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Abelian dominance and quark confinement in Yang-Mills theories. II. Oblique confinement and η' mass

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Making a working hypothesis of Abelian dominance at a long-distance scale, we analyze the U(1) problem as well as the confinement problem in the SU(N) Yang-Mills theory with the vacuum angle θ. We show that quarks are confined only when $\theta/2\pi$ is a rational number such that $\theta/2\pi\neq(1+nN)/mN$, n and m being integers. We also calculate a correlation function of the topological charge density $Q(x)=(16\pi^2)^{-1}\mathrm{Tr}F_{\mu\nu}F^{*}_{\mu\nu}(x)$ at $\theta=0$. We derive that $\int d^4x\langle T\{Q(x)Q(0)\}\rangle=N^2/128\pi^4(N-1)(\alpha')^2$, where α' denotes the Regge slope of mesons. This formula yields $\sim(150\text{ MeV})^4$ in case of SU(3), which gives rise to a mass ~550 MeV to the η' meson through a chiral anomaly in QCD with massless quarks. This numerical result would explain the η' mass reasonably well in the approximation where pions are massless Goldstone bosons.

I. INTRODUCTION

In order to analyze the problem of quark confinement in the Yang-Mills theory, we have recently proposed the hypothesis of Abelian dominance.[1] The hypothesis states that the Abelian components of the gauge field dominate in the Yang-Mills theory at a long-distance scale. We emphasize that the Abelian components are defined as gauge-invariant objects. The crucial result of our analysis is that the vacuum structure is dependent on resolution R, where R is a typical scale length in the system to be considered. We have shown that the vacuum has two phases in R, and that monopole condensation occurs for $R\geq R_c$. Namely, though quarks are essentially free particles at a short-distance scale as asymptotic freedom dictates, they are confined by electric vortices at a long distance scale ($R\geq R_c$). We have estimated that $R_c\approx0.6$ GeV $^{-1}$ in the SU(3) Yang-Mills theory.

The aim of the present paper is to generalize our analysis to the SU(N) Yang-Mills theory with the vacuum angle θ. We shall show that oblique confinement modes predicted by 't Hooft[2] appear when $\theta/2\pi$ is an irrational number or when $\theta/2\pi=(1+nN)/mN$, n and m being integers. We shall also estimate the mass of the η' meson based on the formula due to Witten.[3]

Let us briefly review our program according to which we analyze the Yang-Mills theory. We have proposed[1] to consider the effective Lagrangian $L(R)$ at resolution R, which is defined by integrating out the momentum components of all the field variables larger than R^{-1}. The Lagrangian $L(R)$ is adequate to describe the structure of the system whose typical scale length is R: In fact, semiclassical arguments based on it would be enough to give quite accurate results for the system, since the Lagrangian already contains quantum effects with wavelength $\lambda\leq R$. Such an effective Lagrangian is particularly important when the system has a phase transition in scale length R. This is so because, if the critical point is R_c, this phase transition should occur only by quantum effects with wavelength up to R_c. Namely, for the realization of this phase, it is not necessary to take into account the quantum effects with wavelength longer than R_c. The physics in the new phase would be most conveniently described by the effective Lagrangian $L(R)$ with $R\geq R_c$.

In our previous paper,[1] we have applied this general idea together with the hypothesis of Abelian dominance to the SU(N) Yang-Mills theory with the vacuum angle $\theta=0$. We have first shown that the monopole condensation occurs at $R=R_c$ because the entropy effects become dominant over the self-energies of monopoles. Here, the resolution scale R acts as the lattice spacing which determines the integration measure of monopole excitations since monopoles have no intrinsic scales in the Yang-Mills theory; they are Wu-Yang monopoles in SU(2). Then, based on the effective Lagrangian $L(R)$, $R\geq R_c$, we have demonstrated semiclassically that electric flux is squeezed into vortices. We have related the string tension σ and the vortex width m^{-1} to resolution R:

$$\sigma=\frac{(N-1)g^2(\Lambda R)}{2\pi N R^2}\,,\qquad m^2=\frac{8}{R^2}\,, \tag{1.1}$$

where $g(\Lambda R)$ is the effective coupling constant at resolution R. It is not surprising that the vortex width m^{-1} depends on R, since the effective Lagrangian $L(R)$ does not provide any information on the structure of the system whose scale length is smaller than R. In our scheme we may increase the resolution R up to the critical point R_c without affecting the vortex picture. We thus conclude that the vortex width m^{-1} should be evaluated at the critical point $R=R_c$. On the other hand, the potential energy between two quarks should be calculated on the basis of the effective Lagrangian $L(R)$ by choosing R to be the distance between two quarks. Now, in the confinement phase the effective coupling constant $g(\Lambda R)$ is determined by (1.1), or

$$g^2(\Lambda R)/4\pi=\frac{N}{2(N-1)}\sigma R^2\,, \tag{1.2}$$

in terms of resolution R, because the string tension is independent of the distance between two quarks.

The generalization of our formalism to include the vacuum angle θ is straightforward, as we shall describe in this paper. Our analysis turns out to confirm and amplify the analysis of 't Hooft[2] with respect to quark confinement in the θ vacuum. (We note that a similar analysis has been carried out by Cardy and Rabinovici[4] in an instance of Z_p models.) Let us briefly summarize the results. In our formalism, where the Abelian dominance is postulated, the sole effect of the vacuum angle θ is to provide magnetic monopoles with electric charges.[5] More precisely, a monopole which possesses the magnetic charge $\vec{\eta}_M$ acquires the electric charge $(\theta/2\pi)\vec{\eta}_M$ with $\vec{\eta}_M$ being a vector on the root lattice of SU(N).[6] Recall that monopoles are labeled by the root lattice of SU(N). It is interesting to remark that the periodicity of the system in the vacuum angle θ follows in our effective Abelian theory, though topological excitations have apparently no electric charges at $\theta=0$ but have electric charges $\vec{\eta}_M$ at $\theta=2\pi$. As suggested by 't Hooft,[2] the periodicity is realized by considering dyon-gluon pairs which become pure monopoles at $\theta=2\pi$. Consequently, it is necessary to consider not only the condensation of dyons but also the condensation of dyon-gluon pairs in the θ vacuum. We call the resulting phases EM condensation phases since the condensate consists of electric charges as well as magnetic charges in general. The phase structure is relatively simple in the weak-coupling regime; the system is always in the Coulomb phase for $g<g_1$ or $R<R_1$. In case of SU(3), we estimate $R_1\approx 0.6\ \text{GeV}^{-1}$ which depends on θ only weakly. Then, at this point, the condensation of pure dyons occurs. With respect to the strong-coupling phase, it is necessary to consider two cases separately: (i) $\theta/2\pi$ is a rational number, and (ii) it is an irrational number. In case (i), there is a phase-transition point beyond which the condensate is pure magnetic monopoles; these monopoles are dyon-gluon pairs with no electric charges. In case (ii), there arises an infinite sequence of EM condensation phases through which the system passes as the effective coupling constant $g(\Lambda R)$ increases. In any case, the vacuum is a magnetic superconductor in the strong-coupling regime and electric flux is squeezed into vortices. However, the confinement of quarks does not necessarily follow, because quark charges may be screened by dyon charges or gluon charges. We obtain the following conclusion: Quarks are not confined when $\theta/2\pi$ is an irrational number or when $\theta/2\pi=(1+nN)/mN$, n and m being integers.

In the previous paper,[1] as phenomenological applications of our formalism, we have derived the relation between the Regge slopes of mesons and gluonia and we have also estimated the bag constant. In this paper, we wish to estimate the mass of the η' meson. We make use of the observation that the η' meson is a would-be Goldstone boson associated with the chiral U(1) symmetry. Then it has been argued[3] that the η' mass is calculable in the leading order of $1/N$ by examining the θ dependence of the vacuum energy of the pure Yang-Mills theory. In case of SU(3) we derive the η' mass in the chiral limit as

$$m_{\eta'}{}^2=\frac{27}{128\pi^4 f_\pi{}^2(\alpha')^2}\,, \tag{1.3}$$

where f_π and α' denote the pion decay constant and the Regge slope of mesons, respectively. This formula amounts to $m_{\eta'}\sim 0.55$ GeV, where we have used $f_\pi=0.095$ GeV and $\alpha'=0.9\ \text{GeV}^{-2}$. The agreement with the experimental data (~ 0.96 GeV) would be reasonable; note that pions are massless in the chiral-limit approximation. This fact together with similar results derived previously suggests that (i) the U(1) problem as well as the confinement problem is solvable by considering the condensation of dyons, and (ii) the hypothesis of Abelian dominance is basically correct.

In this paper, we shall make an extensive use of weight vectors $\vec{\epsilon}$ and root vectors $\vec{\eta}$ of SU(N). For readers' convenience, we list the minimum formulas with respect to these vectors.[7] Let $\{\lambda_H; H=1,\ldots,N-1\}$ be a set of diagonal Gell-Mann matrices of SU(N) normalized such that $\mathrm{Tr}(\lambda_H\lambda_K)=2\delta_{HK}$. Then, we define the elementary weight vectors $\vec{\epsilon}_i=(\epsilon_i^1,\ldots,\epsilon_i^{N-1})$ by setting $\epsilon_i^H=(\lambda_H/2)_{ii}$. There are N such vectors, among which $N-1$ vectors are independent. They characterize a quantity which transforms according to the fundamental representation of SU(N). On the other hand, we define the elementary root vectors $\vec{\eta}_{ij}$ by $\vec{\eta}_{ij}=\vec{\epsilon}_i-\vec{\epsilon}_j$. There are $N(N-1)$ nontrivial vectors, among which $N-1$ vectors are independent. They characterize a quantity which transforms according to the adjoint representation of SU(N). These vectors satisfy

$$\vec{\epsilon}_i\cdot\vec{\epsilon}_j=-\frac{1}{2N}+\frac{1}{2}\delta_{ij}\ ,\qquad \vec{\eta}_{iN}\cdot\vec{\eta}_{jN}=\tfrac{1}{2}+\tfrac{1}{2}\delta_{ij}\quad (i\neq N, j\neq N)\ . \tag{1.4}$$

In general, weight vectors and root vectors are constructed by

$$\vec{\epsilon}=\sum_{i=1}^{N-1} n_i\vec{\epsilon}_i\ ,\qquad \vec{\eta}=\sum_{i=1}^{N-1} m_i\vec{\eta}_{iN} \tag{1.5}$$

with n_i and m_i being integers. These vectors constitute the weight lattice and the root lattice of SU(N), respectively. Now, generalized Poisson resummation formulas are obtained[7] :

$$\sum_{\vec{\eta}}\exp[-4\pi i\,\vec{\eta}\cdot\vec{\mathrm{B}}]=(\tfrac{1}{2})^{N-1}\sum_{\vec{\epsilon}}\delta(\vec{\mathrm{B}}-\vec{\epsilon})\ ,\qquad \sum_{\vec{\epsilon}}\exp[-4\pi i\,\vec{\epsilon}\cdot\vec{\mathrm{B}}]=(\tfrac{1}{2})^{N-1}\sum_{\vec{\eta}}\delta(\vec{\mathrm{B}}-\vec{\eta})\ , \tag{1.6}$$

where $\sum_{\vec{\eta}}$ and $\sum_{\vec{\epsilon}}$ indicate the summations over all points on the root lattice and the weight lattice of SU(N), respectively. These relations are used when we discuss dual transformations which relate magnetic excitations labeled by the root (weight) lattice of SU(N) to electric excitations labeled by the weight (root) lattice of SU(N).

