PHYSICAL REVIEW D VOLUME 26, NUMBER 2 15 JULY 1982

Choice of a gauge in the light of Dirac quantization

A. Burnel
Physique Théorique et Mathématique, Université de Liège, Institut de Physique au Sart Tilman, Bâtiment B.5, B-4000 Liège 1, Belgium
(Received 22 June 1981)

We discuss the choice of a gauge from the point of view of Dirac quantization of constrained systems. In order to illustrate the discussion, we build up a mechanical example of a gauge theory. It consists of the motion on a straight line $y=0$ imbedded in a plane (x,y). Gauging the model introduces a third Cartesian coordinate z. We show there exist three classes of gauge conditions: (I) gauge conditions involving only x,y and the momenta p_x and p_y; (II) gauge conditions involving the z coordinate; (III) gauge conditions involving the time derivative of the z coordinate. Class I allows, in general, the decoupling between physical and unphysical variables and the existence of an effective Hamiltonian depending only on x and p_x. It corresponds to the choice of a unique representative for each gauge-group orbit. Class II is a generalization of the ungauged model of a particle moving on a straight line imbedded in a plane. Separation between physical (x) and unphysical (y) degrees of freedom is not directly possible. External constraints must be imposed on physical states as well as on the measure defining the physical Hilbert space. Class III keeps all the unphysical degrees of freedom y and z. An indefinite-metric formalism is needed to introduce a cancellation between these unphysical degrees of freedom. Application to Abelian and non-Abelian Yang-Mills theory is easily done by the correspondence $x,y \leftrightarrow A_k^\alpha$, $z \leftrightarrow A_0^\alpha$. Usual gauges are discussed according to this classification. In particular, we try to derive the class-II gauge conditions $n^\mu A_\mu^\alpha$, $n_0 \neq 0$ from a suitable class-I condition. This is not possible in the non-Abelian case. In the Abelian case, the lightlike gauge $n^2=0$ does not lead to an effective Hamiltonian depending only on physical degrees of freedom.

I. INTRODUCTION

Quantization of gauge theories is affected by various difficulties: indefinite metric, Faddeev-Popov[1] ghosts, gauge ambiguities,[2] etc. The origin of these difficulties rests on the fundamental question in the quantization of a gauge theory—the choice of a gauge condition. In this paper, we try to clarify how this choice should be made in the best possible way.

Dirac[3] quantization is the best tool to study this problem. Since the method applies to any constrained system, it is interesting to try to build up a mechanical example of a gauge theory, i.e., a gauge theory involving only a finite number of degrees of freedom instead of fields. The simplest possible example is a particle on a plane, constrained to move only along a straight line that we take as the x axis. The requirement of Lagrangian invariance for time-dependent translation along the y axis introduces a new variable z, without $\dot{z}$ dependence in the Lagrangian. The Euler-Lagrange equation corresponding to this additional variable leads to the desired constraint, the vanishing of the variable canonically conjugate to y. We get a Lagrangian depending on three variables but only one of them is independent. Analogy with the free electromagnetic field is evident since only two of the four components of A_μ are independent.

From the point of view of Dirac quantization, the choice of a gauge consists in the replacement, in the Hamiltonian, of an arbitrary function by a well-defined one. It fixes the time evolution of the additional variable by giving $\dot{z}$. This time evolution can, however, result from a condition on z and the other coordinates and momenta. In the same way, the condition involving z can, with the help of the equation of motion, result from a condition involving only x, y, p_x, and p_y. The gauge condition can therefore be classified into three classes that we will give explicitly in the framework of Maxwell or Yang-Mills gauge theories where the role of z is played by the time components of the potentials.

Class I: gauge conditions involving only A_k^α and their canonically conjugate momenta π_k^α;
Class II: gauge conditions involving also A_0^α;
Class III: gauge conditions involving $\partial_0 A_0^\alpha$.

Class III is the most general one. Any gauge condition of class I or II leads to a gauge condition of class III. Physical degrees of freedom cannot be directly separated from unphysical ones. An indefinite metric is necessary.

Class I is the most interesting one. The separation between physical and unphysical degrees of freedom is possible. When it only depends on coordinates, it consists in a choice of a unique representative for each gauge-group orbit. With the help of a gauge condition of this class, it is in general possible to get an effective Hamiltonian in terms of physical degrees of freedom only.

Although frequently used, class II is annoying. In our example, the variable z can be eliminated and the Hamiltonian depends on two variables and their canonically conjugate momenta. The physics is not contained in the sole Hamiltonian. The constraint must be imposed independently and the measure of the Hilbert space $L^2(\mathrm{R}^2, d\mu)$ must be manipulated in order to reduce to $L^2(\mathrm{R}^1, dx)$. There is no natural mathematical way to realize this reduction, except by introducing by hand a δ function $\delta(y-y_0)$ inside d^2x. For these reasons, and although the procedure is quite consistent, we think class-II gauge conditions should be avoided. Two methods are possible. First, we could work with the resulting class-III gauge condition and the underlying indefinite-metric formalism. Second, we could try to find a class-I gauge condition implying the given class-II condition.

The problem of classification of gauges will be discussed for free Maxwell as well as pure Yang-Mills gauge theories. In the case of electromagnetic field theory, essentially two class-I gauges are known: the Coulomb gauge

$$\partial_k A_k = 0 \tag{1}$$

and the axial gauge

$$\vec{n}\cdot\vec{A} = 0 . \tag{2}$$

Generalizations of the axial gauge given by

$$n^\mu A_\mu = 0 , \quad n_0 \neq 0 \tag{3}$$

are class II. We show that they can be deduced from class-I generalizations. For lightlike

$$n^2 = 0 , \tag{4}$$

however, it is not possible to find an effective Hamiltonian allowing the quantization in term of physical degrees of freedom only.

Relativistic gauges involve $\partial_0 A_0$. They are of class III and need therefore an indefinite-metric formalism. We also give the Lagrangian for the indefinite-metric formalism of class-I, -II, and -III gauge conditions. In the case of the axial gauge (2), the Lagrangian is nonlocal and contains second-order derivatives.

For Yang-Mills gauge theory, it is known that the Coulomb gauge is not suitable. It gives rise to Gribov[2] ambiguities. Only the axial gauge and the Izergin, Korepin, Semenov-Tian-Shansky and Faddeev[4] gauge given in the SU(2) case by

$$\partial_k A_k^3 = 0 , \quad \vec{n}\cdot\vec{\pi}^1 = \vec{n}\cdot\vec{\pi}^2 = 0 \tag{5}$$

are suitable class-I gauges. The temporal gauge condition

$$A_0^\alpha = 0 \tag{6}$$

or, most generally, a class-II gauge condition

$$n^\mu A_\mu^\alpha = 0 , \quad n_0 \neq 0 \tag{7}$$

cannot be deduced from a suitable class-I gauge condition. An indefinite-metric formulation is however always possible.

