Operator-product expansion and vacuum instability

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This paper examines the operator product using the example of scalar field theories with unstable vacuums. We find that an operator-product expansion about the unstable vacuum, with the additional assumption that nontrivial operators subtracted with respect to this vacuum have nonvanishing expectation value in the physical vacuum, does not reproduce the predictions of the operator-product expansion about the stable vacuum, except for the leading-twist contribution. We discuss the implications of this for applications of the operator-product expansion in QCD.

I. INTRODUCTION

The operator-product expansion and its generalizations are a basic tool in the analysis of QCD effects. In such analyses it is assumed that whatever nonperturbative effects occur can be absorbed into operator matrix elements, and that the calculation of large-Q behavior of coefficients can be done using renormalization-group-improved perturbation theory.[1] It is also a widely held belief that the nonperturbative effects modify the vacuum—that is to say, that the physical vacuum differs significantly from the vacuum defined order by order in perturbation theory. One signal of this difference is that composite operators such as $\bar{\psi}(x)\psi(x)$ or $F_{\mu\nu}(x)F^{\mu\nu}(x)$ may acquire nonvanishing expectation values in the physical vacuum, even though their expectation values in the perturbative vacuum have been defined to zero via subtractions. The nonvanishing of $\langle\bar{\psi}(x)\psi(x)\rangle$ has long been a feature of the PCAC (partial conservation of axial-vector current) understanding of the pion mass via the relation

$$m_\pi{}^2 f_\pi{}^2 = m_q \langle \bar{\psi}(x)\psi(x)\rangle \ .$$

Nonvanishing vacuum expectation values for other operators have also been much discussed in recent literature.[2]

The purpose of this paper is to investigate the question of whether these two viewpoints are mutually consistent. The operator-product expansion involves subtracted operators and is made with reference to a particular choice of vacuum. The question studied here is whether an operator-product expansion about an unphysical vacuum can reproduce the results of an expansion about the correct vacuum simply by allowing nontrivial vacuum expectation values for the various operators of the theory. By "results" we mean in particular the predictions for Q^2 evolution of physical processes.

We use the case of spontaneously broken scalar theories to investigate this point. In such theories, as is well known, one can perform a shift of variables and rewrite the Lagrangian in terms of variables which are fluctuations about the classical vacuum. If one performs an operator-product expansion for this shifted theory one can evaluate the Q^2 behavior of coefficients for any physical process. These results we consider the correct, or physical, answers for this theory. However, one can also consider the operator-product expansion in terms of the variables of the original unshifted theory. We compare these two expansions and show that, even when the operators appearing in the expansion of the unshifted theory are allowed to acquire vacuum expectation values, the results for Q^2 evolution of the nonleading-twist contributions differ. Mathematically the reason for this is quite clear. The process of shifting variables and the renormalization-group improvement of the operator coefficients both involve summation of infinite sets of perturbation-theory graphs. The reordering of these summation processes, together with the subtraction of divergent loop graphs, can certainly change the answer.

Section II of this paper contains the details of these calculations for real scalar field theory. We show that the operator expansion about the unstable vacuum does not reproduce the results given by the shifted theory for the next-to-leading (or higher) twist terms. Section III contains a similar discussion for the case of the operator $j(x)j(0)$ in a complex scalar field theory and in Sec. IV we examine the effect for matrix elements between states other than the vacuum state. In all cases we find the two approaches do not agree beyond the leading-twist term.

In Sec. V we turn to a discussion of the implication of these results for the QCD case. We suggest that the problems observed in our example, for which the instability of the vacuum is observable even at the classical level, will also occur in a case where the instability is due to nonperturbative effects. However, in this latter case we know of no way of performing the equivalent of the shifted scalar field calculations, that is, of defining a consistent expansion about the physical vacuum, so that we cannot directly check our suggestion.

II. REAL SCALAR FIELD THEORY

We begin by analyzing the propagator in a real scalar field theory with a negative M^2 parameter. The Lagrangian is

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi_0)^2-\tfrac{1}{2}M_0{}^2\phi_0{}^2-\frac{\lambda_0}{4!}\phi_0{}^4 . \qquad (2.1)$$

We will renormalize the theory by introducing the rescalings

$$\phi_0=Z^{1/2}\phi,\ \ M_0=Z_m M,\ \ \lambda_0=\lambda\mu^\epsilon Z_\lambda \qquad (2.2)$$

and defining all Feynman integrals by dimensional continuation. The counterterms will be fixed by Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) subtraction.[3]