This paper is composed as follows. In Sec. II, we analyze the compatibility of the hypothesis of Abelian dominance and the periodicity of the system in θ. In Sec. III, we discuss the condensation of dyon-gluon pairs which produces various EM phases. We also argue that the system is in the Coulomb phase in the weak coupling regime. In Sec. IV, we derive effective Lagrangians describing these EM phases. We also discuss the screening of quark charges by gluon charges or dyon charges which leads to oblique confinement modes. In Sec. V, we estimate the mass of the η' meson. Sec. VI is devoted to our conclusions.

II. ABELIAN DOMINANCE IN θ VACUUMS

Through the study of instantons, it has been recognized that QCD possesses a previously unknown parameter, the vacuum angle θ. The pure Yang-Mills theory reads effectively as

$$L(\theta)=\frac{1}{2{g_0}^2}\mathrm{Tr}F_{\mu\nu}F_{\mu\nu}-i\frac{\theta}{16\pi^2}\mathrm{Tr}F_{\mu\nu}F^*_{\mu\nu} \tag{2.1}$$

in the Euclidean metric. Let us consider the Wilson-loop amplitude

$$\langle W(C)\rangle=Z(\theta)^{-1}\int[dF_{\mu\nu}]\delta(D_\mu F^*_{\mu\nu})W(C,F_{\mu\nu})\times\exp\left[-\int L(\theta)\right]\ , \tag{2.2}$$

where the field-strength formulation has been adopted.[8] In our previous paper,[1] we have analyzed the Wilson-loop amplitude for $\theta=0$, where we have proposed to decompose the field strength $F_{\mu\nu}$ into the Abelian component $\hat{F}_{\mu\nu}$ and the non-Abelian component $T_{\mu\nu}$ by diagonalizing $F_{\mu\nu}$ as

$$F_{\mu\nu}=T_{\mu\nu}\hat{F}_{\mu\nu}{T_{\mu\nu}}^{-1}\ . \tag{2.3}$$

We have called $\hat{F}_{\mu\nu}$ the Abelian component since it takes values in the Cartan subalgebra of SU(N), which is the maximal Abelian subalgebra of SU(N). We emphasize that $\hat{F}_{\mu\nu}$ is a diagonal matrix field whose elements, being the eigenvalues of $F_{\mu\nu}$, are gauge invariant. We do this diagonalization because classical configurations of magnetic monopoles are constructed within the Cartan subalgebra of SU(N) and because these monopoles are believed to be the essential agent which leads to the confinement of quarks.

According to the program described in the Introduction, we wish to derive the effective Lagrangian of the Yang-Mills theory at resolution R. In principle, we are able to do this by integrating out the momentum components of all field variables with $p\geq R^{-1}$. Actually, it is almost impossible to carry out such integrations explicitly by the techniques currently available. As in the previous paper,[1] to simplify the problem without losing the essential topological structure of the Yang-Mills theory, we make a working hypothesis of Abelian dominance. Namely, at a resolution beyond a certain mass scale, we assume that the non-Abelian component does not contribute to the effective Lagrangian. Thus, the effective Lagrangian must involve solely the Abelian component $\hat{F}_{\mu\nu}$; the only effects of the non-Abelian component is to smear out the short-distance behaviors of the theory inclusive of monopole singularities. Recall that classical monopole solutions have pointlike singularities in the Yang-Mills theory; they are Wu-Yang monopoles in SU(2). In this way it would be possible to construct an effective Abelian gauge theory of the SU(N) Yang-Mills theory at a long-distance scale R. The symmetry group of the effective Abelian theory is given by $W(\mathrm{SU}(N))\times T(\mathrm{SU}(N))$, where $W(\mathrm{SU}(N))$ and $T(\mathrm{SU}(N))$ stand for the Weyl group and the maximal torus of SU(N), respectively. The Weyl group corresponds to the freedom permutating the diagonal elements of $\hat{F}_{\mu\nu}$.

More precisely, we assume that the Yang-Mills Lagrangian (2.1) would be approximated in terms of Abelian fields as

$$L(\theta)=\frac{1}{4g^2(\Lambda R)}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu}-\frac{i\theta}{32\pi^2}\vec{F}_{\mu\nu}\cdot\vec{F}^*_{\mu\nu} \quad (2.4)$$

at a long-distance scale R, where $g(\Lambda R)$ is the effective coupling constant at R with Λ being a renormalization mass parameter. Here, $\vec{F}_{\mu\nu}$ denotes an $(N-1)$-dimensional vector $\vec{F}_{\mu\nu}=(F^1_{\mu\nu},\ldots,F^{N-1}_{\mu\nu})$ which corresponds to the diagonal matrix $\hat{F}_{\mu\nu}=\sum_{H=1}^{N-1}F^H_{\mu\nu}\lambda^H/2$. We also assume that the Bianchi identity

$$D_\mu F^*_{\mu\nu}=0 \quad (2.5)$$

would read

$$\partial_\mu\vec{F}^*_{\mu\nu}=\vec{k}_\nu \quad (2.6)$$

at a long-distance scale, where $\vec{k}_\mu$ represents the configuration of magnetic-monopole excitations.

We go on to review the topological structure of the SU(N) Yang-Mills theory.

(i) The Z_N topology[7]; topological excitations are magnetic vortices labeled by the weight lattice of SU(N) and magnetic monopoles labeled by the root lattice of SU(N).

(ii) The θ periodicity; the generating functional is periodic in the vacuum angle θ because the Pontryagin index,[9]

$$Q=\frac{1}{16\pi^2}\int \mathrm{Tr}F_{\mu\nu}F^*_{\mu\nu}\ , \quad (2.7)$$

is an integer for arbitrary field configurations with appropriate boundary conditions.

We have studied the Z_N topology in detail in previous papers.[1,7] As we have shown therein, the monopole current $\vec{k}_\mu$ in (2.6) must be parametrized by

$$\vec{k}_\mu(x)=4\pi\sum_M\vec{\eta}_M\int d\tau\delta^{(4)}(x-z^M)\dot{z}^M_\mu\ , \quad (2.8a)$$

where $\vec{\eta}_M$ assumes arbitrary root vectors of SU(N). For later convenience, we introduce magnetic Dirac strings attached to the magnetic monopoles by solving

$$\partial_\mu\vec{\rho}^{M*}_{\mu\nu}=\vec{k}_\nu\ . \quad (2.9)$$

We may write down $\vec{\rho}^{M*}_{\mu\nu}$ explicitly as

$$\vec{\rho}^{M*}_{\mu\nu}=4\pi\sum_M\vec{\eta}_M\int d^2\tau\delta^{(4)}(x-z^M)[z^M_\mu,z^M_\nu]\ , \quad (2.8b)$$

where

$$[z_\mu,z_\nu]=\partial(z_\mu,z_\nu)/\partial(\tau_1,\tau_2)\ .$$

We now analyze the θ periodicity. The necessary and sufficient condition is that the Pontryagin index (2.7) is an integer for arbitrary field configuration. Let us solve the Bianchi identity (2.6):

$$\vec{F}_{\mu\nu}=\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu+\vec{\rho}^{M}_{\mu\nu} \quad (2.10)$$

with (2.8) and (2.9). Then, the Pontryagin index (2.7) reads

$$\begin{aligned}Q&=\frac{1}{32\pi^2}\int\vec{F}_{\mu\nu}\cdot\vec{F}^*_{\mu\nu}\\&=\frac{1}{16\pi^2}\int\vec{F}_{\mu\nu}\cdot\vec{\rho}^{M*}_{\mu\nu}\\&=-\frac{1}{8\pi^2}\int\vec{A}_\mu\cdot\vec{k}_\mu\ .\end{aligned} \quad (2.11)$$

It follows that the magnetic flux passing through arbitrary monopole loops must be quantized. This is possible only if the magnetic flux is composed of an ensemble of quantized vortex loops. Indeed, by making use of relations (1.4), it is easy to prove that the Pontryagin index (2.11) is an integer if the magnetic flux is described by quantized vortices:

$$\vec{\rho}^{V*}_{\mu\nu}=4\pi\sum_V\vec{\epsilon}_V\int d^2\tau\delta^{(4)}(x-z^V)[z^V_\mu,z^V_\nu] \quad (2.12a)$$

with $\vec{\epsilon}_V$ being weight vectors of SU(N). Moreover, since

$$\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu = \vec{\rho}^{\,V}_{\mu\nu} , \tag{2.13}$$

the quantized vortices satisfy

$$\partial_\mu \vec{\rho}^{\,V*}_{\mu\nu} = 0 . \tag{2.12b}$$

Namely, quantized vortices may exist only as closed loops. Note that they are precisely those magnetic vortices implied by the Z_N topology of SU(N) gauge theories.[7] Combining (2.10) and (2.13), we obtain

$$\vec{F}_{\mu\nu} = \vec{\rho}^{\,V}_{\mu\nu} + \vec{\rho}^{\,M}_{\mu\nu} . \tag{2.14}$$

Hence, the electromagnetic fields are composed of quantized vortex loops and monopoles; both of them are magnetic excitations. This is the condition that should assure the θ periodicity of the system.