Our paper is organized as follows. In Sec. II, we will develop our simple example of a particle in a plane, constrained to move on a straight line. The question of choosing a gauge will be treated with several illustrative examples for each case. In Sec. III, we will attack this problem for the free electromagnetic field. In Sec. IV, the same will be done for the pure Yang-Mills field. And finally, in Sec. V, we will conclude with some remarks.

II. MOTION ON A STRAIGHT LINE AS A GAUGE THEORY

Let us start from the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{8}$$

describing a free particle in the x, y plane. We could add a potential to L. It will, however, play no role in the story provided it depends only on x. L is invariant for translations along both axes

$$x \to x + a , \tag{9a}$$

$$y \to y + b . \tag{9b}$$

If we allow the translation along the y axis to be time dependent, we need an additional variable z in order to restore the invariance of the Lagrangian. z transforms as

$$z \to z + \dot{b} \tag{9c}$$

and L becomes

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \dot{y}z + \tfrac{1}{2}z^2 . \tag{10}$$

In contrast to field theory, no new interaction is introduced and no kinetic term for the z variable is allowed. The Euler-Lagrange equation corresponding to z is given by

$$\frac{\partial L}{\partial z} = -(\dot{y} - z) = 0 . \tag{11}$$

In terms of canonically conjugate momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} , \quad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} - z , \tag{12}$$

it reads

$$p_y = 0 . \tag{13}$$

Among the three variables x, y, z only one is independent since p_y and p_z vanish. The Lagrangian (10) describes a theory with one degree of freedom with the help of three variables as the Maxwell-Lagrangian describes a theory in terms of four fields, only one of them being independent. We have then a simple model for a gauge theory.

We now apply the Dirac quantization method for constrained systems. The nonvanishing canonical

Poisson brackets

$$\{x,p_x\}=\{y,p_y\}=\{z,p_z\}=1 \tag{14}$$

are evidently incompatible with the constraints $p_z=p_y=0$, which should be taken in a weak sense. This means that p_y and p_z vanish only when all the Poisson brackets have been calculated. In quantum theory, the corresponding operators do not vanish on their whole domain but only on a part of it, which is taken as the physical Hilbert space. From Eq. (10), the Hamiltonian is

$$H=\tfrac{1}{2}(p_x{}^2+p_y{}^2)+zp_y\,, \tag{15}$$

and can be generalized by

$$H_T=\tfrac{1}{2}(p_x{}^2+p_y{}^2)+zp_y+\lambda p_z\,, \tag{16}$$

where λ is an arbitrary function of coordinates and momenta. H_T is weakly equal to H, which we denote by the symbol $\approx$, i.e., H_T reduces to H if the constraints (11) and (13) are taken in their strong sense. H_T gives rise to the same equations of motion as those deduced from L but in the weak sense defined above. In addition, we get

$$\lambda\approx\dot{z}\ . \tag{17}$$

λ characterizes the gauge freedom and the choice of a gauge consists, generally, in a choice of a given λ. This choice can be made directly on λ or λ can be determined from other conditions. For instance, if we give a relation between the coordinates x,y,z

$$f(x,y,z)\approx 0\,, \tag{18}$$

it is time independent. Therefore

$$\dot{f}\approx 0 \tag{19}$$

and, from the equation of motion and the definition of canonical momenta,

$$p_x\frac{\partial f}{\partial x}+z\frac{\partial f}{\partial y}+\dot{z}\,\frac{\partial f}{\partial z}\approx 0\,. \tag{20}$$

Equation (20) can be solved with respect to $\dot{z}$, which is therefore fixed by (18).

In the same way,

$$f(x,y)\approx 0 \tag{21}$$

gives, by time derivation and the use of equations of motion,

$$p_x\frac{\partial f}{\partial x}+z\frac{\partial f}{\partial y}\approx 0\,, \tag{22}$$

i.e., z can be determined from a condition on x and y and more generally on x,y,p_x. Time derivation of (22) will give $\lambda\approx\dot{z}$.

We can therefore classify the gauge conditions into three classes.

Class I. Gauge conditions involving only x, y and, more generally, x, y, p_x, and p_y. Since p_y weakly vanishes, we can even drop it from the gauge condition for which only the weak form is required. The Poisson brackets of a gauge condition of this class with the constraint p_y do not vanish. The Poisson brackets with p_z do. In Dirac terminology, p_z is a primary and p_y a secondary constraint, i.e., it results from the primary constraint by time derivation and use of equations of motion. A class-I gauge condition can essentially be taken as the variable canonically conjugate to the secondary constraint. By time derivation, it gives a class-II gauge condition.

Class II. Gauge conditions involving x, y, p_x, and also z. The Poisson brackets with the primary constraint do not vanish. A class-II gauge condition can be taken as the variable canonically conjugate to this primary constraint. By time derivation, it gives a class-III gauge condition.

Class III. Gauge conditions involving $\dot{z}$, i.e., conditions fixing the arbitrary functions λ in the generalized Hamiltonian (16).

Some examples will clarify the classification.

1. We can take

$$y\approx 0 \tag{23}$$

as a gauge condition. By time derivation and use of Eq. (11), z and also $\dot{z}$ vanish. This is the natural class-I gauge condition. It allows a direct physical interpretation of the theory as describing the motion of a particle on the x axis.

2. Another class-I gauge condition can be

$$y-f(x)\approx 0\,. \tag{24}$$

It implies, by time derivation and use of the equations of motion,

$$z-p_xf'\approx 0\,, \tag{25}$$

i.e., y and z are fixed in term of the physical degree of freedom x and its canonically conjugate variable p_x. The gauge-group orbits, i.e., the points of the configuration space related one to the other by a gauge transformation, are plane orthogonal to the x axis. The choice of a class-I gauge condition depending only on coordinates clearly consists in a choice of a representative for each orbit, all the points of which are physically equivalent.

3. We can generalize the condition (24) into

$$f(x,y)\approx 0\,. \tag{26}$$

If $\partial f/\partial y\neq 0$, z is also fixed but the representative of the orbit is not generally fixed unambiguously since many solutions of Eq. (26) may exist. When the curve (26) intersects the projection of a gauge-group orbit on the $z=0$ plane in many points, $\dot{z}$, as a function of x and p_x, remains also ambiguous and the gauge is not fixed. This is nothing other than the Gribov ambiguity. A condition for the absence of such ambiguity is that $\partial f/\partial y$ be of constant sign. For instance, $y^2-x\approx 0$ is not a suitable gauge condition, in contrast to $y^3-x\approx 0$, which is quite equivalent to $y-x^{1/3}\approx 0$, since, for reasons of self-adjointness, only real solutions are taken. With $y^3-x\approx 0$, however, we have, as the resulting class-II gauge condition

$$y^2 z - p_x \approx 0\,. \tag{27}$$

At $x=0$, $y=0$, and from Eq. (27), we should get $(p_x)_{x=0}=0$, which is generally nonsense. The accidental vanishing of $\partial f/\partial y$ has not to be taken into account and (27) must be solved with respect to z in order to get

$$z \approx \frac{p_x}{y^2} \tag{28}$$

even at the origin. This value of z is exactly the same as the one given by the equivalent class-I gauge condition $y-x^{1/3}\approx 0$.

