+(1+Cf^{(1)}_{\theta\theta}+Af^{(2)}_{\theta\theta})r^2d\theta^2 \\ &+[2(A-2C)f_{\theta\phi}]r^2\sin\theta\,d\theta\,d\phi+(1+Cf^{(1)}_{\phi\phi}+Af^{(2)}_{\phi\phi})r^2\sin^2\theta\,d\phi^2 \,. \end{aligned} \tag{5}$$

Here

$$\begin{aligned} A&=3\left[\frac{F^{(2)}}{r^3}+\frac{3F^{(1)}}{r^4}+\frac{3F}{r^5}\right], \\ B&=-\left[\frac{F^{(3)}}{r^2}+\frac{3F^{(2)}}{r^3}+\frac{6F^{(1)}}{r^4}+\frac{6F}{r^5}\right], \\ C&=\frac{1}{4}\left[\frac{F^{(4)}}{r}+\frac{2F^{(3)}}{r^2}+\frac{9F^{(2)}}{r^3}+\frac{21F^{(1)}}{r^4}+\frac{21F}{r^5}\right], \\ F&=F(t-r),\quad F^{(n)}\equiv\left[\frac{d^nF(x)}{dx^n}\right]_{x=t-r}. \end{aligned} \tag{6}$$

The angular functions f_{ij} are listed below in the order $M=\pm 2,\pm 1,0$:

$$\begin{aligned} f_{rr}&=\sin^2\theta\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ 2\sin\theta\cos\theta\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix},\ 2-3\sin^2\theta\,, \\ f_{r\theta}&=\sin\theta\cos\theta\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ (\cos^2\theta-\sin^2\theta)\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix},\ -3\sin\theta\cos\theta\,, \\ f_{r\phi}&=\sin\theta\begin{bmatrix}-\sin2\phi\\ \cos2\phi\end{bmatrix},\ \cos\theta\begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix},\ 0\,, \\ f^{(1)}_{\theta\theta}&=(1+\cos^2\theta)\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ 2\sin\theta\cos\theta\begin{bmatrix}-\cos\phi\\ -\sin\phi\end{bmatrix},\ 3\sin^2\theta\,, \\ f^{(2)}_{\theta\theta}&=\begin{bmatrix}-\cos2\phi\\ -\sin2\phi\end{bmatrix},\ 0,\ -1\,, \\ f_{\theta\phi}&=\cos\theta\begin{bmatrix}\sin2\phi\\ -\cos2\phi\end{bmatrix},\ \sin\theta\begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix},\ 0\,, \\ f^{(1)}_{\phi\phi}&=-f^{(1)}_{\theta\theta}\,, \\ f^{(2)}_{\phi\phi}&=\cos^2\theta\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ 2\sin\theta\cos\theta\begin{bmatrix}-\cos\phi\\ -\sin\phi\end{bmatrix},\ 3\sin^2\theta-1\,. \end{aligned} \tag{7}$$

One can also construct an ingoing solution by replacing $F(t-r)$ with some function of $t+r$, changing the sign in front of the odd derivatives in Eqs. (6). Of course, the superposition of ingoing and outgoing solutions is also a solution.

The odd-parity metric is

$$ds^2=-dt^2+dr^2+(2Kd_{r\theta})r\,dr\,d\theta+(2K\,d_{r\phi})r\sin\theta\,dr\,d\phi+(1+L\,d_{\theta\theta})r^2d\theta^2$$
$$+(2L\,d_{\theta\phi})r^2\sin\theta\,d\theta\,d\phi+(1+Ld_{\phi\phi})r^2\sin^2\theta\,d\phi^2\ , \tag{8}$$

where

$$K=\frac{G^{(2)}}{r^2}+\frac{3G^{(1)}}{r^3}+\frac{3G}{r^4}\ ,$$
$$L=\frac{G^{(3)}}{r}+\frac{2G^{(2)}}{r^2}+\frac{3G^{(1)}}{r^3}+\frac{3G}{r^4}\ , \tag{9}$$
$$G=G(t-r)\ ,$$

and where

$$d_{r\theta}=4\sin\theta\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ -2\cos\theta\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix},\ 0\ ,$$
$$d_{r\phi}=-4\sin\theta\cos\theta\begin{bmatrix}\sin2\phi\\ -\cos2\phi\end{bmatrix},\ 2(\cos^2\theta-\sin^2\theta)\begin{bmatrix}\sin\phi\\ -\cos\phi\end{bmatrix},\ -4\cos\theta\sin\theta\ ,$$
$$d_{\theta\theta}=-2\cos\theta\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ -\sin\theta\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix},\ 0\ , \tag{10}$$
$$d_{\theta\phi}=(2-\sin^2\theta)\begin{bmatrix}\sin2\phi\\ -\cos2\phi\end{bmatrix},\ \cos\theta\sin\theta\begin{bmatrix}\sin\phi\\ -\cos\phi\end{bmatrix},\ -\sin^2\theta\ ,$$
$$d_{\phi\phi}=2\cos\theta\begin{bmatrix}\cos2\phi\\ \sin2\phi\end{bmatrix},\ \sin\theta\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix},\ 0\ .$$