We have briefly stated the hypothesis of Abelian dominance in the presence of the vacuum angle θ. According to the hypothesis, after integrating out all field variables with the momentum range $p > R^{-1}$, we assume that the Wilson-loop amplitude (2.2) leads to

$$\langle W(C) \rangle = Z(\theta)^{-1} \sum_k \sum_\rho \int [d\vec{F}_{\mu\nu}] W(C, \vec{F}_{\mu\nu}) \delta(\vec{F}_{\mu\nu} - \vec{\rho}^{\,V}_{\mu\nu} - \vec{\rho}^{\,M}_{\mu\nu}) \delta(\partial_\mu \vec{F}^*_{\mu\nu} - \vec{k}_\nu) \delta(\partial_\mu \vec{k}_\mu)$$
$$\times \exp\left\{ - \int \left[\frac{1}{4g^2(\Lambda R)} \vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu} - \frac{i\theta}{32\pi^2} \vec{F}_{\mu\nu} \cdot \vec{F}^*_{\mu\nu} \right] \right\} , \tag{2.15}$$

where it is understood that the integrations are to be made only for the momentum components with $p < R^{-1}$, which implies that the effective Lagrangian derived from (2.15) has resolution R. Here, the symbols ($\sum_k$) and ($\sum_\rho$) indicate the integrations over all configurations of magnetic-monopole excitations $\vec{k}_\mu$ and all magnetic vortex-loop excitations $\vec{\rho}^{\,V}_{\mu\nu}$, respectively. Note that the resolution scale R acts as the lattice spacing which determines the integration measure of these excitations.

In our previous paper,[1] where we have analyzed the Wilson-loop amplitude in the $\theta = 0$ vacuum, we have neglected the vortex-loop excitations. We shall later argue that these vortex loops always condense. Then, it is possible to investigate the effects of the vortex condensation by integrating over all possible configurations of vortex loops described by $\vec{\rho}^{\,V}_{\mu\nu}$:

$$\sum_\rho \delta(\vec{F}_{\mu\nu} - \vec{\rho}^{\,V}_{\mu\nu} - \vec{\rho}^{\,M}_{\mu\nu}) = \sum_\rho \int [d\vec{C}_{\mu\nu}] \exp[i(\vec{F}_{\mu\nu} - \vec{\rho}^{\,M}_{\mu\nu}) \cdot \vec{C}_{\mu\nu} - i\vec{\rho}^{\,V}_{\mu\nu} \cdot \vec{C}_{\mu\nu}] . \tag{2.16}$$

To evaluate (2.16), we make use of the Poisson resummation formula (1.6), which reads

$$\sum_\rho \exp(-i\vec{\rho}^{\,V}_{\mu\nu} \cdot \vec{C}_{\mu\nu}) = \sum_j \delta(\vec{C}_{\mu\nu} - \tfrac{1}{2} \vec{j}_{\mu\nu}) \tag{2.17}$$

in the present context. We then obtain

$$\sum_\rho \delta(\vec{F}_{\mu\nu} - \vec{\rho}^{\,V}_{\mu\nu} - \rho^M_{\mu\nu}) = \sum_j \exp\left[\frac{i}{2} (\vec{F}_{\mu\nu} - \rho^M_{\mu\nu}) \cdot \vec{j}_{\mu\nu} \right] , \tag{2.18}$$

where

$$\vec{j}_{\mu\nu}(x) = \sum_E \vec{\eta}_E \int d^2\tau \delta^{(4)}(x - z^E) [z^E_\mu, z^E_\nu] , \tag{2.19a}$$

with

$$\vec{j}_\nu(x) \equiv \partial_\mu \vec{j}_{\mu\nu}(x) = \sum_E \vec{\eta}_E \int d\tau \delta^{(4)}(x - z^E) \dot{z}^E_\nu . \tag{2.19b}$$

Here, $\vec{j}_\mu$ represents electric charges as is obvious from the coupling of $\vec{j}_{\mu\nu}$ with the gauge field $\vec{F}_{\mu\nu}$ in (2.18). We have shown that, as a result of the condensation of magnetic vortex loops labeled by weight vectors of SU(N), there appears an incoherent plasma of electric charges labeled by root vectors of SU(N). We call them gluon charges, though they also include the charges of bound states composed of gluons. We emphasize that gluon charges have been introduced naturally into our effective Abelian theory for the realization of the θ periodicity.

We substitute (2.18) into (2.15) to obtain

$$\langle W(C)\rangle = Z(\theta)^{-1}\sum_k\sum_j \int [d\vec{F}_{\mu\nu}][d\vec{B}_\mu][d\vec{\chi}]W(C,\vec{F}_{\mu\nu}) \times \exp\left\{-\int\left[\frac{1}{4g^2}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu}+\frac{i}{2}\vec{F}_{\mu\nu}\cdot\left[\vec{G}^*_{\mu\nu}-\frac{\theta}{16\pi^2}\vec{F}^*_{\mu\nu}-\vec{j}_{\mu\nu}\right] + i(\vec{B}_\mu+\partial_\mu\vec{\chi})\cdot\vec{k}_\mu\right]\right\} \tag{2.20}$$

with $\vec{G}_{\mu\nu}=\partial_\mu\vec{B}_\nu-\partial_\nu\vec{B}_\mu$, where use was made of the identities

$$\delta(\partial_\mu\vec{F}^*_{\mu\nu}-\vec{k}_\nu)=\int[d\vec{B}_\nu]\exp\left[i\int\vec{B}_\nu\cdot(\partial_\mu\vec{F}^*_{\mu\nu}-\vec{k}_\nu)\right], \tag{2.21}$$

$$\delta(\partial_\mu\vec{k}_\mu)=\int[d\vec{\chi}]\exp\left[i\int\vec{\chi}\cdot\partial_\mu\vec{k}_\mu\right]. \tag{2.22}$$

The saddle-point equations read

$$\partial_\mu\vec{F}^*_{\mu\nu}=\vec{k}_\nu, \tag{2.23a}$$

$$\partial_\mu\vec{F}_{\mu\nu}=ig^2\left[\vec{j}_\nu+\frac{\theta}{8\pi^2}\vec{k}_\nu\right], \tag{2.23b}$$

where we have neglected the contribution from the Wilson loop $W(C,\vec{F}_{\mu\nu})$. These saddle-point equations imply that a monopole with the magnetic charge $\vec{\eta}_M$ carries the electric charge $(\theta/2\pi)\vec{\eta}_M$ as well. Therefore, topological excitations are actually dyons in the θ vacuum.

It is also useful to rewrite (2.20) as

$$\langle W(C)\rangle = Z(\theta)^{-1}\sum_k\sum_j \int [d\vec{F}_{\mu\nu}][d\vec{B}_\mu][d\vec{\chi}]W(C,\vec{F}_{\mu\nu}) \times \exp\left\{-\int\left[\frac{1}{4g^2}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu}+\frac{i}{2}\vec{F}_{\mu\nu}\cdot\left[\vec{G}^*_{\mu\nu}-\frac{\theta}{8\pi^2}\vec{\rho}^{M*}_{\mu\nu}-\vec{j}_{\mu\nu}\right] + i(\vec{B}_\mu+\partial_\mu\vec{\chi})\cdot\vec{k}_\mu\right]\right\}, \tag{2.24}$$

where use was made of (2.11). In this formula, the periodicity of the system is clear. Indeed, a shift in θ by 2π produces the factor $(1/8\pi)\vec{F}_{\mu\nu}\cdot\vec{\rho}^{M*}_{\mu\nu}$ in the action, but this effect may be compensated by a change of integration variable $\vec{j}_{\mu\nu}$ as

$$\vec{j}_{\mu\nu}\rightarrow\vec{j}_{\mu\nu}+\frac{1}{4\pi}\vec{\rho}^{M*}_{\mu\nu}. \tag{2.25}$$

Such a change of variable is allowed because the charge current $\vec{j}_{\mu\nu}$ and the monopole current $(1/4\pi)\vec{\rho}^{M*}_{\mu\nu}$ have formally the same representations; see (2.8) and (2.19). We emphasize that the periodicity is recovered because of the cancellation between the electric charge of dyons and gluons. This suggests that we should consider dyon-gluon pairs as basic topological excitations. We shall come back to this point in Sec. III when we discuss the self-energies of topological excitations.

III. EM CONDENSATION MODES

In the previous section, we derived the formula (2.20) for the Wilson-loop amplitude, where the condensation of magnetic vortex loops have been taken into account. We proceed to analyze the condensation of dyons in this formula. We also consider the condensation of dyon-gluon pairs. We call them EM condensation modes since the condensate consists of electric charges in addition to magnetic charges in general. In order to analyze these modes, it is necessary to evaluate the self-energies of each topological excitation. This is so because the condensation takes place as a result of the balance between the entropy effects and the self-energies. Because the Wilson loop $W(C,\vec{F}_{\mu\nu})$

does not play any roles, we neglect the contribution from it in this section.

The self-energy of a topological excitation may be calculated on the basis of formula (2.20) as follows. First, making a change of variables

$$\vec{F}_{\mu\nu}\to\vec{F}_{\mu\nu}-g^2\frac{i(\vec{G}^*_{\mu\nu}-\vec{j}^*_{\mu\nu})-(\theta g^2/8\pi^2)(\vec{G}_{\mu\nu}-\vec{j}_{\mu\nu})}{1+\theta^2g^4/64\pi^4}, \quad (3.1)$$

we integrate out $F_{\mu\nu}$. We thus obtain

$$Z(\theta)=\sum_k\sum_j\int[d\vec{B}_\mu][d\vec{\chi}]\exp\left\{-\frac{g^2(\theta)}{4}\int\left[\vec{G}_{\mu\nu}{}^2-i\frac{\theta g^2}{4\pi^2}\vec{G}_{\mu\nu}\cdot\vec{j}_{\mu\nu}+\vec{j}_{\mu\nu}{}^2\right]-i\int(\vec{B}_\mu+\partial_\mu\vec{\chi})\cdot\vec{k}_\mu\right\}, \quad (3.2)$$

where

$$g^2(\theta)=g^2\Big/\left[1+\frac{\theta^2g^4}{64\pi^4}\right]. \quad (3.3)$$

We now integrate over $\vec{B}_\mu$ and $\vec{\chi}$. The result is given by

$$\begin{aligned}Z(\theta)=\sum_k\sum_j&\delta(\partial_\mu\vec{k}_\mu)\delta(\partial_\mu\vec{j}_\mu)\\ &\times\exp\left\{-\frac{1}{2g^2}\int\vec{k}_\mu(x)\Delta_{\mu\nu}(x-y)\vec{k}_\nu(y)\right.\\ &\left.-\frac{g^2}{2}\int\left[\vec{j}_\mu(x)+\frac{\theta}{8\pi^2}\vec{k}_\mu(x)\right]\Delta_{\mu\nu}(x-y)\left[\vec{j}_\nu(y)+\frac{\theta}{8\pi^2}\vec{k}_\nu(y)\right]\right\},\end{aligned} \quad (3.4)$$

where $\Delta_{\mu\nu}$ stands for the massless propagator with the momentum cutoff R^{-1}. This represents a system of plasmas composed of dyons and gluons: dyons are specified by magnetic current $\vec{k}_\mu$ and electric current $(\theta/8\pi^2)\vec{k}_\mu$. A gluon charge current $\vec{j}_\mu$ has appeared as a result of the condensation of magnetic vortex loops, as we have argued in Sec. II.