4. The simplest class-II gauge condition is $z\approx 0$. From the equations of motion, it results that y is a constant which is not fixed in contrast to a class-I gauge condition. From the physical point of view, there is, however, no fundamental difference between the class I $y=\text{constant}$ and the class II $z=0$. The motion is on a straight line. Only the choice of the origin on the y axis remains arbitrary for class II while it is fixed for class I. There is a physical equivalence between all classes of gauges, as there should be. Only the formalism is quite different.

5. A class-III gauge condition fixes the value of λ in H_T. We rewrite

$$H_T = \tfrac{1}{2}(p_x{}^2 + p_y{}^2) + z p_y + \lambda p_z - \tfrac{1}{2} p_z{}^2 \tag{29}$$

by adding a term quadratic in the primary constraint. This addition does not modify the weak form of the equations of motion. H_T is now a Hamiltonian with three independent degrees of freedom. The corresponding Lagrangian is

$$L = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(\dot{y} - z)^2 - \tfrac{1}{2}(\dot{z} - \lambda)^2 \tag{30}$$

with an arbitrary motion along the z axis. y and z degrees of freedom should cancel and an indefinite metric is needed to assure this cancellation. We do not insist here on this procedure which is better known in the free Maxwell theory.

Class-I and class-II gauge conditions can be considered as constraints and added to the original ones p_y and p_z. A class-II gauge condition and the primary constraint no longer first-class, i.e., their Poisson brackets are not weakly zero. Following Dirac,[3] the Poisson brackets must be modified according to the rule

$$\{A, B\}_D = \{A, B\} - \{A, \phi_\alpha\} C_{\alpha\beta}{}^{-1} \{\phi_\beta, B\}\,, \tag{31}$$

where $\{\,,\}_D$ are the new Dirac brackets, ϕ_α are the second-class constraints, and

$$C_{\alpha\beta} = (\{\phi_\alpha, \phi_\beta\}) \tag{32}$$

is a nonsingular antisymmetric matrix. The use of Dirac instead of Poisson brackets allows the use of second-class constraints as strong equations. If they can be decoupled from the other degrees of freedom, an effective Hamiltonian results. In our example, the class-II gauge condition $z=0$ allows us to set z and p_z strictly equal to zero in the Hamiltonian. We get

$$H = \tfrac{1}{2}(p_x{}^2 + p_y{}^2)\,, \tag{33}$$

i.e., a Hamiltonian describing a two-dimensional motion. The secondary constraint is no longer implied by the Hamiltonian (33) which therefore does not contain the whole physics of the problem. We must select the allowed states by imposing the condition $p_y|\text{phys}\rangle = 0$. In addition, the measure of the Hilbert space $L^2(R^2, d^2x)$ must be modified in order to get only $L^2(R^1, dx)$ as the physical Hilbert space.

This difficulty can be avoided when a class-I gauge condition is imposed. The secondary constraint becomes also second-class and can be decoupled in the same way. The effective Hamiltonian becomes

$$H = \tfrac{1}{2} p_x{}^2\,, \tag{34}$$

which is the usual one for the description of a free one-dimensional motion.

In our example, the decoupling of the constraints and gauge conditions is trivial. It is no longer the case for more general gauge conditions or when nonlinear constraints occur. In principle, we should find a canonical transformation such that the constraints and gauge conditions appear as pairs of new canonically conjugate variables whose Poisson brackets with the physical degrees of freedom are strongly vanishing. Such a transformation should, in principle, solve the problems of operator ordering and configuration-space representation of momenta, which can occur in quantum theory. The example of a motion on a half-line where the second problem occurs has been given elsewhere.[5]

III. THE CHOICE OF A GAUGE FOR THE FREE ELECTROMAGNETIC FIELD THEORY

Let us now turn to the free Maxwell field theory described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{35}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ . \tag{36}$$

The field equations are

$$\partial^\mu F_{\mu\nu}=0\ , \tag{37}$$

while canonical momenta are given by

$$\pi^\mu=\frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)}=F^{\mu 0}\ . \tag{38}$$

It is clear that there is no variable canonically conjugate to $A_0(x)$, which therefore corresponds to the variable z in our mechanical example. A secondary constraint is given by the field Eq. (37) for $\nu=0$. It reads

$$\partial_k\pi_k=0 \tag{39}$$

and corresponds to the vanishing of p_y.

The canonical Hamiltonian is

$$H=\int d^3x\big[\tfrac{1}{2}|\vec\pi(x)|^2+\tfrac{1}{2}|\vec{B}(x)|^2 -\pi_k(x)\partial_k A_0(x)\big]\ , \tag{40}$$

where

$$B_i=\tfrac{1}{2}\epsilon_{ijk}F_{jk}\ . \tag{41}$$

It can be generalized into

$$H_T=H+\int d^3x\ \Lambda(x)\pi_0(x)\ , \tag{42}$$

where Λ is an arbitrary function. H_T clearly reduces weakly to H when the primary constraint

$$\pi_0=0 \tag{43}$$

is taken into account. In the following, we will take the constraints (39) and (43) only in a weak sense and assume the Poisson brackets

$$\{A_\mu(x),\pi_\nu(y)\}_{x_0=y_0}=g_{\mu\nu}\delta^{(3)}(\vec{x}-\vec{y})\ . \tag{44}$$

Using these Poisson brackets, the constraints (43) and (39) are first-class. The generalized Hamiltonian H_T gives rise to the following equations of motion:

$$\partial_0 A_i=\{A_i,H_T\}\approx-\pi_i+\partial_i A_0\ , \tag{45a}$$

$$\partial_0\pi_i=\{\pi_i,H_T\}\approx-\partial_j F_{ji}\ , \tag{45b}$$

$$\partial_0 A_0=\{A_0,H_T\}\approx\Lambda\ . \tag{45c}$$

Any generalized Hamiltonian H_T with a fixed Λ leads to the same weak equations although the strong form of these equations may be quite different. The physics is described by the weak equations while the strong equations are an integral part of the mathematical formulation. Different Λ's lead to different mathematical theories describing the same physical situation. This is the gauge freedom. According to the previous section, there are three classes of gauges. Class I is characterized by gauge conditions involving only $A_k(x)$ and $\pi_k(x)$. For class II, the gauge conditions involve also $A_0(x)$, while in class III $\Lambda(x)$ is given directly. Let us now look at the different usual gauges in light of this classification.