Let us first consider the usual treatment where we introduce a shift to the classically stable vacuum

$$\phi=v+\rho \ \text{ with } \tfrac{1}{6}\lambda v^2=-M^2 . \qquad (2.3)$$

The Lagrangian can then be written as

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\rho)^2-\tfrac{1}{2}m^2\rho^2-\frac{gm\sqrt{3}}{3!}\rho^3 -\frac{g^2}{4!}\rho^4+\mathscr{L}_{\text{c.t.}} , \qquad (2.4)$$

where we have introduced the notations

$$m^2=-2M^2,\ \ \lambda=g^2 ,$$

and the counterterms are given by

$$\begin{aligned}\mathscr{L}_{\text{c.t.}}=&\ \tfrac{1}{2}(Z-1)(\partial_\mu\rho)^2-\frac{m^3\sqrt{3}}{2g}(Z_\lambda Z^2-Z_m{}^2Z)\rho\\ &-\tfrac{1}{2}m^2(\tfrac{3}{2}Z_\lambda Z^2-\tfrac{1}{2}Z_m{}^2Z-1)\rho^2\\ &-\frac{mg\sqrt{3}}{3!}(Z_\lambda Z^2-1)\rho^3-\frac{g^2}{4!}(Z_\lambda Z^2-1)\rho^4 .\end{aligned} \qquad (2.5)$$

FIG. 1. The combination of counterterms which always appear together as a mass counterterm.

Notice that the mass counterterms in this theory always occur in the combination given in Fig. 1 which contributes exactly $(Z_m{}^2Z-1)m^2$.

The quantity $Z_m{}^2-1$ is thus given, to lowest order, from the diagrams of Fig. 2 ($Z=1$ to this order). We notice that the diagrams of Fig. 2(a) occur only in the shifted theory and that their contribution to Z_m and hence to γ_m in this theory is nonzero. In fact in the BPHZ scheme the only contribution to γ_m at this order comes from the second graph of Fig. 2(a). Using the usual renormalization-group arguments one can show that for large q^2 the propagator to leading order in λ has the form

$$\begin{aligned}d(q^2)&=\frac{1}{q^2-m^2(q^2)}\\ &=\frac{1}{q^2}+\frac{m^2(q_0{}^2)}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{2\gamma_m}\\ &\quad+O\left[\lambda^2;\left[\frac{m^2}{q^2}\right]^2\frac{1}{q^2}\right],\end{aligned} \qquad (2.6)$$

where

$$\gamma_m=-\mu^2\frac{\partial}{\partial\mu^2}\bigg|_{\lambda_0,\epsilon}\ln Z_m . \qquad (2.7)$$

We note that (2.6) could also be derived using the operator-product expansion, with the usual Zimmermann[3] prescription for $N_2(\phi^2)$, which defines $\langle N_2(\phi^2)\rangle=0$.

We consider the result (2.6) to be the correct re-

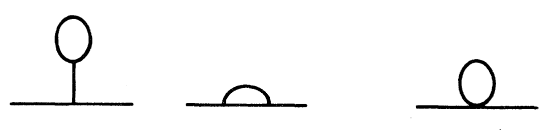

FIG. 2. (a) Lowest-order diagrams which contribute to the mass counterterm and contain three-point interactions. (b) Lowest-order diagram which contributes to the mass counterterm and has no three-point interactions.

sult to this order. We now examine whether this same result is obtained if, instead of proceeding directly to the shifted theory, we calculate $d(q^2)$ from the operator-product expansion in the unshifted theory, but then allow the nontrivial operators to acquire a nonvanishing vacuum expectation value. Thus we will study the quantity

$$F(x,q)=\int d^4\xi\, e^{iq\xi}T\left[\phi\left(\frac{x+\xi}{2}\right)\phi\left(\frac{x-\xi}{2}\right)\right]. \tag{2.8}$$

The usual operator-product expansion for this quantity, using the Lagrangian (2.1) and ignoring temporarily the negative value of M^2, is

$$F(x,q)=\hat{c}_0(q^2,M^2(q^2))\mathbb{1}+\hat{c}_2(q^2,M^2(q^2))N_2(\phi^2(x))+\text{ operators of dimension } \geq 4\,, \tag{2.9}$$

where $\hat{M}(q^2)=M(q_0{}^2)(q^2/q_0{}^2)^{\hat{\gamma}_m}$. The quantity $N(\phi^2(x))$ in (2.9) denotes a composite operator subtracted as an operator of dimension 2 with respect to the naive perturbative vacuum (that is, in lowest order the state such that $\phi\,|\,v\rangle=0$). Now we take the physical vacuum expectation value of (2.9) to find the propagator, and again use a renormalization-group-improved analysis. This gives, to the same order as kept in (2.6),

$$D(q^2)=\frac{1}{q^2}+\frac{\hat{M}^2(q_0{}^2)}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{2\gamma_m}+\frac{\tilde{c}}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{\gamma_{\phi^2}}\langle N_2(\phi^2(x))\rangle$$
$$+\text{ higher-order corrections}\,. \tag{2.10}$$