First, let us consider a single dyon sitting at the origin:

$$\vec{k}_\mu(x)=4\pi\vec{\eta}_M\delta^{(3)}(x)\delta_{\mu 0}. \quad (3.5)$$

Then, the self-energy may be calculated[1] from (3.4) as

$$M_d=\frac{\pi}{4R}\left[\frac{4\pi}{g^2}+\frac{g^2}{4\pi}\left[\frac{\theta}{2\pi}\right]^2\right]\vec{\eta}_M{}^2, \quad (3.6)$$

which is composed of two terms: the term π^2/g^2R due to the magnetic charge and the term $g^2\theta^2/64\pi^2R$ due to the electric charge. We next consider a pair of a dyon labeled by $\vec{\eta}_M$ and a gluon labeled by $\vec{\eta}_E$, placing both of them at the origin. Then, the self-energy is given by

$$M_b=\frac{\pi}{4R}\left[\frac{4\pi}{g^2}\vec{\eta}_M{}^2+\frac{g^2}{4\pi}\left[\vec{\eta}_E+\frac{\theta}{2\pi}\vec{\eta}_M\right]^2\right]. \quad (3.7)$$

Note that (3.7) is periodic in θ when we choose $\vec{\eta}_E$ appropriately in each period of θ; that is, (3.7) is invariant under the simultaneous transformations $\theta\to\theta+2\pi$ and $\vec{\eta}_E\to\vec{\eta}_E-\vec{\eta}_M$. Comments on this mass formula will be given at the end of this section.

In the $\theta=0$ vacuum, since it follows that $M_d<M_b$, the condensation of dyons (which are actually monopoles) occurs regardless of the existence of gluon charges. Thus, the analysis we have made in the previous paper[1] needs no modifications, though we have neglected gluon charges entirely. The sole role that gluon charges play is to make it possible that a vortex with the flux labeled by root vectors may split into shorter vortices.

However, in the θ vacuum, the self-energy of a dyon may become smaller ($M_d>M_b$) when it composes a bound state with a gluon. Then, it is preferable for dyons to compose bound states with gluons and then to condense. The condensation of dyon-gluon pairs may occur when their entropy effects become dominant over the self-energies, that is,[10]

$$RM_b\le\ln(2d), \quad (3.8)$$

where d is the space-time dimension: here, $d=4$.

Substituting (3.7) into (3.8), we obtain the condition

$$F(\vec{\eta}_M,\vec{\eta}_E)\equiv \frac{4\pi}{g^2}\vec{\eta}_M{}^2+\frac{g^2}{4\pi}\left[\vec{\eta}_E+\frac{\theta}{2\pi}\vec{\eta}_M\right]^2 \le \frac{4\ln 8}{\pi}\ . \tag{3.9}$$

This inequality gives rise to two critical coupling constants g_w and g_s provided that

$$\vec{\eta}_M{}^2\left[\vec{\eta}_E+\frac{\theta}{2\pi}\vec{\eta}_M\right]^2 \le \left[\frac{2\ln 8}{\pi}\right]^2 . \tag{3.10}$$

Then, the condensation may occur for $g_w \le g \le g_s$. Approximate solutions are given by

$$\begin{aligned} &g_w{}^2/4\pi \approx 0.38\vec{\eta}_M{}^2\ ,\\ &g_s{}^2/4\pi \approx 2.6\left[\vec{\eta}_E+\frac{\theta}{2\pi}\vec{\eta}_M\right]^{-2} . \end{aligned} \tag{3.11}$$

In general, corresponding to the sets of charge vectors $(\vec{\eta}_M,\vec{\eta}_E)$ satisfying (3.10), a variety of EM condensation modes follow. However, not all of them realize physically. It must be those dyon-gluon pairs which have the minimum self-energy that give rise to an actual phase. Hence, in order to determine the phase at resolution R, it is necessary to search for the set of charge vectors $(\vec{\eta}_M,\vec{\eta}_E)$ which minimizes $F(\vec{\eta}_M,\vec{\eta}_E)$ at the effective coupling constant $g=g(\Lambda R)$.

In order to analyze $F(\vec{\eta}_M,\vec{\eta}_E)$ explicitly, we substitute (1.5) into (3.9). Namely, by setting

$$\begin{aligned} &\vec{\eta}_M=\sum_{i=1}^{N-1} m_i\vec{\eta}_{iN}\ ,\\ &\vec{\eta}_E=\sum_{i=1}^{N-1} n_i\vec{\eta}_{iN}\ , \end{aligned} \tag{3.12}$$

we obtain

$$F(\vec{\eta}_M,\vec{\eta}_E)=\frac{1}{2x}\left[\left[\sum_i m_i\right]^2+\sum_i m_i{}^2\right] +\frac{x}{2}\left[\left[\sum_i n_i+\frac{\theta}{2\pi}\sum_i m_i\right]^2 +\sum_i\left[n_i+\frac{\theta}{2\pi}m_i\right]^2\right], \tag{3.13}$$

where $x=g^2/4\pi$. Without loss of generality we assume θ to lie in the interval $[-\pi,\pi]$. Then, for fixed values of x and θ, we search for the sets of nonzero integers $\{m_i\}$ and $\{n_i\}$ which minimize (3.13). The corresponding root vectors (3.12) turn out to determine the phase at the effective coupling constant $g(\Lambda R)$ for a fixed value of θ: the system is in the EM condensation phase if

$$F(\vec{\eta}_M,\vec{\eta}_E)\le 4\ln 8/\pi,\quad (\vec{\eta}_M,\vec{\eta}_E)\neq 0;$$

otherwise the system is in the Coulomb phase.

For the sake of definiteness, let us solve the above problem when $\theta/2\pi=1/p$, p being an integer. In this case, (3.13) reads

$$F(\vec{\eta}_M,\vec{\eta}_E)=\frac{1}{2x}\left[\left[\sum_i m_i\right]^2+\sum_i m_i{}^2\right] +\frac{x}{2p^2}\left[\left[\sum_i m_i+p\sum_i n_i\right]^2 +\sum_i (m_i+pn_i)^2\right] . \tag{3.14}$$

By assuming $\vec{\eta}_M\neq 0$, we deduce the following results.

(i) When $\vec{\eta}_M=-p\vec{\eta}_E$ with $\vec{\eta}_E$ being an elementary root vector, we obtain

$$F(\vec{\eta}_M,\vec{\eta}_E)=\frac{p^2}{x}\ . \tag{3.15a}$$

(ii) Otherwise we obtain

$$F(\vec{\eta}_M,\vec{\eta}_E)\ge\frac{1}{x}+\frac{x}{p^2}\ . \tag{3.15b}$$

The equality in (3.15b) holds if $\vec{\eta}_E=0$ and $\vec{\eta}_M$ is an elementary root vector. Therefore, when $\theta/2\pi=1/p$, there are only two EM condensation phases. The critical coupling constants are given by

$$\begin{aligned} &g_1{}^2/4\pi\approx 0.38\ ,\\ &g_2{}^2/4\pi=p(p^2-1)^{1/2}\ . \end{aligned} \tag{3.16}$$

We note that the condensate consists of pure dyons for $g_1<g<g_2$ and that it consists of pure monopoles for $g_2<g$; here, monopoles are dyon-gluon pairs with no electric charges. As we shall argue later, the system is in the Coulomb phase for $g<g_1$. We have depicted the phase structure as well as the functions $F(\vec{\eta}_M,\vec{\eta}_E)$ for $\theta/2\pi=\frac{1}{2}$ in the case of SU(2) in Fig. 1(a). We have also indicated the condensates on the electric-magnetic charge lattice of SU(2) in Fig. 2(a).

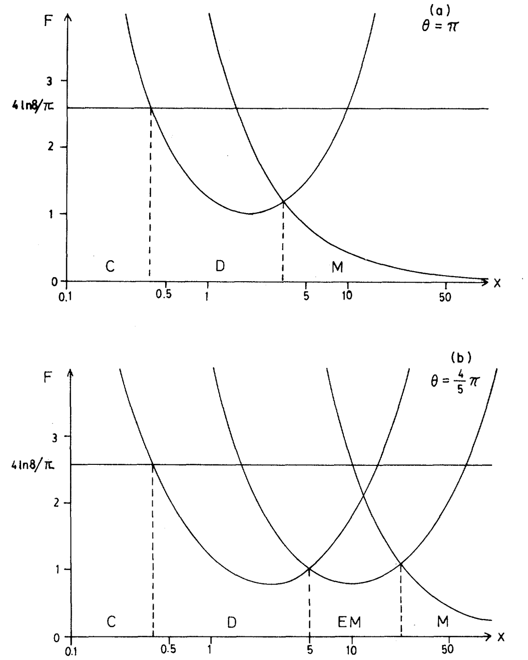

FIG. 1. Phase diagrams in SU(2); Fig. 1(a) for $\theta=\pi$ and Fig. 1(b) for $\theta=4\pi/5$. Curves represent the "self-energy" F of topological excitations (dyon, dyon-gluon pair and monopole) as a function of effective coupling constant $x=g^2(\Lambda R)/4\pi$. These topological excitations condense when $F\leq 4\ \ln 8/\pi$ [see (3.9)]. Initials *C*, *D*, EM, and *M* on the *x* axis stand for the Coulomb, dyon, EM, and monopole condensation phases, respectively.