A. Coulomb gauge

The natural class-I gauge condition is the Coulomb condition

$$\partial_k A_k(x)=0\ . \tag{46}$$

Using the equation of motion (45a) and the constraint (39), it clearly implies the class-II gauge condition

$$\Delta A_0(x)\approx 0\ , \tag{47a}$$

which reduces to

$$A_0(x)\approx 0 \tag{47b}$$

if we assume the vanishing of the potentials at infinity.

The constraints being linear, the canonical transformation allowing the decoupling of physical degrees of freedom from constraints and the use of these as strong equations is simply given by transforming A_i into first-class quantities according to the Dirac prescription

$$A'=A-\{A,\phi_\alpha\}C_{\alpha\beta}^{-1}\phi_\beta\ . \tag{48}$$

Since the Coulomb gauge example has been treated explicitly in Ref. 6, we will not give here the details of the procedure and direct the interested reader to this reference. An explicit example which is not found in the literature will be given later. We get here

$$A_i^T(x)=[(\delta_{ij}-\partial_i\Delta^{-1}\partial_j)A_j](x)\ , \tag{49}$$

where the operator Δ^{-1} is given by

$$(\Delta^{-1}f)(x)=-\frac{1}{4\pi}\int\frac{d^3y}{|\vec{x}-\vec{y}|}f(\vec{y})\ . \tag{50}$$

In order to get a canonical transformation, we must add the following transformation giving the unphysical degrees of freedom in terms of original A_μ's:

$$A_i^L(x)=(\partial_i\,\Delta^{-1}\partial_j A_j)(x)\ , \tag{51a}$$

$$A_0'(x)=A_0(x)\ . \tag{51b}$$

This implies the following transformation on π_μ:

$$\pi_i^T(x)=[(\delta_{ij}-\partial_i\Delta^{-1}\partial_j)\pi_j](x)\ , \tag{52a}$$

$$\pi_i^L(x)=(\partial_i\,\Delta^{-1}\partial_j\pi_j)(x)\ , \tag{52b}$$

$$\pi_0'(x)=\pi_0(x)\ . \tag{52c}$$

The Poisson brackets between transverse (T) quantities are identical to the Dirac brackets between original A_i's and π_i's:

$$\{A_i^T(x), \pi_j^T(y)\}_{x_0=y_0} = \{A_i(x), \pi_j(y)\}_{D,\, x_0=y_0} = [(\delta_{ij} - \partial_i \Delta^{-1} \partial_j)\, \delta^{(3)}]\,(\vec{x}-\vec{y})\,. \quad (53)$$

Equation (53) is the basis of the canonical quantization of the free electromagnetic field in the Coulomb gauge.

On the other hand, we have

$$\{A_i^L(x), \pi_j^L(y)\}_{x_0=y_0} = (\partial_i \Delta^{-1} \partial_j\, \delta^{(3)})(\vec{x}-\vec{y})\,, \quad (54a)$$

$$\{A_0'(x), \pi_0'(y)\}_{x_0=y_0} = \delta^{(3)}(\vec{x}-\vec{y})\,, \quad (54b)$$

while the Dirac brackets involving these quantities are all vanishing.

The linearity of the canonical transformation implies the absence of any ordering problems, which are evidently absent from the generalized Hamiltonian H_T and the transformations (49), (51), and (52) can easily be inverted in order to obtain the effective Hamiltonian in the Coulomb gauge

$$H = \tfrac{1}{2} \int d^3x\, (|\vec{\pi}^T|^2 + |\mathrm{curl}\, \vec{A}^T|^2)\,. \quad (55)$$

H is clearly nonlocal. This fact simply results from the appearance of derivatives in the secondary constraint (39).

It may be interesting to look at the indefinite-metric formulation corresponding to the Coulomb gauge. It consists in forgetting the class-I and class-II gauge conditions and taking as fundamental the resulting class-III gauge condition which reads

$$\partial_0 A_0 \approx 0\,. \quad (56)$$

The generalized Hamiltonian density of the Coulomb gauge is

$$\mathcal{H}_C = \tfrac{1}{2}(|\vec{\pi}|^2 + |\vec{B}|^2) - \pi_k \partial_k A_0\,. \quad (57)$$

The corresponding Lagrangian density is given by

$$\mathcal{L}_C = \pi^k \partial_0 A_k + \pi_0 \partial_0 A_0 - \mathcal{H}_G = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \pi_0 \partial_0 A_0\,. \quad (58)$$

Such a Lagrangian allows a direct quantization without the necessity of resorting to the Hamiltonian formalism. The field equations are

$$\partial^\mu F_{\mu\nu} - \partial_0 \pi_0 \delta_{\nu 0} = 0\,, \quad (59)$$

$$\partial_0 A_0 = 0\,, \quad (60)$$

while the canonical commutation relations are

$$[A_\mu(x), \pi_\nu(y)]_{x_0=y_0} = i g_{\mu\nu} \delta^{(3)}(\vec{x}-\vec{y})\,. \quad (61)$$

The theory develops as in an ordinary relativistic gauge. A generalization is possible if we take (56) only in a weak sense. This will be done later (see Sec. III C).

Here a remark is in order. There are three possible formulations of the Coulomb gauge according to which class gauge condition is taken as fundamental. The class-III indefinite-metric formulation is clearly the most general one. It keeps all the unphysical degrees of freedom and introduces a cancellation between them. This cancellation depends on the chosen formulation and also on the chosen gauge. It is not possible for the class-II gauge condition where the scalar temporal degree of freedom is ignored. For the class-I gauge condition, the cancellation is trivial. Both scalar and longitudinal degrees of freedom are ignored.

B. Axial gauge

Another frequently used class-I gauge condition, known as the axial gauge, is

$$\vec{n}\cdot\vec{A} = 0\,, \quad |\vec{n}|^2 = 1\,. \quad (62)$$

Using the field equation (45a), it implies the class-II gauge condition

$$\vec{n}\cdot\vec{\pi} - \vec{n}\cdot\vec{\partial} A_0 \approx 0\,. \quad (63)$$