In (2.10) we have defined

$$\hat{c}_2(q^2,M^2(q^2))=\frac{\tilde{c}}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{\gamma_{\phi}{}^2}\left[1+O\left[\frac{M^2(q^2)}{q^2}\right]\right]. \tag{2.11}$$

The quantity $(\hat{Z}_m{}^2-1)$ comes from the diagram of Fig. 2(b) only, so that it is clear that $\hat{\gamma}_m\neq\gamma_m$. In fact $\hat{\gamma}_\phi{}^2=\hat{\gamma}_m=0$ to this order. Thus it is clear that, although the leading terms in (2.6) and (2.11) are the same, the q^2 evolution of the terms of next-to-leading twist is different. [Clearly these terms can be made to match at any one $q_0{}^2$ by a choice of $\langle N_2(\phi^2(x))\rangle$.] This discrepancy obviously will not be improved by performing a higher-order calculation. Similar discrepancies will also occur in the q^2 evolution of all higher-twist contributions.

The discrepancies between (2.6) and (2.11) can be understood in a straightforward fashion. As shown in Fig. 3(a) the zeroth-order propagator of the shifted theory corresponds to an infinite sum of diagrams, which contribute to all n-point functions in the unshifted theory. Similarly, as in the example shown in Fig. 2(b), any higher-order diagram of the shifted theory can be related to an infinite sum of diagrams of the unshifted theory. Thus the perturbation expansion of the shifted theory arises from summing terms like $\lambda^n\sum_m(\lambda\phi^2)^m$ in the unshifted theory. This resumming of an infinite number of graphs which are higher order in λ in the unshifted theory can clearly change which diagrams are identified as leading-logarithmic corrections in the two cases. Thus the differences between the various γ's are understandable. The collection of diagrams summed by the inclusion of anomalous-dimension effects is simply different in the two cases. Furthermore, discrepancies can arise because many diagrams which are unsubtracted in the unshifted theory are first summed in the shifted theory and then the summed graph is subtracted. Our point here is that our results should not be regarded as peculiar; mathematically they are not unexpected. They indicate that simply allowing nontrivial

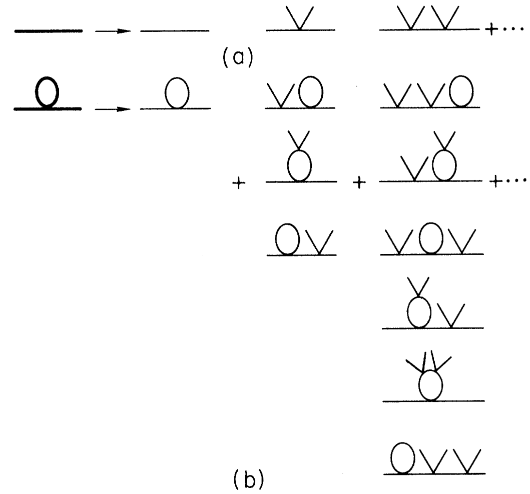

FIG. 3. (a) Expansion of shifted zeroth-order propagator in terms of diagrams of unshifted theory. (b) Expansion of shifted one-loop graph in terms of diagrams of unshifted theory.

operators to acquire vacuum expectation values does not achieve all the resummations necessary to turn the unshifted theory into the shifted theory.

III. COMPLEX SCALAR FIELD THEORY

The phenomenon discovered for the propagator in the previous section persists for other Green's functions. Let us consider a slightly more interesting theory, the complex scalar field theory

$$\mathscr{L}=\partial\phi^*\partial\phi-M^2\phi^*\phi-\lambda(\phi^*\phi)^2 \tag{3.1}$$

again with $M^2<0$. In theory we can define a current

$$j_\mu=\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^* , \tag{3.2}$$

which is conserved for $M^2>0$ but not in the broken theory. However, since the breaking is soft the anomalous dimension associated with this current is zero even in the $M^2<0$ case. Now let us consider the operator-product expansion of the quantity

$$F(q,x)=\int d^4\xi\, e^{iq\xi}j_\mu\left[\frac{x+\xi}{2}\right]j_\nu\left[\frac{x-\xi}{2}\right] . \tag{3.3}$$