We are able to derive the phase structure, though the derivation is somewhat tedious, for other values of θ. Here, we only give general features of the phase structure. Let us start with the weak-coupling regime. As is obvious from (3.11), the first critical point is given when $\vec{\eta}_M$ is an elementary root vector ($\vec{\eta}_M{}^2=1$). In this case, the formula (3.13) may be minimized by choosing $\vec{\eta}_E=0$. Namely, the first EM condensation phase is always given by the condensation of pure dyons. We call this the dyon condensation phase. It is also possible to show that the next critical point exists at $g_2{}^2/4\pi=O(\theta^{-2})$ when $|\theta/2\pi|\ll 1$. Thus, $g_2\rightarrow\infty$ as $\theta\rightarrow 0$. This implies that other EM phases disappear in the limit $\theta\rightarrow 0$. With respect to the strong-coupling regime, we need to study two cases separately.

(i) $\theta/2\pi$ is a rational number, i.e., $\theta/2\pi=q/p$ with q and p being integers. It is possible to find a set of vectors such that $\vec{\eta}_E+(\theta/2\pi)\vec{\eta}_M=0$. The corresponding EM phase covers all the strong-coupling regime beyond a certain critical coupling constant. We call this the monopole condensation

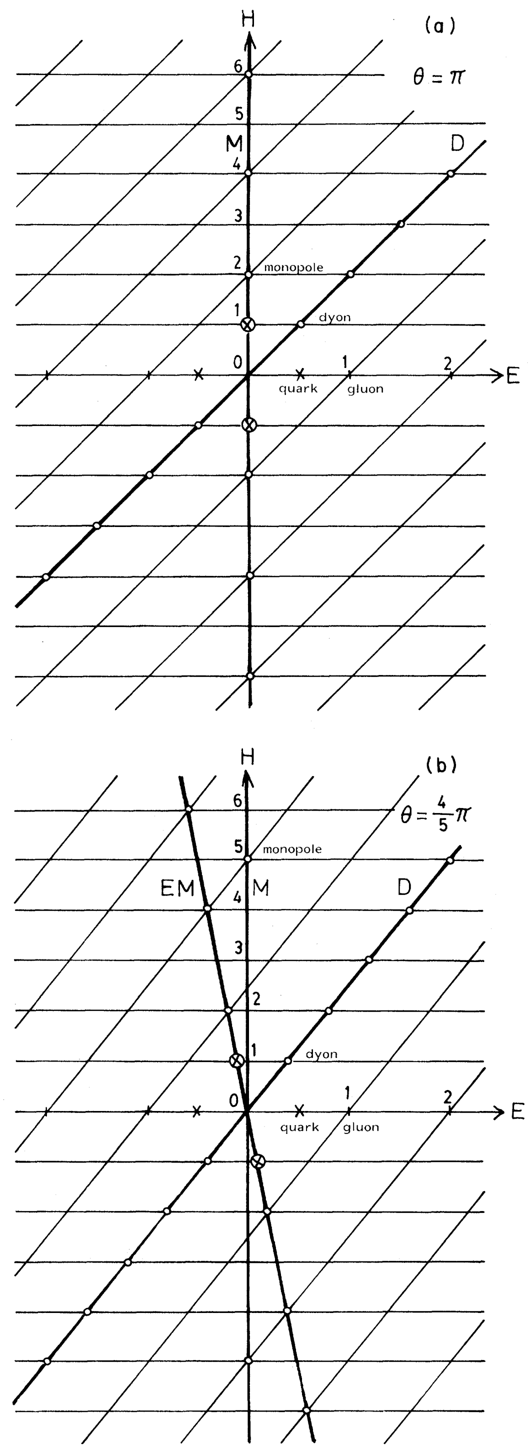

FIG. 2. Electric-magnetic charge lattice in SU(2); Fig. 2(a) for $\theta=\pi$ and Fig. 2(b) for $\theta=4\pi/5$. Lattice sites represent dyon-gluon pairs in general. Points on the heavy lines marked *D*, EM, and *M* stand for the dyons, dyon-gluon pairs and monopoles which are due to condense and compose the dyon, EM, and monopole condensation phases, respectively. The symbol ⊗ denotes dyon-quark pairs.

phase since the condensate is pure monopoles. It is obvious that there are only a finite number of EM condensation phases when $\theta/2\pi$ is a rational number. We have depicted the phase structure for $\theta/2\pi=\frac{2}{5}$ in case of SU(2) in Fig. 1(b). See also Fig. 2(b).

(ii) $\theta/2\pi$ is an irrational number. It is impossible to find a set of vectors such that $\vec{\eta}_E+(\theta/2\pi)\vec{\eta}_M=0$, but it is possible to choose a set of vectors that makes $[\vec{\eta}_E+(\theta/2\pi)\vec{\eta}_M]^2$ arbitrarily small. For definiteness, let us choose $\vec{\eta}_M=p\vec{\eta}$ and $\vec{\eta}_E=q\vec{\eta}$ with $\vec{\eta}$ being an elementary root vector. Now, a theorem in the theory of numbers states that there is an integer q for a sufficiently large p such that

$$\left[q+\frac{\theta}{2\pi}p\right]^2 \lesssim \frac{1}{2p^2} \; . \tag{3.17}$$

From this theorem it follows that the condition (3.10) is satisfied for the set of vectors $(p\vec{\eta},q\vec{\eta})$. Furthermore, it is easy to prove that for a sufficiently large g there is always a set of vectors $(p\vec{\eta},q\vec{\eta})$ that satisfy the inequality (3.9). Therefore, the Coulomb phase never realizes in the strong-coupling regime. We conclude that the system passes through an infinite number of EM condensation phases as the coupling constant increases.

In this section, based on the mass formula (3.7), we have analyzed the condensation of dyons as well as dyon-gluon pairs. We have tacitly assumed that charged gluons never condense only by themselves, which would otherwise lead to the Higgs phase in the weak-coupling regime of the Yang-Mills theory. We wish to argue why this is possible within our formalism. For this purpose, let us first make comments on the mass formula (3.7). In deriving this we have not considered dynamical mechanisms which bind dyons and gluons. Rather, we have simply placed both of them at the same point and evaluated the self-energy due to the electromagnetic fields. The essential point is that, contrary to (3.6), (3.7) is periodic in θ when we choose $\vec{\eta}_E$ appropriately in each periods of θ. Therefore, under the hypothesis of Abelian dominance, we should regard (3.7) as the correct mass formula of topological excitations in the θ vacuum. We may interpret this fact by saying that the periodicity is realized by considering dyon-gluon pairs in the effective Abelian theory. This interpretation is consistent with the proof of the θ periodicity we have made in Sec. II.

On the contrary, we should not evaluate the masses of gluons by setting $\vec{\eta}_M=0$ in the formula (3.7). First of all, there are no reasons to do so. Note that charged gluons belong to the non-Abelian component $T_{\mu\nu}$ in our scheme since they are not invariant under the Abelian gauge transformations, but that dyons are topological excitations in the Abelian component $\hat{F}_{\mu\nu}$ [see (2.3)]. Furthermore, the hypothesis of Abelian dominance implies that the non-Abelian component $T_{\mu\nu}$ is quite "heavy" and hence so must be charged gluons. Therefore, the mass of gluons would be given by

$$M_g=\frac{g^2}{16R}\vec{\eta}_E{}^2+m(\vec{\eta}_E,R) \; , \tag{3.18}$$

where $m \gg R^{-1}$. Then, the condition (3.8) is always violated and charged gluons never condense by themselves. We have shown in Sec. II that charged gluons appear if magnetic vortex loops condense. Because of duality, the above fact is equivalent to saying that the condensation of magnetic vortex loops indeed occurs. We conclude that the hypothesis of Abelian dominance also implies the absence of the Higgs phase in the Yang-Mills theory. We wish to discuss this problem in detail in a future publication.

IV. OBLIQUE CONFINEMENT AND EFFECTIVE LAGRANGIANS

In the previous section, we have analyzed the condition for topological excitations to condense and compose EM phases. The aim of this section is to evaluate the Wilson-loop amplitude, explicitly in each of these phases. We do this by performing a dual transformation in the formula (2.20) or (2.24). Equivalently, we integrate over all possible configurations of topological excitations which are due to condense.

When the gauge field is restricted in the Cartan subalgebra, the fundamental-representation Wilson-loop operator reads[1]

$$W(C,\vec{\mathrm{F}}_{\mu\nu})=\sum_{i=1}^{N}\exp\left[\frac{i}{2}\int \vec{\epsilon}_i\cdot\vec{\mathrm{F}}_{\mu\nu}J_{\mu\nu}\right] , \tag{4.1}$$

where $\vec{\epsilon}_i$ are the elementary weight vectors of SU(N). Here

$$J_{\mu\nu}(x)=\int d^2\tau\,\delta^{(4)}(x-z)[z_\mu,z_\nu] \tag{4.2a}$$

parametrizes a surface whose boundary is given by the loop C. The loop C itself is parametrized by

$$J_\nu = \partial_\mu J_{\mu\nu} \ . \tag{4.2b}$$

In what follows, we analyze (2.20) or (2.24) together with (4.1). We study the dyon condensation phase and the monopole condensation phase in detail. Then, we briefly describe EM condensation phases in general.