Applying Eq. (48), we easily get the canonical transformation allowing the decoupling of physical from unphysical degrees of freedom. It reads

$$A_i^\perp = [\delta_{ij} - \partial_i n_j (\vec{n}\cdot\vec{\partial})^{-1}] A_j + [-\partial_i (\vec{n}\cdot\vec{\partial})^{-1} + n_i](\vec{n}\cdot\vec{\partial})^{-1} \pi_0\,, \quad (64a)$$

$$A_i^\parallel = \partial_i n_j (\vec{n}\cdot\vec{\partial})^{-1} A_j - \partial_i (\vec{n}\cdot\vec{\partial})^{-2} \pi_0\,, \quad (64b)$$

$$A_0' = A_0 - (\vec{n}\cdot\vec{\partial})^{-1}\, \vec{n}\cdot\vec{\pi}\,, \quad (64c)$$

$$\pi_i^\perp = \{\delta_{ij} - n_i (\vec{n}\cdot\vec{\partial})^{-1} \partial_j\} \pi_j\,, \quad (64d)$$

$$\pi_i^\parallel = n_i (\vec{n}\cdot\vec{\partial})^{-1} \partial_j \pi_j\,, \quad (64e)$$

$$\pi_0' = \pi_0\,, \quad (64f)$$

where the terms depending on π_0 in (64a) and (64b) are added in order to have

$$\{A_i^{\perp,\parallel}, A_0'\} = 0\,.$$

For the particular case

$$\vec{n} = (0,0,1)\,, \quad (65)$$

$(\vec{n}\cdot\vec{\partial})^{-1}$ is given by

$$(\partial_3^{-1} f)(x) = \tfrac{1}{2} \int d^3y\, \delta(x_1 - y_1)\, \delta(x_2 - y_2)\, \epsilon(x_3 - y_3) f(y)\,. \quad (66)$$

The Poisson brackets between the physical variables A_i and π_i are identical to the Dirac brackets between the original A_i's and π_i's. They read

$$\{A_i(x),\pi_j(y)\}_{x_0=y_0}=\{A_i(x),\pi_j(y)\}_{D,\,x_0=y_0}$$
$$=[\delta_{ij}-\partial_i n_j(\vec{n}\cdot\vec{\partial})^{-1}]\delta^{(3)}(\vec{x}-\vec{y}).\qquad(67)$$

For $\vec{n}$ given by Eq. (65), the effective Hamiltonian in the axial gauge is

$$H=\tfrac{1}{2}\int d^3x\left[\pi_1{}^2+\pi_2{}^2+\left(\partial_3{}^{-1}\sum_{i=1}^{2}\partial_i\pi_i\right)^2+(\partial_1A_2)^2+(\partial_3A_2)^2+(\partial_1A_2-\partial_2A_1)^2\right].\qquad(68)$$

It is also nonlocal, as it should be, since Gauss's law (39) contains derivatives.

It is interesting to look at the indefinite-metric formulation corresponding to the axial gauge. The resulting class-III gauge condition, which will be taken as the fundamental one, reads

$$n_i\partial_jF_{ji}+\vec{n}\cdot\vec{\partial}\,\partial_0A_0\approx 0.\qquad(69)$$

Its solution with respect to ∂_0A_0 is

$$\partial_0A_0\approx-(\vec{n}\cdot\vec{\partial})^{-1}n_i\partial_jF_{ji},\qquad(70)$$

which is nonlocal. Equation (70) can also be written as

$$\partial^\mu A_\mu+(\vec{n}\cdot\vec{\partial})^{-1}\Delta\vec{n}\cdot\vec{A}\approx 0.\qquad(71)$$

The generalized Hamiltonian density is

$$\mathcal{H}_{\text{axial}}=\tfrac{1}{2}(|\vec{E}|^2+|\vec{B}|^2)-\pi_k\partial_kA_0+\pi_0[\partial_kA_k-(\vec{n}\cdot\vec{\partial})^{-1}\Delta\vec{n}\cdot\vec{A}].\qquad(72)$$

The corresponding Lagrangian density

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\pi_0[\partial^\mu A_\mu+(\vec{n}\cdot\vec{\partial})^{-1}\Delta\vec{n}\cdot\vec{A}]\qquad(73)$$

is nonlocal and contains second-order derivatives.

C. $n^\mu A_\mu=0$ gauges

The case $n_0=0$ has been discussed in the previous section. We now assume $n_0\neq 0$, which means that we have a class-II gauge condition. Theories with class-II gauge conditions suffer from the fact that the physics is not contained in the sole effective Hamiltonian. External constraints must be imposed on the physical states and a manipulation of the Hilbert-space measure must be done in order to get the desired physical space. We can avoid the occurrence of class-II gauge conditions either by trying to derive them from appropriate class-I conditions or by using as fundamental the resulting class-III conditions.

Let us first try to derive the class-II gauge condition

$$n^\mu A_\mu=0\qquad(74)$$

from an appropriate class-I condition. Since Eq. (74) contains, as a particular case, $A_0=0$, which can result from the Coulomb gauge condition, the appropriate linear and local class-I condition must be of the form

$$\alpha\partial_iA_i+\beta n_iA_i+\gamma n_i\pi_i=0,\qquad(75)$$

where α, β, and γ are operators and $\alpha\neq 0$. Using Eq. (45a), (75) implies the class-II condition

$$-\alpha\partial_i\pi_i+\alpha\Delta A_0-\beta n_i\pi_i+\beta\vec{n}\cdot\vec{\partial}A_0-\gamma\Delta\vec{n}\cdot\vec{A}+\gamma\vec{n}\cdot\vec{\partial}\partial_jA_j\approx 0\qquad(76)$$

which must be equivalent to (74). Assuming $\alpha=n_0$ for normalization, solving (75) with respect to ∂_iA_i and substituting it into (76), the equivalence gives

$$n_0\left(\Delta+\beta\frac{\vec{n}\cdot\vec{\partial}}{n_0}\right)A_0-\left(\beta+\gamma^2\frac{\vec{n}\cdot\vec{\partial}}{n_0}\right)\vec{n}\cdot\vec{\pi}-\gamma\left(\Delta+\beta\frac{\vec{n}\cdot\vec{\partial}}{n_0}\right)\vec{n}\cdot\vec{A}\approx\kappa(n_0A_0-\vec{n}\cdot\vec{A}).\qquad(77)$$

It results in

$$\kappa=\Delta+\beta\frac{\vec{n}\cdot\vec{\partial}}{n_0},\quad\beta=-\gamma^2\frac{\vec{n}\cdot\vec{\partial}}{n_0},\quad\gamma=1.$$

Therefore the class-I gauge condition implying (74) is

$$n_0{}^2\partial_iA_i-\vec{n}\cdot\vec{\partial}\,\vec{n}\cdot\vec{A}+n_0\vec{n}\cdot\vec{\pi}=0.\qquad(78)$$

It contains the Coulomb gauge and the axial gauge as particular cases. The axial gauge must, however, be treated separately.