As before we can introduce shifted fields

$$\phi=\frac{v+\rho+i\chi}{\sqrt{2}} \tag{3.4}$$

with

$$\lambda v^2=-M^2$$

and then, as before, define the shifted Lagrangian. (Renormalization is dealt with as in the previous example.) The operator-product expansion for the product of two currents then takes the form

$$\begin{aligned}F(x,q^2)&=c_0(q^2)\mathbb{1}+c_1(q^2)N_1(\rho(x))\\&\quad+\frac{c_2(q^2)}{q^2}N_2(\rho^2(x))\\&\quad+\frac{c_2'(q^2)}{q^2}N_2(\chi^2(x))\\&\quad+\text{ operators of dimension 3}\\&\qquad\text{and higher ,}\end{aligned} \tag{3.5}$$

whereas for the unshifted theory we have

$$\begin{aligned}\hat F(q^2,x)&=\hat c_0(q^2)\mathbb{1}+\frac{\hat c_2}{q^2}N_2(\phi^*\phi(x))\\&\quad+\text{ operators of dimension 4}\\&\qquad\text{and higher .}\end{aligned} \tag{3.6}$$

As in the previous example we allow $\langle N(\phi^*\phi(x))\rangle$ to be nonzero for the unshifted theory. We obtain

$$\langle F(q^2,x)\rangle=a+\frac{bm^2(q_0{}^2)}{q^2}\left[\frac{q^2}{q_0{}^2}\right]^{2\gamma_m} \tag{3.7}$$

for the shifted theory, whereas for the unshifted theory we find

$$\begin{aligned}\langle\hat F(q^2,x)\rangle&=a+\frac{\hat b\hat M^2(q_0{}^2)}{q^2}\left[\frac{q^2}{q_0{}^2}\right]^{2\hat\gamma_m}\\&\quad+\frac{\hat c_2}{q^2}\langle N_2(\phi^*\phi(x))\rangle\left[\frac{q^2}{q_0{}^2}\right]^{\hat\gamma_{\phi^2}}\\&\quad+\text{ higher-order corrections .}\end{aligned} \tag{3.8}$$

The constants a, b, $\hat b$, and $\hat c_2$ in (3.7) and (3.8) are numbers calculated in perturbation theory in the usual way; their values need not concern us here. The essential point is that the leading-twist contributions to (3.7) and (3.8) are the same, but as before the quantities γ_m and $\hat\gamma_m$ are different and $\hat\gamma_m=\hat\gamma_{\phi^2}=0$ to order λ. Hence again the q^2 evolution of the nonleading-twist terms obtained in the two calculations are different.

IV. OTHER MATRIX ELEMENTS

The operator-product expansion is valuable precisely because of the fact that the coefficients are independent of the matrix elements, and hence the q^2 evolution of different processes can be related. So far we have discussed the vacuum-to-vacuum matrix elements. Let us now discuss some external particle states to see whether these fare any better.

Consider, for example, a single-particle state of momentum k, which we will denote by $|k\rangle$. Consider the connected Green's function

$$G(q^2)=\int d^4x\, e^{iq\cdot x}\langle k\,|\,j_\mu(x)j_\nu(0)\,|\,k\rangle . \tag{4.1}$$

For large q^2, in the shifted theory one finds $G(q^2)$ dominated by the terms

$$G(q^2)=\frac{c_2(q^2)}{q^2}\langle k\,|\,N_2(\rho^2(0))\,|\,k\rangle + \frac{c_2'(q^2)}{q^2}\langle k\,|\,N_2(\chi^2)\,|\,k\rangle$$

$$+\text{ terms suppressed by further powers of } q^2 \text{ and } O(\lambda^2)\;. \tag{4.2}$$

In the unshifted theory the same matrix element of a product of two currents is given, to this order in $1/q^2$ by

$$\hat{G}=\frac{\hat{c}_2(q^2)}{q^2}\langle k\,|\,N_2(\phi^*\phi(0))\,|\,k\rangle$$

$$+\text{ terms suppressed by further powers of } q^2\;. \tag{4.3}$$

To leading order (4.2) and (4.3) give the same answer for G. (Note to this order $\hat{\gamma}_{\phi^2}=\gamma_{\rho^2}=\gamma_{\chi^2}=0$.) Once again discrepancies appear when we look at the next-to-leading contributions. Contributions which in the shifted theory come from higher-order terms in c_2 or c_2' involving three point vertices, e.g., Fig. 4(a) will give terms of order

$$\Delta G \propto \frac{m^2(q_0{}^2)}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{(\gamma_m+\gamma_{\rho^2})}\langle k\,|\,N_2(\rho^2)\,|\,k\rangle$$

or (4.4)

$$(\rho\to\chi)\;,$$

whereas in the unshifted theory such terms will arise only as part of the coefficient of the $N_4((\phi^*\phi)^2)$ operator as in Fig. 4(b) and hence will appear as

$$\Delta\hat{G}\propto\frac{\langle k\,|\,N_4(\phi^*\phi)^2\,|\,k\rangle}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{\hat{\gamma}_{\phi^4}}. \tag{4.5}$$