A. The dyon condensation phase

We evaluate the Wilson loop in the dyon condensation phase, where the condensate consists of pure dyons. This is the first EM phase that appears as scale length R increases. See Figs. 1 and 2 for illustration; dyons are indicated by points on the heavy line D in Fig. 2. To describe this phase, it is more convenient to use formula (2.20) than (2.24). We start with

$$\begin{aligned}\langle W(C)\rangle = Z(\theta)^{-1}\sum_{i=1}^{N} \sum\!\!\!\!\!\!\int_k \sum\!\!\!\!\!\!\int_j \int [d\vec{\mathrm{F}}_{\mu\nu}][d\vec{\mathrm{B}}_\mu][d\vec{\chi}] \\ \times \exp\Bigg\{ -\int \Bigg[\frac{1}{4g^2}\vec{\mathrm{F}}_{\mu\nu}\cdot\vec{\mathrm{F}}_{\mu\nu} \\ + \frac{i}{2}\vec{\mathrm{F}}_{\mu\nu}\cdot \left[\vec{\mathrm{G}}^*_{\mu\nu} - \frac{\theta}{16\pi^2}\vec{\mathrm{F}}^*_{\mu\nu} - \vec{\mathrm{j}}_{\mu\nu} - \vec{\epsilon}_i J_{\mu\nu}\right] \\ + i(\vec{\mathrm{B}}_\mu + \partial_\mu\vec{\chi})\cdot\vec{\mathrm{k}}_\mu \Bigg]\Bigg\} \ .\end{aligned} \tag{4.3}$$

Hereafter we set $\vec{\mathrm{j}}_{\mu\nu}=0$ since we consider the condensation of pure dyons and since gluon charges do not play any essential roles. For instance, gluon charges never screen the quark charge as we can see easily.

By making a change of variables

$$\vec{\mathrm{F}}_{\mu\nu} \to \vec{\mathrm{F}}_{\mu\nu} - g^2(\theta)\left[i(\vec{\mathrm{G}}^*_{\mu\nu} - \vec{\epsilon}_i J^*_{\mu\nu}) - \frac{\theta g^2}{8\pi^2}(\vec{\mathrm{G}}_{\mu\nu} - \vec{\epsilon}_i J_{\mu\nu})\right] , \tag{4.4}$$

it is possible to decouple the integration over $\vec{\mathrm{F}}_{\mu\nu}$ from the rest of the system. Thus, we obtain

$$\begin{aligned}\langle W(C)\rangle = Z(\theta)^{-1}\sum_{i=1}^{N} \int [d\vec{\mathrm{B}}_\mu][d\vec{\chi}] \\ \times \exp\left\{ -\frac{g^2(\theta)}{4}\int \left[\left[\vec{\mathrm{G}}_{\mu\nu} - \frac{i\theta}{2\pi}\frac{g^2}{4\pi}\vec{\epsilon}_i J_{\mu\nu} - \vec{\epsilon}_i J^*_{\mu\nu}\right]^2 + \left[\frac{\theta}{2\pi}\right]^2 \left[\frac{g^2}{4\pi}\right]^2 \vec{\epsilon}_i^{\,2} J_{\mu\nu}^{\ 2}\right]\right\} \\ \times \sum\!\!\!\!\!\!\int_k \exp\left[-i\int \vec{\mathrm{k}}_\mu\cdot(\vec{\mathrm{B}}_\mu + \partial_\mu\vec{\chi}) - \mathcal{M}(x)\right] ,\end{aligned} \tag{4.5}$$

where $g(\theta)$ has been defined by (3.3), and we have separated the self-energy density of dyons,

$$\mathcal{M}(x) = \frac{\pi^2}{Rg^2(\theta)}\sum_M \vec{\eta}_M^{\,2}\int d\tau\,\delta^{(4)}(x - z^M)(\dot{z}^M_\mu \dot{z}^M_\mu)^{1/2} \ . \tag{4.6}$$

This formula is easily derived from (3.4); see also (3.6).

The physical meanings of the effective coupling constant $g(\theta)$ and the field tensor $\vec{\mathrm{G}}_{\mu\nu}$ are made clear as follows. The dyons have electric charges $(\theta/2\pi)\vec{\eta}_M$ and magnetic charges $\vec{\eta}_M$. We rotate the $(\vec{E},\vec{H})$ space so that dyons become pure monopoles. The electromagnetic field tensor in the new frame is denoted by $\vec{G}_{\mu\nu}$ with $\vec{B}_\mu$ being the magnetic potential. Hence,

$$g(\theta)\vec{G}_{\mu\nu}=i\cos\phi\,\vec{F}^*_{\mu\nu}/g-\sin\phi\,\vec{F}_{\mu\nu}/g\ . \tag{4.7}$$

Here the rotation angle ϕ is defined by

$$\cos\phi=g(\theta)/g\ , \tag{4.8}$$

where g and $g(\theta)$ are the coupling constants in the old frame and in the new frame, respectively. These coupling constants are related one to another by the equation

$$F(\vec{\eta}_M,0)=\frac{4\pi}{g^2}\vec{\eta}_M{}^2+\frac{g^2}{4\pi}\left[\frac{\theta}{2\pi}\vec{\eta}_M\right]^2 =\frac{4\pi}{g^2(\theta)}\vec{\eta}_M{}^2\ , \tag{4.9}$$

which states that the self-energy of a dyon is invariant by the rotation. We remark that the transformation (4.7) is equivalent to the saddle-point equation of (4.3) apart from $\vec{j}_{\mu\nu}$ and $J_{\mu\nu}$, or

$$\frac{1}{2g^2}\vec{F}_{\mu\nu}=-\frac{i}{2}\left[\vec{G}^*_{\mu\nu}-\frac{\theta}{8\pi^2}\vec{F}^*-\vec{j}_{\mu\nu}-\vec{\epsilon}_iJ_{\mu\nu}\right]\ , \tag{4.10}$$

and that the formula (4.9) is reduced to the definition (3.3) of $g(\theta)$ (see Fig. 2 for illustration).

Now we are able to perform a dual transformation in formula (4.5), or equivalently to change the integration over magnetic monopoles ($\sum\!\!\!\!\!\!\int_k$) labeled by the root lattice of SU(N) into the integration over electric vortices ($\sum\!\!\!\!\!\!\int_\sigma$) labeled by the weight lattice of SU(N).[1,6] Here, "electric" and "magnetic" refer to the rotated frame. The result is

$$\langle W(C)\rangle=Z(\theta)^{-1}\sum_{i=1}^{N}\sum\!\!\!\!\!\!\int_\sigma\int[d\vec{B}_\mu][d\vec{\chi}]\times\exp\left\{-\frac{g^2(\theta)}{4}\int\left[\left[\vec{G}_{\mu\nu}-\frac{i\theta g^2}{8\pi^2}\vec{\epsilon}_iJ^*_{\mu\nu}-\vec{\epsilon}_iJ^*_{\mu\nu}-\vec{\sigma}^*_{\mu\nu}\right]^2+2m^2(\vec{B}_\mu+\partial_\mu\vec{\chi})^2+\frac{\theta^2g^4}{64\pi^4}\vec{\epsilon}_i{}^2J_{\mu\nu}{}^2\right]\right\}\ , \tag{4.11}$$

where $m^2=8/R^2$ and

$$\vec{\sigma}_{\mu\nu}(x)=\sum_E\vec{\epsilon}_E\int d^2\tau\,\delta^{(4)}(x-z^E)[z^E_\mu,z^E_\nu] \tag{4.12}$$

with $\partial_\mu\vec{\sigma}_{\mu\nu}=0$. Here, $\vec{\sigma}_{\mu\nu}$ stands for electric strings which are closed upon themselves. The minimum flux is given by $\vec{\epsilon}_E=\vec{\epsilon}$, $\vec{\epsilon}$ being an elementary weight vector, in (4.12). These strings describe topological excitations of electric vortex loops labeled by the weight lattice of SU(N) in the dyon condensation phase. We may interpret that, as a result of the dyon condensation, the magnetic gauge symmetry

$$\vec{B}_\mu\to\vec{B}_\mu+\partial_\mu\vec{f} \tag{4.13}$$

is spontaneously broken and the mass of $\vec{B}_\mu$ is generated. Consequently, stable electric vortices have appeared as topological excitations in the dyon condensed phase.

We may extract the effective Lagrangian from (4.11),

$$L_{\rm eff}=\frac{g^2(\theta)}{4}\left[\left[\vec{G}_{\mu\nu}-\vec{\epsilon}J^*_{\mu\nu}-\frac{i\theta g^2}{8\pi^2}\vec{\epsilon}J_{\mu\nu}\right]^2+2m^2(\vec{B}_\mu+\partial_\mu\vec{\chi})^2\right]\ , \tag{4.14}$$

in the presence of external quarks, where $\vec{\epsilon}$ denotes generically the elementary weight vector of SU(N). From this effective Lagrangian it is possible to derive the string tension

$$\sigma(\theta,R)=\frac{\vec{\epsilon}^{\,2}}{8\pi}g^2(\theta)m^2(R) =\sigma\left[1+\left[\frac{\sigma R^2}{4\vec{\epsilon}^{\,2}}\right]^2\left[\frac{\theta}{2\pi}\right]^2\right]^{-1}\ , \tag{4.15}$$

where use was made of (1.1), (3.3), and (4.11); here, σ is the string tension in the $\theta=0$ vacuum. It is remarkable that the string tension is a monotonically decreasing function of R, R being the distance between two quarks. We emphasize that this formula is only valid in the dyon condensation phase.

B. The monopole condensation phase

We next consider the monopole condensation phase, where the condensate consists of pure monopoles that are dyon-gluon pairs with no electric charges, i.e.,

$$\vec{\eta}_E+\frac{\theta}{2\pi}\vec{\eta}_M=0 \,. \tag{4.16}$$

(See Figs. 1 and 2 for illustration, where pure monopoles are indicated by points on the heavy line M.) This phase is realized in the strong-coupling regime when $\theta/2\pi$ is a rational number, $\theta/2\pi=q/p$ with p and q being integers. When $\theta/2\pi$ is an irrational number, as the effective coupling constant $g(\Lambda R)$ increases, the system passes through an infinite sequence of EM phases and approaches to the monopole condensation phase.

To discuss this phase, it is convenient to use formula (2.24), or

$$\langle W(C)\rangle=Z(\theta)^{-1}\sum_k\sum_j\int[d\vec{F}_{\mu\nu}][d\vec{B}_\mu][d\vec{\chi}] \times\exp\left\{-\int\left[\frac{1}{4g^2}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu}+\frac{i}{2}\vec{F}_{\mu\nu}\cdot\left[\vec{G}^*_{\mu\nu}-\frac{\theta}{8\pi^2}\vec{\rho}^{M*}_{\mu\nu}-\vec{j}_{\mu\nu}-\vec{\epsilon}_iJ_{\mu\nu}\right] + i(\vec{B}_\mu+\partial_\mu\vec{\chi})\cdot\vec{k}_\mu\right]\right\} . \tag{4.17}$$

Here, the condition (4.16) or, equivalently,

$$\frac{\theta}{8\pi^2}\vec{\rho}^{M*}_{\mu\nu}+\vec{j}_{\mu\nu}=0 \,, \tag{4.18}$$

is satisfied by the dyon-gluon pairs which are due to condense. However, before setting (4.18) in (4.17), it is necessary to analyze whether topological excitations may screen the quark charge $\vec{\epsilon}_i$.