Let us now proceed with some details of the Dirac quantization. We first consider only the class-II gauge condition and the primary constraint. They are second-class constraints such that $\{\phi_1,\phi_2\}=n_0$. Using Eq. (48), we transform the class-I gauge condition and the secondary constraint into quantities where Poisson brackets with the other pair of constraints vanish. This gives us the four constraints

$$\begin{aligned}&\phi_1=n_0A_0-\vec{n}\cdot\vec{A},\quad\phi_2=\pi_0,\\&\phi_3=n_0{}^2\partial_iA_i-\vec{n}\cdot\vec{\partial}\,\vec{n}\cdot\vec{A}+n_0\vec{n}\cdot\vec{\pi}-|\vec{n}|^2\pi_0,\\&\phi_4=\partial_k\pi_k-\frac{\vec{n}\cdot\vec{\partial}}{n_0}\pi_0\end{aligned}\qquad(79)$$

for which the $C_{\alpha\beta}$ matrix is

$$C_{\alpha\beta}(x,y)=\{\phi_\alpha(x),\phi_\beta(y)\}_{x_0=y_0}$$
$$=\begin{pmatrix}0&n_0&0&0\\-n_0&0&0&0\\0&0&0&\Delta'\\0&0&-\Delta'&0\end{pmatrix}\delta^{(3)}(\vec{x}-\vec{y}),\qquad(80)$$

where

$$\Delta' = n_0^2\Delta - (\vec{n}\cdot\vec{\partial})^{+2}. \tag{81}$$

Letting Δ'^{-1} be the solution of

$$[n_0^2\Delta - (\vec{n}\cdot\vec{\partial})^{+2}]\,\Delta'^{-1}(\vec{x}-\vec{y}) = \delta^{(3)}(\vec{x}-\vec{y}), \tag{82}$$

we easily get the inverse of the C matrix. It reads

$$C_{\alpha\beta}^{-1}(x,y) = \begin{bmatrix} 0 & -\frac{1}{n_0}\delta^{(3)}(\vec{x}-\vec{y}) & 0 & 0 \\ \frac{1}{n_0}\delta^{(3)}(\vec{x}-\vec{y}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Delta'^{-1}(\vec{x}-\vec{y}) \\ 0 & 0 & \Delta'^{-1}(\vec{x}-\vec{y}) & 0 \end{bmatrix}. \tag{83}$$

Applying Eq. (31), we get the following nonvanishing Dirac brackets:

$$\{A_i(x), A_j(y)\}_{D,\,x_0=y_0} = -n_0(n_i\partial_j + n_j\partial_i)\Delta'^{-1}(\vec{x}-\vec{y}), \tag{84a}$$

$$\{A_i(x), \pi_j(y)\}_{D,\,x_0=y_0} = -\delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) - \partial_i(\vec{n}\cdot\vec{\partial}n_j - n_0^2\partial_j)\Delta'^{-1}(\vec{x}-\vec{y}), \tag{84b}$$

$$\{A_0(x), A_j(y)\}_{D,\,x_0=y_0} = -(|\vec{n}|^2\partial_j + n_j\vec{n}\cdot\vec{\partial})\Delta'^{-1}(\vec{x}-\vec{y}), \tag{84c}$$

$$\{A_0(x), A_0(y)\}_{D,\,x_0=y_0} = -\frac{2|\vec{n}|^2}{n_0}\vec{n}\cdot\vec{\partial}\,\Delta'^{-1}(\vec{x}-\vec{y}), \tag{84d}$$

$$\{A_0(x), \pi_j(y)\}_{D,\,x_0=y_0} = -\frac{n_j}{n_0}\delta^{(3)}(\vec{x}-\vec{y}) - \frac{\vec{n}\cdot\vec{\partial}}{n_0}(\vec{n}\cdot\vec{\partial}n_j - n_0^2\partial_j)\Delta'^{-1}(\vec{x}-\vec{y}), \tag{84e}$$

which are transformed into commutators by quantization.

Let us now take

$$\vec{n} = (0,0,n_3). \tag{85}$$

Equation (78) can be solved with respect to A_3 except if $n_0^2 = n_3^2$, which corresponds to the lightlike gauges. When A_3 can be expressed in terms of A_1 and A_2, these two fields can be kept as fundamental and there is no problem for getting an effective Hamiltonian depending only on physical degrees of freedom. For the lightlike gauges, A_3 must be kept as a fundamental field. However, by (84a),

$$\{A_3(x), A_3(y)\}_{D,\,x_0=y_0} \neq 0.$$

It is clear that such brackets cannot be the canonical Poisson brackets of a theory with an effective Hamiltonian having A_3 as a canonical variable. There is no effective Hamiltonian in the lightlike gauges.

The indefinite metric formulation corresponding to $n^\mu A_\mu = 0$ gauges can easily be obtained by taking the class-III gauge condition

$$n_0\partial_0 A_0 \approx \vec{n}\cdot\vec{\partial}A_0 - \vec{n}\cdot\vec{\pi} \tag{86}$$

which results from (74) by time derivation and use of the equations of motion (45), as the fundamental gauge condition. The Hamiltonian density is, from (40) and (42),

$$\mathcal{H} = \tfrac{1}{2}(|\vec{\pi}|^2 + |\vec{B}|^2) - \pi_k\partial_k A_0 + \pi_0\frac{\vec{n}\cdot\vec{\partial}A_0 - \vec{n}\cdot\vec{\pi}}{n_0}. \tag{87}$$

The corresponding Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{n_0}\pi_0\partial_0(n^\mu A_\mu). \tag{88}$$

It allows a direct quantization based on the field equations

$$\partial^\mu F_{\mu\nu} - \partial_0\pi_0\frac{n_\nu}{n_0} = 0, \tag{89}$$

$$\partial_0(n^\mu A_\mu) = 0 \tag{90}$$

and the canonical commutation relations

$$[A_\mu(x), \pi_\nu(y)]_{x_0=y_0} = ig_{\mu\nu}\delta^{(3)}(\vec{x}-\vec{y}). \tag{91}$$

The Lagrangian (88) can be generalized in order to give Eq. (90) not as a strong equation but as a weak one. This is done by adding a term

$$-\frac{\alpha\pi_0^2}{2}$$

to the Lagrangian density. The strong form of the resulting gauge condition is

$$\partial_0(n^\mu A_\mu) = \alpha\pi_0 \tag{90'}$$

which weakly reduces to (86). The term $\alpha\pi_0^2/2$ must also be added to the Hamiltonian (87). Such terms quadratic in the constraints play no role at all in the derivation of the weak equation since the Poisson brackets of π_0^2 with any quantity are always weakly zero. That is the reason why they have not been taken into account up to now. They are, however, integral parts of the indefinite-metric formulation.

D. Relativistic gauges

Relativistic gauge conditions are written in weak form as

$$\partial^\mu A_\mu \approx 0\,. \tag{92}$$

Involving $\partial_0 A_0$, they are of class III and therefore need an indefinite-metric formulation. As at the end of the previous section, we can transform Eq. (92) into a strong equation by introducing an additional scalar field which should be identified with π_0. We take as a strong class-III relativistic gauge condition

$$\partial^\mu A_\mu = aS\,, \tag{93}$$

where S is a scalar field identified with $-\pi_0$.