The components of the Riemann tensor for a linearized wave in the TT gauge can be computed from the expressions

$$R_{0i0j}=-\tfrac{1}{2}h_{ij,00}\ ,$$
$$R_{0ijk}=\tfrac{1}{2}(h_{ij\,|\,k}-h_{ik\,|\,j})_{,0}\ , \tag{11}$$
$$R_{ijkl}=\tfrac{1}{2}(h_{il\,|\,jk}-h_{jl\,|\,ik}-h_{ik\,|\,jl}+h_{jk\,|\,il})\ .$$

There are algebraic relations between the components because of the vanishing of the Ricci tensor. The components for the even-parity $M=0$ wave are listed in the Appendix in a convenient form.

In computer solutions of Einstein's equations, one often wishes to calculate the amount of energy radiated in the form of gravitational waves. One gains much insight into such calculations by comparison with analytic wave solutions.[8] Accordingly, we also list in the Appendix expressions for the Isaacson energy flux vector, $T_{0i}^{(\mathrm{GW})}$, and the Bel-Robinson flux vector, P^i. These quantities are given by the following expressions in the TT gauge:

$$T_{0i}^{(\mathrm{GW})}=\frac{1}{32\pi}h_{jk,0}h^{jk}{}_{|\,i}\ ,$$
$$P^i=R_{0jk0}R_0{}^{jki}\ . \tag{12}$$

(Strictly speaking, $T_{0i}^{(\mathrm{GW})}$ should be averaged over several wavelengths.)

III. MOTION OF A TEST PARTICLE IN A LINEARIZED WAVE

The equation of motion of a test particle in a linearized wave is simply the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}=0\ , \tag{13}$$

where τ is the proper time of the particle. By the chain rule for partial derivatives,

$$\frac{d^2x^i}{dt^2}=\left[\frac{d\tau}{dt}\right]^2\left[\frac{d^2x^i}{d\tau^2}-v^i\frac{d^2t}{d\tau^2}\right], \tag{14}$$

where $v^i \equiv dx^i/dt$. Substituting from Eq. (13) on the right-hand side of Eq. (14), we obtain

$$\frac{d^2x^i}{dt^2}=-\Gamma^i{}_{00}+v^i(\Gamma^0{}_{00}+2\Gamma^0{}_{0j}v^j+\Gamma^0{}_{jk}v^jv^k)$$
$$-2\Gamma^i{}_{0j}v^j-\Gamma^i{}_{jk}v^jv^k\,. \tag{15}$$

Now the key point is that the first three terms on the right-hand side of Eq. (15) are identically zero in the TT gauge:

$$\frac{d^2x^i}{dt^2}=\Gamma^0{}_{jk}v^iv^jv^k-2\Gamma^i{}_{0j}v^j$$
$$-\Gamma^i{}_{jk}v^jv^k \text{ (TT gauge)}\,. \tag{16}$$

Thus a test particle initially at rest ($v^i=0$) remains at rest and absorbs no energy from the wave.

The reader may object that this result is coordinate dependent, since we used the TT gauge. We now show that this is not the case.

The multipole expansion is made with respect to some spherical coordinate system in flat spacetime. Take the global inertial frame of an observer at rest with respect to the center of symmetry to be the TT coordinate system in the absence of the wave. Consider the case where a test particle is at rest in this inertial frame before the wave arrives. During the dynamical phase of the passage of the wave, the concepts "at rest" and "absorbs no energy" are not physically meaningful (coordinate independent).[9] After the wave has passed by, the particle is again in flat spacetime and since the TT coordinate system matches smoothly to the inertial frame of our central observer, it is meaningful to say that the particle is at rest and has absorbed no energy.

One further point must be clarified. After the wave has passed by, the particle finds itself in flat spacetime again. How do we know that this is the "same" flat spacetime as before the wave arrived? Is it possible that the two flat spacetimes are related by a boost, and so the particle has a velocity with respect to the original Lorentz frame? Intuitively, it seems impossible to sandwich a region of arbitrary multipole wave between two regions of flat spacetime related by a boost. We show that this is indeed so.