We make a change of variables

$$\vec{j}_{\mu\nu}\to\vec{j}_{\mu\nu}+\vec{\eta}\,'_E J_{\mu\nu} \,, \tag{4.19a}$$

$$\vec{k}_\mu\to\vec{k}_\mu+4\pi\vec{\eta}\,'_M J_\mu \,, \tag{4.19b}$$

which also implies

$$\vec{\rho}^{M*}_{\mu\nu}\to\vec{\rho}^{M*}_{\mu\nu}+4\pi\vec{\eta}\,'_M J_{\mu\nu} \,. \tag{4.19c}$$

Then (4.17) yields

$$\langle W(C)\rangle=Z(\theta)^{-1}\sum_k\sum_j\int[d\vec{F}_{\mu\nu}][d\vec{B}_\mu][d\vec{\chi}] \times\exp\left\{-\int\left[\frac{1}{4g^2}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu}+\frac{i}{2}\vec{F}_{\mu\nu}\cdot\left[\vec{G}^*_{\mu\nu}-\frac{\theta}{8\pi^2}\vec{\rho}^{M*}_{\mu\nu}-\vec{j}_{\mu\nu}-\vec{\epsilon}\,'_iJ_{\mu\nu}\right] + i(\vec{B}_\mu+\partial_\mu\vec{\chi})\cdot\vec{k}_\mu-2\pi i\,\vec{\eta}\,'_M\cdot\vec{G}_{\mu\nu}J_{\mu\nu}\right]\right\} , \tag{4.20}$$

where

$$\vec{\epsilon}\,'_i=\vec{\epsilon}_i+\vec{\eta}\,'_E+\frac{\theta}{2\pi}\vec{\eta}\,'_M \,. \tag{4.21}$$

Provided that $\vec{\epsilon}\,'^2_i<\vec{\epsilon}_i^{\,2}$, the quark charge is screened. We come back to this problem after the effective Lagrangian is derived.

We now substitute (4.18) into (4.20), and then we integrate over $\vec{F}_{\mu\nu}$. We obtain

$$\langle W(C)\rangle = Z(\theta)^{-1}\sum_{i=1}^{N}\int [d\vec{B}_\mu][d\vec{\chi}]$$
$$\times\exp\left\{-\frac{g^2}{4}\int\left[\left[\vec{G}_{\mu\nu}-\vec{\epsilon}\,'_i J^*_{\mu\nu}+\frac{4\pi i}{g^2}\vec{\eta}\,'_M J_{\mu\nu}\right]^2+\frac{16\pi^2}{g^4}\vec{\eta}\,'^2_M J_{\mu\nu}^2\right]\right\}$$
$$\times \sum_k \exp\left[-i\int \vec{k}_\mu\cdot(\vec{B}_\mu+\partial_\mu\vec{\chi})-\mathcal{M}(x)\right], \tag{4.22}$$

where we have separated explicitly the self-energy density of monopoles,

$$\mathcal{M}(x)=\frac{\pi^2}{Rg^2}\sum_M \vec{\eta}_M^{\,2}\int d\tau\,\delta^{(4)}(x-z^M)(\dot{z}^M_\mu\dot{z}^M_\mu)^{1/2}. \tag{4.23}$$

This term is easily extracted from (3.4).

The dual transformation may be performed exactly as before in formula (4.22). Here, there is an important remark. As we have described in Sec. III, the minimum vector $\vec{\eta}_M$ that satisfies $\vec{\eta}_E+(q/p)\vec{\eta}_M=0$ is given by $\vec{\eta}_M=p\vec{\eta}$, $\vec{\eta}$ being an elementary root vector. Accordingly, the dual transformation yields a similar formula to (4.11) with (4.12), but the minimum electric flux therein is given by $\vec{\epsilon}_E=(1/p)\vec{\epsilon}$, $\vec{\epsilon}$ being an elementary weight vector. We derive the effective Lagrangian as

$$L=\frac{g^2}{4}\left[\left[\vec{G}_{\mu\nu}-\vec{\epsilon}\,'J^*_{\mu\nu}-\frac{4\pi i}{g^2}\vec{\eta}\,'_M J_{\mu\nu}\right]^2+2m^2(\vec{B}_\mu+\partial_\mu\vec{\chi})^2\right], \tag{4.24}$$

which leads to the string tension

$$\sigma_M(\theta)=\frac{(\vec{\epsilon}\,')^2}{8\pi}g^2m^2=\frac{(\vec{\epsilon}\,')^2}{\vec{\epsilon}^{\,2}}\sigma, \tag{4.25}$$

where σ is the string tension in the $\theta=0$ vacuum.

As is expected, the screened charge $\vec{\epsilon}\,'$ has appeared in the string tension. We now determine the minimum value of $(\vec{\epsilon}\,')^2$. By substituting (3.12) into (4.21), it is straightforward to prove that the minimum value is given by $\vec{\epsilon}\,'=0$ if

$$\theta/2\pi=(1+nN)/mN,\quad n,m \text{ integers} \tag{4.26}$$

and otherwise given by $\vec{\epsilon}\,'=(s/p)\vec{\epsilon}$, s being a certain integer dependent of p and q, $p\geq s\geq 1$. Therefore, when θ satisfies (4.26), no vortices are generated: the Wilson-loop amplitude yields the peripheral law, or quarks are liberated. We have illustrated an example of such screening of quark charges in Fig. 2(b). This mode has been named as oblique confinement by 't Hooft,[2] though quarks are not actually confined in this phase. On the other hand, when θ does not satisfy (4.26), the string tension is nonvanishing; $\sigma_M=(s^2/p^2)\sigma$. The Wilson-loop amplitude yields the area law and quarks are confined.

C. EM condensation phases in general

In the previous subsections we have derived effective Lagrangians in two specific choices of EM phases, that is, in the dyon condensation phase and in the monopole condensation phase. We now wish to analyze EM phases in general. Since we may derive effective Lagrangians by applying exactly the same method we have developed, we keep our description as concise as possible.

We have shown in Sec. III that each EM phase is characterized by a set of root vectors $(\vec{\eta}_M,\vec{\eta}_E)$; the dyon-gluon pairs due to condense carry the electric charge $[\vec{\eta}_E+(\theta/2\pi)\vec{\eta}_M]$ and the magnetic charge $\vec{\eta}_M$. In order to derive an effective Lagrangian, it is convenient to rotate the $(\vec{E},\vec{H})$ space so that these dyon-gluon pairs possess only magnetic charges in the rotated frame. The rotation angle ϕ_H is defined by $\cos\phi_H=g_H(\theta)/g$, $H=1,\ldots,N-1$, where

$$\frac{4\pi}{g_H{}^2(\theta)}(\eta_M^H)^2=\frac{4\pi}{g^2}(\eta_M^H)^2+\frac{g^2}{4\pi}\left[\eta_E^H+\frac{\theta}{2\pi}\eta_M^H\right]^2 \quad (4.27)$$

or

$$g_H{}^2(\theta)=\frac{g^2}{1+(C^H)^2g^4/16\pi^2} \quad (4.28)$$

with

$$C^H=\left[\eta_E^H+\frac{\theta}{2\pi}\eta_M^H\right]\Big/\eta_M^H . \quad (4.29)$$

Here, the rotation angle ϕ_H depends on the component H in general. In the rotated frame a dual transformation is trivially performed as before. As a result we are able to derive an effective Lagrangian L_{eff}

$$L_{\text{eff}}=\sum_H \frac{g_H{}^2(\theta)}{4}\left[\left[G_{\mu\nu}^H-\epsilon'^H J^*_{\mu\nu}+\frac{i}{4\pi}C^H g^2\epsilon'^H J_{\mu\nu}+\frac{4i\pi\eta'^H_M J_{\mu\nu}}{g_H{}^2(\theta)}\right]^2+2m^2(B_\mu^H+\partial_\mu\chi^H)^2\right] , \quad (4.30)$$

where ϵ'^H is the screened quark charge;

$$\epsilon'^H=\epsilon^H+\eta'^H_E+\left[C_i^H+\frac{\theta}{2\pi}\right]\eta'^H_M . \quad (4.31)$$

The string tension may be obtained as

$$\sigma(\theta,R)=\frac{1}{8\pi}\sum_H(\epsilon'^H)^2 g_H{}^2(\theta)m^2(R) , \quad (4.32)$$

which is dependent of resolution R as in the dyon condensation phase.

Finally, we analyze the problem of quark confinement when $\theta/2\pi$ is an irrational number. In this case, it is impossible to screen the quark charge completely. However, it is possible to make the screened charge $\vec{\epsilon}'_i$ arbitrarily small. For instance, we may choose $\vec{\eta}'_M$ and $\vec{\eta}'_E$ so that $(\vec{\epsilon}'_i)^2\approx 1/R$. Then, although $(\vec{\eta}'_M)^2\approx R$ in this case, it is easy to prove by making use of the effective Lagrangian (4.30) that the Wilson-loop amplitude yields the peripheral law. Namely, quarks are not confined.

V. THE η' MASS

As a phenomenological application of our analysis of the confinement problem in the θ vacuum, we wish to estimate the mass of the η' meson. It has been recognized that the η' meson is a Goldstone boson associated with the spontaneous breakdown of the chiral U(1) symmetry. This Goldstone boson has actually received a mass as a result of the chiral anomaly. In fact, in the limit $N=\infty$ the anomaly is switched off and the η' mass is argued to vanish. Witten has observed that the η' mass may be obtained in the leading order of $1/N$ by requiring that the θ dependence of the Yang-Mills vacuum should be removed by the introduction of massless quarks into the system. He has derived a formula

$$m_{\eta'}{}^2=\frac{2N_f}{f_\pi{}^2}\left[\frac{d^2E}{d\theta^2}\right]_{\theta=0} \quad (5.1)$$

with

$$\frac{d^2E}{d\theta^2}=\left[\frac{1}{16\pi^2}\right]^2\times\int d^4x\langle T[\mathrm{Tr}F_{\mu\nu}F^*_{\mu\nu}(x) ,\mathrm{Tr}F_{\mu\nu}F^*_{\mu\nu}(0)]\rangle , \quad (5.2)$$

in the Yang-Mills theory with massless quarks, where N_f is the number of light (massless) quarks ($N_f=3$) and f_π is the pion decay constant ($f_\pi\approx 0.095$ GeV). We may identify $E(\theta)$ as the vacuum energy of the pure Yang-Mills theory with the vacuum angle θ.