Then the generalized Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}(|\vec{\pi}|^2 + |\vec{B}|^2) - \pi_k \partial_k A_0 - S(\partial_k A_k + aS) + \tfrac{1}{2}aS^2\,. \tag{94}$$

We added the term $\frac{1}{2}aS^2$, where a is some constant, in order to find Eq. (93) as resulting from (94). The corresponding Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - S\partial^\mu A_\mu + \tfrac{1}{2}aS^2\,. \tag{95}$$

It is identical to the one used by Lautrup[7] in order to make a unified treatment of relativistic gauges. A consistent quantization can be realized with field equations

$$\partial^\mu F_{\mu\nu} + \partial_\nu S = 0 \tag{96}$$

in addition to Eq. (93) and canonical commutation relations

$$[A_\mu(x), \pi_\nu(y)]_{x_0=y_0} = i g_{\mu\nu}\delta^{(3)}(\vec{x}-\vec{y})\,. \tag{97}$$

Can relativistic gauge conditions be derived from appropriate class-I or -II conditions? The answer is no since it is clear that any of the previously used class-I gauge conditions lead to Eq. (92). We have here an example of a class-III gauge condition which does not result from class-I or class-II conditions.

IV. THE CHOICE OF A GAUGE FOR PURE YANG-MILLS FIELD THEORY

The non-Abelian pure Yang-Mills field theory is characterized by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha\,, \tag{98}$$

where

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g f^\alpha{}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu\,, \tag{99}$$

g is some coupling constant, and the $f^\alpha{}_{\beta\gamma}$'s are the structure constants of a non-Abelian group, say SU(N). $\mathcal{L}$ is invariant under the infinitesimal transformation

$$A^\alpha_\mu \to A^\alpha_\mu + \partial_\mu\theta^\alpha + g f^\alpha{}_{\beta\gamma}\theta^\beta A^\gamma_\mu\,. \tag{100}$$

We have

$$\pi^\mu_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^\alpha_\mu)} = F^{\mu 0}_\alpha \tag{101}$$

so that again A^α_0 has no canonically conjugate variable.

The field equations are

$$D^{\mu\alpha}{}_\beta F^\beta_{\mu\nu} = 0\,, \tag{102}$$

where

$$D^{\mu\alpha}{}_\beta = \partial^\mu\delta^\alpha{}_\beta + g f^\alpha{}_{\beta\gamma}A^{\mu\gamma}\,. \tag{103}$$

For $\nu=0$, Eqs. (102) give the secondary constraints

$$D_k{}^\alpha{}_\beta \pi_k{}^\beta = 0 \tag{104}$$

which are the generalizations of Gauss's law of Maxwell's theory. These constraints as well as the vanishing of π^α_0 are incompatible with the canonical Poisson brackets

$$\{A^\alpha_\mu(x), \pi^\nu_\beta(y)\}_{x_0=y_0} = \delta_\mu{}^\nu\delta^\alpha{}_\beta\delta^{(3)}(\vec{x}-\vec{y}) \tag{105}$$

and will therefore be taken in a weak sense. We have

$$\{D^\alpha_{k\beta}\pi^\beta_k(x), D^\gamma_{l\tau}\pi^\tau_l(y)\}_{x_0=y_0} = g f^{\alpha\gamma}{}_\lambda D^\lambda_{k\beta}\pi^\beta_k \tag{106}$$

so that the constraints are first-class since the right-hand side of Eq. (106) weakly vanishes.

We write the generalized Hamiltonian as

$$H_T = \int d^3x\,[\tfrac{1}{2}\vec{\pi}^\alpha\cdot\vec{\pi}_\alpha + \tfrac{1}{2}\vec{B}^\alpha\cdot\vec{B}_\alpha + \Lambda_\alpha D^\alpha_{k\beta}\pi^\beta_k + \Lambda_\alpha\pi^\alpha_0]\,, \tag{107}$$

where Λ_α are N^2-1 arbitrary functions and

$$B^\alpha_i = \tfrac{1}{2}\epsilon_{ijk}F^\alpha_{jk}\,. \tag{108}$$

The resulting equations of motion are

$$\partial_0 A^\alpha_i = \{A^\alpha_i, H_T\} \approx -\pi^\alpha_i + D_i{}^\alpha{}_\beta A^\beta_0\,, \tag{109a}$$

$$\partial_0\pi^\alpha_i = \{\pi^\alpha_i, H_T\} \approx -D_j{}^\alpha{}_\beta F^\beta_{ji} - g f^\alpha{}_{\beta\gamma}A^\beta_0\pi^\gamma_i\,, \tag{109b}$$

$$\partial_0 A_0 = \{A^\alpha_0, H_T\} \approx \Lambda^\alpha\,. \tag{109c}$$

Owing to the self-interaction of Yang-Mills fields, the choice of a suitable gauge is a particularly difficult task. Before we discuss the usual gauges, let us assume a set of class-I gauge conditions

$$f^\alpha(A^\beta_i, \pi^\beta_i) = 0\,, \quad \alpha = 1, \ldots, N^2-1\,. \tag{110}$$

The resulting class-II conditions are, from Eqs. (109),

$$\frac{\partial f^\alpha}{\partial A^\gamma_i}(-\pi^\gamma_i + D_i{}^\gamma{}_\beta A^\beta_0) + \frac{\partial f^\alpha}{\partial \pi^\gamma_i}(-D_j{}^\gamma{}_\beta F^\beta_{ji} - g f^\gamma{}_{\beta\lambda}A^\beta_0\pi^\lambda_i) \approx 0\,, \tag{111}$$

where it should be kept in mind that $\partial f^\alpha/\partial A_i^\gamma$ and $\partial f^\alpha/\partial \pi_i^\gamma$ may be operators. Equation (111) may be written as

$$\kappa^\alpha{}_\beta A_0^\beta - \frac{\partial f^\alpha}{\partial A_i^\gamma}\pi_i^\gamma - \frac{\partial f^\alpha}{\partial \pi_i^\gamma} D_j{}^\gamma{}_\beta F_{ji}^\beta \approx 0 , \tag{112}$$

where the operator

$$\kappa^\alpha{}_\beta = \frac{\partial f^\alpha}{\partial A_i^\gamma} D_i{}^\gamma{}_\beta - g f^\gamma{}_{\beta\lambda} \frac{\partial f^\alpha}{\partial \pi_i^\gamma}\pi_i^\lambda \tag{113}$$

is also given by

$$\{f_\alpha(x), (D_k{}^\beta{}_\gamma \pi_k^\gamma)(y)\}_{x_0=y_0} = \kappa_\alpha{}^\beta \delta^{(3)}(\vec{x}-\vec{y}) . \tag{114}$$

We have seen that class-I gauge conditions should appear as variables canonically conjugate to the secondary constraints. Therefore, $(\kappa^\alpha{}_\beta)$ should be a constant diagonal matrix. We may however alleviate this too restrictive constraint in order to accept gauges analogous to the $y^3 - x = 0$ gauge condition in our mechanical example. We impose only a unique intersection of class-I gauge conditions with the global ungauged-group orbits. In other words, Eq. (110) considered as an equation on the group elements must have a unique solution. This will assure that the functions Λ^α are univocally fixed in terms of physical fields. From Eq. (112), it is clear that a necessary condition for this is the nonvanishing of $\det\kappa$. An accidental vanishing of $\det\kappa$ can, however, be allowed. We have already encountered this problem in our mechanical example with the gauge $y^3 - x = 0$ where $\det \kappa = y^2$ vanishes at $x = 0$. We solved Eq. (27) which corresponds to Eq. (112) for nonvanishing values of y^2 and took the limit $y = 0$ in the solution. The same job should be done here.