A region of spacetime is flat if the Riemann tensor is zero. From Eq. (A1), we see that this holds for our wave solution if

$$\ddot{A}=\ddot{B}=\ddot{C}=0\,, \tag{17}$$

where a dot denotes a time derivative. From Eq. (6), this is equivalent to

$$F^{(2)}=0,\ \ i.e.,\ F=a(t-r)+b\,, \tag{18}$$

where a and b are constants. The metric (5) for this choice of F is (we take the $M=0$ even-parity solution, for example)

$$ds^2=-dt^2+\left[1+\frac{9(at+b)}{r^5}(2-3\sin^2\theta)\right]dr^2+\frac{36(at+b)}{r^5}\sin\theta\cos\theta\, r\, dr\, d\theta$$
$$+\left[1+\frac{9(at+b)}{r^5}(\tfrac{7}{4}\sin^2\theta-1)\right]r^2d\theta^2+\left[1+\frac{9(at+b)}{r^5}(\tfrac{5}{4}\sin^2\theta-1)\right]r^2\sin^2\theta\, d\phi^2\,. \tag{19}$$

We can verify that this is indeed flat spacetime by finding the coordinate transformation which transforms the metric to the usual Minkowski form. An infinitesimal coordinate transformation $x^\mu \to x^\mu+\xi^\mu$ preserves the TT gauge condition provided

$$\xi_{0,0}=0,\ \ \nabla^2\xi_0=0\,,$$
$$\xi_{i,0}=-\xi_{0,i},\ \ \xi^i{}_{|i}=0\,. \tag{20}$$

(These conditions follow from the transformation law

$$h_{\mu\nu}\to h_{\mu\nu}+\xi_{\mu|\nu}+\xi_{\nu|\mu}\,.)$$

One can easily verify that the $M=0$ even-parity solution of these equations is equivalent to

$$t'=t-\alpha(2-3\sin^2\theta)/r^3\,,$$
$$r'=r+3(\alpha t+\beta)(2-3\sin^2\theta)/r^4\,, \tag{21}$$
$$\theta'=\theta+6(\alpha t+\beta)\sin\theta\cos\theta/r^5\,,$$

where α and β are constants. If we choose $\alpha=-\frac{3}{8}a$ and $\beta=-\frac{3}{8}b$, then Eq. (19) becomes the Minkowski metric

$$ds^2=-dt'^2+dr'^2+r'^2d\theta'^2+r'^2\sin^2\theta' d\phi^2\,. \tag{22}$$

The question is now whether the metric "left over" after the passage of the wave can be of the form (19), when before the wave arrived it was of the form (22) in unprimed coordinates. One way of seeing that this is impossible is to consider the solution of the linearized Einstein equations[4] *including* the source term:

$$\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta} \ . \tag{23}$$

Here

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} h^{\gamma}{}_{\gamma} \tag{24}$$

and the Lorentz gauge condition

$$\bar{h}^{\alpha\beta}{}_{|\beta} = 0 \tag{25}$$

has been imposed. Infinitesimal coordinate transformations of the form $x^{\mu} \rightarrow x^{\mu} + \xi^{\mu}$ preserve the gauge condition (25) provided

$$\Box \xi^{\mu} = 0 \ . \tag{26}$$

Suppose one carries out such a coordinate transformation to put the solution $\bar{h}_{\alpha\beta}$ into the TT gauge outside the source. One will then be left with the freedom (20). Clearly, making use of this freedom to introduce or remove terms of the form (21) has nothing to do with the source terms, which were already fully accounted for in the solution $\bar{h}_{\alpha\beta}$. Without loss of generality, one can choose $a = b = 0$ globally.

Another way of seeing this result is to recall that the gravitational field outside a source is given *uniquely* by the multipole moments of the source.[10] For example, for a "slow motion" source,

$$F^{(4)}(t-r) \propto I^{(2)}(t-r) \ . \tag{27}$$

Eq. (27) is simply the proportionality between the $1/r$ piece of the radiation field and the second time derivative of the quadrupole moment of the source evaluated at retarded time. From Eq. (27) we see that the source I does not determine the part of F which is of the form $a(t-r)+b$.

IV. DISCUSSION

We have shown that in linearized theory, a test particle initially at rest remains at rest after the passage of a gravitational wave. This still leaves open the possiblity that a particle with initial velocity might absorb energy. This question could be explored by integrating Eq. (16) for various choices of v^i(initial).

Interactions between particles will allow wave energy to be absorbed. In the local inertial frame of some particle, a neighboring particle oscillates as the wave passes by.[4] If there are no interactions, a neighbor initially at rest returns to rest after the passage of the wave. Suppose, however, there is a collision with a third particle while the wave is acting. This transfers energy to the third particle which is not recovered during subsequent oscillations.

