Brief Reports

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Spontaneous symmetry breakdown and symmetric spaces

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Spontaneous breakdown of a (compact) internal-symmetry group G to a (unbroken) subgroup H is analyzed using the procedure developed in a previous paper. It is found that the desired analysis can be carried through only if the pair (G,H) is such that the corresponding coset space G/H is a symmetric space (a Riemannian globally symmetric space).

In a recent publication,[1] we analyzed the structure of spontaneous symmetry breaking for the limiting case where the symmetry is broken completely. Here we wish to discuss the more general and physically more interesting case where the symmetry breaking is not complete and a subsymmetry of the original symmetry remains unbroken. Throughout this paper, by symmetry we mean a global (not gauge) symmetry group, which is assumed to be a simple, compact, connected Lie group to be denoted by G. We consider the breaking of G, by the vacuum, down to the unbroken closed (Lie) subgroup H of G. We find that the desired extension of the analysis of Ref. 1 to the present situation is possible only when the coset space $M=G/H$ is a symmetric space (a Riemannian globally symmetric space).

Consider symmetry breaking triggered by the nonvanishing vacuum expectation value (VEV) of some scalar field (Higgs field) $\Phi(x)$. For the VEV, we assume the form[2,3]

$$\langle 0|\Phi(x)|0\rangle = NT(g)v_0 \ , \tag{1}$$

where v_0 is a fixed element belonging to a normed vector space V in which the group G acts according to some unitary or orthogonal representation T. Here $T(g)$ is the representation of the element g of G. We have chosen the norm of the fixed vector v_0 to be unity, and N is some real parameter giving the overall strength of the symmetry breaking. Let H denote the stability group of v_0, then it is a closed subgroup of G. The orbit of the VEV, or equivalently that of the vacuum, is the set $T(g)v_0$, where g runs over G, and can be identified with the coset space $M=G/H$. For $g\in G$, we may write $g=ch$, where $h\in H$, and setting $q=T(c)$ we rewrite Eq. (1) in the form

$$\langle 0|\Phi|0\rangle = Nqv_0 \ . \tag{2}$$

Starting with any one of these vacuum states $|qv_0\rangle$, the standard GNS (Gelfand-Naimark-Segal) construction of field theory will give us a (separable) Hilbert space with a cyclic (and unique) vacuum. Thus we have an assignment of Hilbert spaces for each point q of the coset space M; we look upon this as a Hilbert bundle based on M. This is one description of the state space.

With respect to the invariant measure μ on M, normalized as $\mu(M)=1$, let us now construct[1] a Hilbert space H which is the *direct integral* over the preceding Hilbert bundle. Thus we have the vacuum state $|\Omega\rangle$ in H which is the direct integral

$$|\Omega\rangle = \int^{\oplus} |qv_0\rangle \ . \tag{3}$$

With the aid of q that is at our disposal via Eq. (2), we now construct the operator U (in H):

$$U=\int^{\oplus} qI_{qv_0} \ , \tag{4}$$

where I_{qv_0} is the identity operator for the Hilbert space (fiber of the Hilbert bundle) that corresponds to the vacuum $|qv_0\rangle$. Then it is clear that U commutes with all the operators that reside in the Hilbert space H, in particular, with the operators that represent the elements of the Poincaré group. Therefore, acting on $|\Omega\rangle$, the operator U produces another vacuum state $|\Omega_1\rangle$ which also is in H. From the analysis of Ref. 1, we expect the existence of a whole string $|\Omega_n\rangle$ of vacuum states in H, obtained as

$$|\Omega_n\rangle = U^n|\Omega\rangle \ , \tag{5}$$

where n is an integer. Let us inquire under what circumstances our expectation might be fulfilled. First, we infer from Eq. (4) the direct-integral decomposition

$$U^n = \int^{\oplus} q^n I_{qv_0} \quad , \tag{6}$$

and now we are faced with a problem: The right-hand side of Eq. (6) has no meaning for a general homogeneous space M. To see the nature of the difficulty, as well as the procedure that should be adopted to resolve it, let us first consider the special case $n=-1$. What we are now seeking is an assignment for each q of its inverse, that is, a map $q \rightarrow q^{-1}$. Again, such a *map* does not, in general, exist even though q is given here as a unitary (orthogonal) matrix. To see this we proceed as follows. Let $\mathcal{G}$ and $\mathcal{H}$ denote the Lie algebras of G and H, respectively, and let $\mathcal{P}$ be the complementary vector space in the direct-sum decomposition.

$$\mathcal{G} = \mathcal{H} + \mathcal{P} \ . \tag{7}$$

Let t denote the representation of the Lie algebra $\mathcal{G}$ that corresponds to the representation T of the group G. Then every q has the shape

$$q = \exp[t(P)] \quad , \tag{8}$$

for some, $P \in \mathcal{P}$. Here $t(P)$ is the representative of the point P. Thus taking the inverse $q \rightarrow q^{-1}$ is the same as multiplying $t(P)$ by minus one, $t(P) \rightarrow -t(P)$. While this procedure allows us to find the inverse of one given point q, it does not necessarily permit us to assign an inverse to *every* q in M. Let P_1 and P_2 both belong to $\mathcal{P}$ and consider the commutator

$$[t(P_1), t(P_2)] = t(P_3) \ . \tag{9}$$

Under the transformation $t(P_i) \rightarrow -t(P_i)$ $(i=1,2)$, the term $t(P_3)$ remains unchanged. Hence, if P_3 also belongs to $\mathcal{P}$, then the map $q \rightarrow q^{-1}$ does not exist. Quite clearly, the condition for the existence of the desired map is that the *commutator of every pair of elements from $\mathcal{P}$, if nonvanishing, must belong to* $\mathcal{H}$. In other words, we have the commutation relations[4]

$$[\mathcal{P}, \mathcal{P}] \subset \mathcal{H} \ . \tag{10a}$$

The same argument that led to the above, now gives us the further relation

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P} \ , \tag{10b}$$

since the subalgebra $\mathcal{H}$, according to Eq. (10a), is invariant (pointwise) under the transformation considered. And, of course, we have also