As we have demonstrated in the previous sections, the phase structure in resolution R is dependent of θ. Namely, the system is in the Coulomb phase for $R<R_1$. Then, at $R=R_1$, the dyon condensation occurs and this phase continues up to the second critical point R_2. However, we have shown in Sec. III that $R_2\to\infty$ as $\theta\to 0$. Therefore, in evaluating the correlation function (5.2) at $\theta=0$, it is just enough to analyze the vacuum energy $E(\theta)$ of the dyon condensation phase.

In order to determine the vacuum energy $E(\theta)$ of the dyon condensation phase, we write down an effective Landau-Ginzburg Lagrangian,

$$L=\sum_{H=1}^{N-1}\left[\tfrac{1}{4}(G'^H_{\mu\nu})^2+\left|\left[\partial_\mu+i\frac{4\pi}{g}B'^H_\mu\right]\phi^H\right|^2+V(|\phi^H|^2;\theta)\right] , \quad (5.3)$$

where $\vec{\phi}=(\phi^1,\ldots,\phi^{N-1})$ stands for the dyon field operator. Here, we have taken the potential term so that the effective Lagrangian is symmetric under the symmetry group S_N, which is isomorphic to the Weyl group W(SU(N)) of SU(N), as is expected from the arguments of Sec. 1. We are able to derive an explicit form of the potential term as follows. For this purpose, we examine some properties of the effective potential.

First, when we expand it in powers of $|\phi^H|^2$,

$$V(|\phi^H|^2;\theta)=a_0+a_2|\phi^H|^2+a_4|\phi^H|^4+\cdots, \tag{5.4}$$

it must be that[9]

$$a_2=\frac{RM(\theta)-\ln 8}{R^2}, \tag{5.5}$$

where

$$M(\theta)=\frac{\pi^2}{Rg^2(\theta)} \tag{5.6}$$

is the mass of a dyon. Then, in the condensed phase of dyons, the dyon field ϕ^H must develop a vacuum expectation value such that

$$\langle\phi^H\rangle=v(\theta), \tag{5.7}$$

with

$$v^2(\theta)=\lambda a_2, \tag{5.8}$$

where λ is a certain function to be determined later. Hence, it is required that

$$V'(|\phi^H|^2;\theta)=0 \text{ at } |\phi^H|^2=v^2(\theta). \tag{5.9}$$

In fact, it is based on this potential that the condensation of dyons has been argued to occur for $a_2<0$; this condition is nothing but the condition (3.8) which we have extensively used in Sec. III.

Secondly, we require that the Landau-Ginzburg Lagrangian (5.3) should be reduced to the effective Lagrangian (4.14) at the minimum point of $V(|\phi^H|^2;\theta)$. Here, a comment is in order. The effective Lagrangian (4.14) has been derived from the Wilson-loop amplitude (4.11) by neglecting the electric vortex excitations described by $\sigma_{\mu\nu}$. This implies that the effective Lagrangian becomes accurate only for the case $R\gg R_1$ in which vortices are quite heavy. Now, it is trivial to see that (5.3) is equivalent to (4.14), except for the potential term, if we set $B'^H_\mu=g(\theta)B^H_\mu$ and $\phi^H=\hat{v}(\theta)\exp(-i\chi^H)$, where

$$\hat{v}^2(\theta)=\frac{m^2g^2(\theta)}{32\pi^2}=\frac{g^2(\theta)}{4\pi^2R^2}. \tag{5.10}$$

Hence, the vacuum expectation value (5.8) should approach to (5.10) for $R\gg R_1$, from which it follows that

$$\lambda=-\frac{g^2(\theta)}{4\pi^2\ln 8} \tag{5.11}$$

or

$$v^2(\theta)=\frac{g^2(\theta)}{4\pi^2\ln 8}\left[\frac{\ln 8-RM(\theta)}{R^2}\right] \tag{5.12}$$

with (3.3).

Thirdly, we assume that the θ dependence of the effective potential $V(|\phi^H|^2;\theta)$ appears only in the coefficient a_2 in the expansion (5.4). This assumption is quite plausible because the effect of the θ vacuum would be only to transform monopoles into dyons with an increase of the self-energy, as we have discussed in Secs. II and III. We may then write

$$V(|\phi^H|^2;\theta)=V(|\phi^H|^2)+\frac{1}{R}\delta M(\theta)|\phi^H|^2, \tag{5.13}$$

with

$$\delta M(\theta)=\frac{\pi^2}{R}\left[\frac{1}{g^2(\theta)}-\frac{1}{g^2}\right], \tag{5.14}$$

where use was made of (5.5) and (5.6).

Now, the condition (5.9) leads to a differential equation

$$\frac{dV(x)}{dx}+R^{-1}\delta M=0, \tag{5.15}$$

where $x=v^2(\theta)$ and

$$R^{-1}\delta M=\frac{1}{R^2}\left[\frac{\ln 8}{1+4R^2x\ln 8}-\frac{\pi^2}{g^2}\right]. \tag{5.16}$$

Integrating (5.15) from x_0 to x, we obtain

$$V(x)=V(x_0)+\frac{\pi^2}{R^2g^2}(x-x_0)-\frac{1}{4R^4}\ln\frac{1+4R^2x\ln 8}{1+4R^2x_0\ln 8}. \tag{5.17}$$

In this way, making reasonable assumptions, we have determined explicitly the potential term $V(|\phi^H|^2;\theta)$ in the effective Landau-Ginzburg Lagrangian (5.3). It is quite easy to check that this potential indeed satisfies all the properties we have required. We remark that, if we normalize the potential such that $V=0$ at $\phi^H=0$, we may choose $V(x_0)=0$ at $x_0=0$ in (5.17).

By making use of a semiclassical approximation, we evaluate the vacuum energy as

$$E(\theta)=(N-1)V(v^2(\theta);\theta)$$
$$=(N-1)[V(v^2(\theta))+R^{-1}v^2(\theta)\delta M(\theta)] . \quad (5.18)$$

Since we have

$$v^2(\theta)=\hat{v}^2-\frac{1}{\pi R^2}\left[\frac{g^2}{4\pi}\right]^3\left[\frac{\theta}{2\pi}\right]^2+O(\theta^4) ,$$
$$\delta M(\theta)=\frac{\pi}{4\pi}\frac{g^2}{4\pi}\left[\frac{\theta}{2\pi}\right]^2+O(\theta^4) , \quad (5.19)$$

for $\theta \ll 1$ and $R \gg R_1$, we obtain

$$\left.\frac{\partial^2 E(\theta)}{\partial\theta^2}\right|_{\theta=0}=\frac{N^2}{128\pi^4(N-1)(\alpha')^2} , \quad (5.20)$$

where use was made of (1.2); $\alpha'=1/2\pi\sigma$ is the Regge slope of mesons. Substituting (5.20) into (5.1), we derive

$$m_{\eta'}{}^2=\frac{N_f N^2}{64\pi^4(N-1)f_\pi{}^2(\alpha')^2} . \quad (5.21)$$

We notice that $m_{\eta'}$ is $O(1/N)$ since $f_\pi{}^2$ is $O(N)$ and α' is $O(N)$ in this formula. This serves as a consistency check of our formalism. Finally, we estimate (5.20) and (5.21) numerically in case of SU(3). We obtain that

$$\left.\frac{\partial^2 E}{\partial\theta^2}\right|_{\theta=0}\approx(0.15\text{ GeV})^4 ,$$
$$m_{\eta'}\approx 0.55\text{ GeV} , \quad (5.22)$$

where $f_\pi=0.095$ GeV and $\alpha'=0.9$ GeV^{-2} have been used.

In this section, we have estimated the mass of the η' meson by making use of Witten's formula (5.1). Our numerical result (5.22) explains the experimental data (mass=0.96 GeV) well for our crude approximation. We believe that the hypothesis of Abelian dominance is basically correct and is very useful.

VI. CONCLUSIONS

In this paper, we have analyzed the confinement problem and the U(1) problem in the SU(N) Yang-Mills theory with the vacuum angle θ. For this purpose we have made the hypothesis of Abelian dominance.[1] Then, the sole effect of the vacuum angle is to provide magnetic monopoles with electric charges. We have shown that in order to recover the θ periodicity of the system, it is necessary to consider dyon-gluon pairs as fundamental topological excitations. When these pairs condense, the vacuum becomes a magnetic superconductor and electric flux is squeezed into vortices. However, we have shown that this does not necessarily lead to the confinement of quarks in the θ vacuum. This is so because quark charges may be screened by those of dyons and gluons. We have obtained the following conclusion: The Wilson-loop amplitude yields the area law if and only if $\theta/2\pi$ is a rational number and $\theta/2\pi \neq (1+mN)/nN$, with n and m being integers. Hence, quarks are not confined if $\theta/2\pi=(1+mN)/nN$ or if it is an irrational number. Such a possibility was first noticed by 't Hooft in the case of $\theta/2\pi=\frac{1}{2}$ for the SU(2) model, and the corresponding phase has been named as the oblique confinement mode.[2]

We have also estimated the mass of the η' meson. We have used the Witten formula[3] which relates the η' mass to the θ dependence of the vacuum energy of the pure Yang-Mills theory. We have obtained that $m_{\eta'}\sim 0.55$ GeV, which explains the experimental data well for our crude approximation. This numerical agreement together with similar results derived in the previous paper,[1] suggests that the U(1) problem as well as the confinement problem is solvable by considering the condensation of dyons, and that the hypothesis of Abelian dominance is basically correct.

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[6]Here and hereafter, when we say electric charge $\vec{\eta}_E$ and magnetic charge $\vec{\eta}_M$, we actually mean electric charge $g\vec{\eta}_E$ and magnetic charge $(4\pi/g)\vec{\eta}_M$.

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