Finally, there is the possibility that for a very strong wave, a particle initially at rest might acquire energy through nonlinear effects. Lacking a realistic analytic solution for a nonlinear gravitational wave from a bounded source, one is unable to investigate this question fully.

There have been discussions in the literature on the motion of test particles under the influence of strong *plane* gravitational waves.[11] However, any prescription for connecting the two flat spacetimes on either side of the wave is *ad hoc* in the case of perfect plane symmetry.

The quadrupole wave solution written down here has been used to check numerical codes developed to solve the full nonlinear Einstein equations. It has also been used to investigate various prescriptions for reading off the energy radiated by gravitational waves in computer calculations. This work is described in Ref. 8.

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APPENDIX

The 12 nonzero components of the Riemann tensor for the $M=0$ even-parity wave are

$$\begin{aligned}
R_{trtr} &= -\frac{1}{r^4\sin^2\theta} R_{\theta\phi\theta\phi} = (\tfrac{3}{2}\sin^2\theta - 1)\ddot{A} \ , \\
R_{trt\theta} &= \frac{1}{r^2\sin^2\theta} R_{r\phi\theta\phi} = \tfrac{3}{2} r \sin\theta \cos\theta \ddot{B} \ , \\
R_{t\theta t\theta} &= -\frac{1}{\sin^2\theta} R_{r\phi r\phi} = -\tfrac{3}{2} r^2 \sin^2\theta \ddot{C} + \tfrac{1}{2} r^2 \ddot{A} \ , \\
R_{t\phi t\phi} &= -\sin^2\theta R_{r\theta r\theta} = \tfrac{3}{2} r^2 \sin^4\theta (\ddot{C} - \ddot{A}) + \tfrac{1}{2} r^2 \sin^2\theta \ddot{A} \ , \\
R_{trr\theta} &= \frac{1}{r^2\sin^2\theta} R_{t\phi\theta\phi} = \tfrac{1}{2} r^2 \sin\theta \cos\theta \dddot{A} \ , \\
R_{t\theta r\theta} &= -\frac{1}{\sin^2\theta} R_{t\phi r\phi} = -\tfrac{1}{8} r^3 \sin^2\theta (3\dddot{B} + \dddot{A}) \ .
\end{aligned} \tag{A1}$$

The Isaacson energy flux is

$$32\pi T_{0r}^{(\mathrm{GW})}=9\sin^4\theta(2C'\dot C-C'\dot A-A'\dot C+2A'\dot A-2B'\dot B)+18\sin^2\theta(B'\dot B-A'\dot A)+6A'\dot A\ ,$$
$$32\pi T_{0\theta}^{(\mathrm{GW})}=18\sin^3\theta\cos\theta[\dot C(2C-A-B)+\dot A(2A-B-C)+\dot B(A+C-2B)]+18\sin\theta\cos\theta(B-A)(\dot B+\dot A)\ . \tag{A2}$$

Here a prime denotes a derivative with respect to r.

The Bel-Robinson flux is

$$P^r=\tfrac{3}{16}r\sin^2\theta[(\dddot A-2\ddot C(3\dddot B+\dddot A)\sin^2\theta-4\dddot A\ddot B\cos^2\theta]\ ,$$
$$P^\theta=\tfrac{3}{16}\sin\theta\cos\theta[(8\dddot A\ddot A-\dddot A\ddot B-4\dddot A\ddot C-3\dddot B\ddot B)\sin^2\theta-4\dddot A\ddot A]\ . \tag{A3}$$

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[1]See, for example, *Sources of Gravitational Radiation,* edited by L. Smarr (Cambridge University, Cambridge, 1979) for a review.

[2]Some of these results were reported in a preliminary fashion in Ref. 1 and will be discussed more fully in Ref. 8.

[3]The case treated by W. E. Couch, R. J. Torrence, A. I. Janis, and E. T. Newman, J. Math. Phys. 9, 484 (1968) is not in the TT gauge and so the results are not as convenient for the problems addressed here.

[4]See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973). We adopt their conventions for tensors and we also set $c=G=1$.

[5]See, for example, F. J. Zerilli, J. Math. Phys. 11, 2203 (1970) or Ref. 10.

[6]W. L. Burke, J. Math. Phys. 12, 401 (1971).

[7]J. Mathews, J. Soc. Indust. Appl. Math. 10, 768 (1962).

[8]K. Eppley, L. Smarr, and S. A. Teukolsky (unpublished).

[9]We mean here in the sense of some global coordinate system. Of course, a particle oscillates in the local inertial frame of a neighboring particle when the wave passes by, even though both are at rest in the TT coordinates.

[10]See, for example, K. S. Thorne, Rev. Mod. Phys. 52, 299 (1980).

[11]See, for example, J. Ehlers and W. Kundt in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962) for a discussion and references.