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \ . \tag{10c}$$

Equations (10a)–(10c) are necessary and sufficient to ensure the existence of the desired map $q \rightarrow q^{-1}$ of M to itself. This map is (1) an involution, (2) an automorphism of G (the derived map preserves the commutation relations of the Lie algebra), and (3) under it the subgroup H remains elementwise fixed. Whenever these conditions are satisfied for a compact, connected, simple Lie group G and the closed subgroup H of G, the corresponding coset space $M = G/H$ is a symmetric space.[5]

Having shown that the symmetric space character of the coset space M follows directly from the study of the special case $n=-1$, we are in a position to consider the remaining case $n>0$ in Eq. (6). We recall the property of a symmetric space: Every point q on a symmetric space M can be made to lie on a maximal torus of M; all maximal torii of M are conjugate under the adjoint action of the stability group of the origin[6] (in our case, the subgroup H). Let us combine this result with the standard property enjoyed by every torus, which is that the map $p_n: q \rightarrow q^n$ exists for q on a torus.[7,8] We thus conclude that the desired power mapping exists on our symmetric space M and Eq. (6) is well defined. This establishes the main result of this paper.

Our method of looking at the power map p_n, by use of the maximal torus, is a straightforward extension of a procedure adopted by Hopf[8] for studying the power map on the manifold of a compact Lie group. This is as it is expected to be since the group manifold is a particular example of a symmetric space.[9] In particular, the conjugacy of maximal torii, as well as the fact that these conjugates cover the manifold, is true also for the group manifold. Thus the *rank* of a (compact) symmetric space is defined exactly as in a (compact) Lie group; namely, as the dimension of the maximal torus. However, there are important differences as well between the two cases. The mod-2 degree of the map p_n on a symmetric space M of rank λ is easily found to be

$$\deg_2(p_n) = n^{\lambda} \pmod 2 \quad , \tag{11}$$

from the fact that the equation $y = x^n$, with y a (regular) point on a λ-dimensional torus, has exactly n^{λ} solutions.[8] When M is, in addition, a group manifold, Eq. (11) continues to be true if we read, in place of $\deg_2(p_n)$, the Brouwer degree.[8] This is, however, not at all true if M is not a group manifold but just a symmetric space. In the latter event, no simple closed expression exists for the Brouwer degree, which has to be computed in a case by case basis, as has been done by Araki,[10] for the restricted values $n=2^m$ (m a positive integer). A careful definition of the power map p_n on a symmetric space for the special case $n=2$ was earlier given by Harris.[11]

The fact that $M = G/H$ is a symmetric space puts a restriction on the allowed pattern of symmetry breaking; namely, that (G,H) has to be a compact symmetric pair. If G is taken to be $SU(n)$, then it follows from Cartan's classification[12] that there are exactly three possibilities for H: (1) $H=SO(n)$; (2) $H=Sp(n/2)$, $n=$even; and (3) $H=SU(n-m)\times SU(m)\times U(1)$. It is remarkable that when H is a unitary group (product of unitary groups) it must have a $U(1)$ factor—just the thing that we need to describe one unbroken universal conservation law (charge conservation). However, we do not understand the significance, if any, of this observation.

Working with an entirely different model, based on the harmonic maps, Misner[13] was earlier led to surmise that (G,H) ought to be a symmetric pair; since the model then contains the least arbitrariness.

On a separate occasion, Professor A. S. Wightman was kind enough to favor me with an instructive communication explaining how *not* to interpret the vacuum degeneracy. For this, I wish to thank him once again. I acknowledge the benefit of fruitful conversations with Professor W. D. McGlinn, Professor C. W. Misner, and Professor T. Nagano of the Notre Dame Department of Mathematics.

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[1]S. K. Bose, Phys. Rev. D 24, 2153 (1981), and references cited therein.

[2]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[3]T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).

[4]These are the abstract commutation relations whose concrete realization is being provided by the representation t, that is, by equations such as Eq. (9).

[5]S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic, New York, 1962), pp. 347–354.

[6]O. Loos, *Symmetric Spaces* (Benjamin, New York, 1969), Vol. 2, p. 56.

[7]It is a classical result due to Kronecker that, on any torus, there is a point x such that the set x^n of points, as n runs over non-negative integers, is dense on the torus. See, for instance, Helgason, in Ref. 5, p. 247.

[8]H. Hopf, Comment. Math. Helv. 13, 119 (1940).

[9]S. Helgason, in Ref. 5, p. 188.

[10]S. Araki, Topology 3, 281 (1965).

[11]B. Harris, Ann. Math. 76, 295 (1962).

[12]S. Helgason, in Ref. 5, p. 354 (Table II).

[13]C. W. Misner, Phys. Rev. D 18, 4510 (1978).