Also at this order there will be terms such as

$$\frac{M^2(q_0{}^2)\langle k\,|\,N_2(\phi^*\phi)\,|\,k\rangle}{q^4}\left[\frac{q^2}{q_0{}^2}\right]^{\hat{\gamma}_m+\hat{\gamma}_{\phi^2}}. \tag{4.6}$$

To the order of accuracy of these expressions (γ to order λ only) $\gamma_m\neq\hat{\gamma}_m$ and

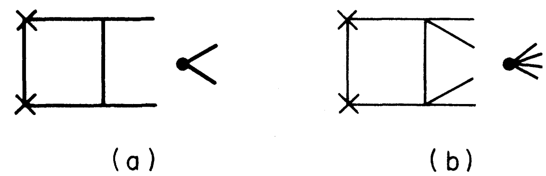

FIG. 4. (a) Contribution to the coefficient of ρ^2 (or χ^2) which contains three-point interactions (ρ and χ lines are not distinguished). (b) Similar diagrams appear as a part of $N_4(\phi^4)$ coefficient in unshifted theory.

$\hat{\gamma}_{\phi^2}=\gamma_{\rho^2}=\gamma_{\chi^2}=\hat{\gamma}_{\phi^4}=\hat{\gamma}_m=0$. Thus we are once again led to conclude that the two series can only match for the leading term in the expansion, with the q^2 evolution for all q^2-suppressed contributions differing in the two cases.

V. COMMENTS AND CONCLUSIONS

The preceding calculations show that, for the case of spontaneously broken symmetry, one does not obtain correct results by making an operator-product expansion of the theory about the unstable vacuum and then allowing the physical vacuum matrix elements of the operators thus defined to be nonzero. The leading-twist term is given correctly but not the higher-twist terms.

We have identified the source of this difference in the infinite-graph resummation involved in going to the shifted theory. The problem arises partly because the vacuum values of an operator can be of order of an inverse coupling constant. This can destroy the perturbative power counting for the unshifted theory, and means that terms which are naively highly suppressed by powers of coupling constant are actually relevant contributions to the correct result. The fact that the propagator itself is modified means that contributions which in the expansion are regarded as suppressed by many powers of q^2 can contribute at next-to-leading twist to an effective shift of a mass scale, and to a change in the anomalous dimension associated with that mass.

Both these effects are relevant to the QCD case. It is clear that nontrivial values for any composite operators will modify the two-point Green's function. Furthermore, it is commonly assumed that the quantity αFF acquires a finite vacuum expectation value. This means that the vacuum-value of the operator FF is assumed to be of order $1/\alpha$ and thus that the kinds of problems encountered in our example are relevant to this case. It is of course clear that our example has not dealt with nonperturbative effects in any way. However, we feel that a method which fails to achieve results which we know are just a resummation of perturbative graphs will not fare any better when the instability of the vacuum arises from nonperturbative effects.

If one believes that the operator-product expansion in QCD cannot be trusted except for the leading-twist terms, what results are changed? The entire perturbative QCD program is based on a proof of factorization which explicitly uses the operator-product approach.[1] Our analysis does not indicate any problem for this approach, since the discrepancies we find would not invalidate the factorization. Thus the majority of QCD perturbative calculations, which discuss only leading-twist effects, are unaffected by this discussion. However, we would suggest that attempts to extract nonleading-twist effects from perturbative QCD calculations[4] may be subject to the diseases found here for the scalar theory.

Finally, this paper would not be complete without some comment on the work of Shifman, Vainshtein, and Zakharov who have led the effort to incorporate the effects of a nontrivial vacuum in QCD calculations. In their work the quantity αFF is assumed to have a vacuum value. However, this leads to the problem discussed above that contributions, which naively are suppressed by additional powers of q and additional factors of α, may resum to alter results.

Note added in proof. It has been pointed out to us by Dr. U. Ellwanger that using minimal subtraction gives $\gamma_m = \hat{\gamma}_m = \frac{1}{2}\hat{\gamma}_{\phi^2}$ to the order at which we work here. This would imply that the results presented here are also scheme dependent, which requires some further study. We thank Dr. Ellwanger for his comments, and for pointing out some numerical errors in our manuscript.

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