The nonvanishing of $\det\kappa$ is not a sufficient condition for unicity of the gauge fixing. Indeed, in our mechanical example, the gauge $y^2 - x = 0$ leads to $\det \kappa = y$, i.e., exactly the same situation as above but there are two solutions $y = \pm\sqrt{x}$ for the gauge condition. This ambiguity problem is avoided if we require that $\det \kappa$ be of constant sign. In the Faddeev[8] functional method, this determinant is called the gauge compensating term and appears as an additional factor in the functional measure. When it depends only on canonical coordinates, it is at the origin of Faddeev-Popov[1] ghosts in the Lagrangian. If it also depends on canonical momenta, it may happen that the functional integration on these momenta cannot be performed. In other words, an effective Lagrangian does not necessarily exist when $\det \kappa$ depends on momenta.

Let us now examine the various usual gauges, as we have done in the Abelian case.

A. Coulomb gauge

The simplest generalization to Yang-Mills field theory of the Coulomb gauge condition is

$$\partial_k A_k^\alpha = 0 , \tag{115}$$

for which

$$\kappa^{C\alpha}{}_\beta = \partial_k D_{k\beta}^\alpha = \Delta\delta^\alpha{}_\beta + g f^\alpha{}_{\beta\gamma}\partial_k A_k^\gamma + g f^\alpha{}_{\beta\gamma} A_k^\gamma \partial_k$$
$$\approx \Delta\delta^\alpha{}_\beta + g f^\alpha{}_{\beta\gamma} A_k^\gamma \partial_k . \tag{116}$$

Since $\det \kappa_C$ may vanish for large fields, there is a problem of unicity of solution of Eqs. (115), known as Gribov[2] gauge ambiguity. Intrinsically, the Coulomb gauge condition (115) is not a suitable one. It can however be used for perturbation theory around $g = 0$. This method keeps only one solution of Eqs. (115). The resulting class-II gauge condition

$$\Delta A_0^\alpha + g f^\alpha{}_{\beta\gamma} A_i^\beta \partial_i A_0^\gamma - \partial_i \pi_i^\alpha = \mathfrak{D}^\alpha{}_\beta A_0^\beta - \partial_i \pi_i^\alpha \approx 0 \tag{117}$$

can be solved, at least formally, by a perturbation expansion as

$$A_0^\alpha(x) \approx (\mathfrak{D}^{-1\alpha}{}_\beta \partial_i \pi_i)(x) . \tag{118}$$

Here Faddeev-Popov ghosts are present; we refer the reader to the traditional literature[9] for more information. The problem of operator ordering also occurs in the Coulomb gauge. It is solved[10,11] by a point-canonical gauge transformation from a gauge without an ordering problem (the temporal gauge, see below) to the Coulomb gauge. This kind of solution consists in decoupling with the help of a canonical transformation, physical from unphysical degrees of freedom. Since in any generalized Hamiltonian there is no ordering problem, the canonical transformation and its unitary quantum partner solve the ordering problem which can occur in a class I gauge.

B. Izergin, Korepin, Semenov-Tian-Shansky, Faddeev[4] gauge

Since Eqs. (115) are not a suitable extension to the non-Abelian case of the Coulomb gauge condition, how must this extension be effected? Assuming locality and the same naive dimensionality for each gauge condition, the most general extension of the Coulomb gauge is

$$f^\alpha = u^\alpha{}_\beta \partial_k A_k^\beta + a^\alpha{}_\beta \vec{n}\cdot\vec{\pi}^\beta = 0 , \tag{119}$$

where $(u^\alpha{}_\beta)$ and $(a^\alpha{}_\beta)$ are matrices not necessarily transforming as tensors under the group action. We get

$$\kappa_{I\beta}^{\alpha} = u^\alpha{}_\beta \Delta + u^\alpha{}_\lambda f^\lambda{}_{\gamma\beta}\partial_i A_i^\gamma + u^\alpha{}_\lambda f^\lambda{}_{\gamma\beta} A_i^\gamma \partial_i$$
$$- g f^\gamma{}_{\beta\lambda} a_\gamma^\alpha \vec{n}\cdot\vec{\pi}^\lambda . \tag{120}$$

We restrict ourselves now to the SU(2) case and, without loss of generality, we may assume that $\kappa^{\alpha}_{I\beta}$ takes the following form:

$$(\kappa^{\alpha}_{I\beta}) = \begin{pmatrix} \kappa^1{}_1 & \kappa^1{}_2 & 0 \\ \kappa^2{}_1 & \kappa^2{}_2 & 0 \\ \kappa^3{}_1 & \kappa^3{}_2 & \kappa^3{}_3 \end{pmatrix} . \tag{121}$$

This gives

$$u^1{}_1 = u^1{}_2 = u^1{}_3 = u^2{}_3 = u^2{}_2 = u^2{}_1 = 0$$

and

$$a^1{}_1 \vec{\mathrm{n}} \cdot \vec{\pi}^2 - a^1{}_2 \vec{\mathrm{n}} \cdot \vec{\pi}^1 = 0 ,$$

$$a^2{}_1 (\vec{\mathrm{n}} \cdot \vec{\pi})^2 - a^2{}_2 (\vec{\mathrm{n}} \cdot \vec{\pi})^1 = 0 .$$

On the other hand, the nonvanishing of κ_{33} leads to

$$u^3{}_1 = u^3{}_2 = 0 ,$$

$$a^3{}_2 \vec{\mathrm{n}} \cdot \vec{\pi}^1 = a^3{}_1 \vec{\mathrm{n}} \cdot \vec{\pi}^2 = 0 .$$

Equations (119) reduce, then, to

$$f^3 = \partial_k A_k{}^3 + a^3{}_\beta \vec{\mathrm{n}} \cdot \vec{\pi}^\beta , \tag{122a}$$

$$f^1 = \vec{\mathrm{n}} \cdot \vec{\pi}^1 + a^1{}_3 \vec{\mathrm{n}} \cdot \vec{\pi}^3 , \tag{122b}$$

$$f^2 = (\vec{\mathrm{n}} \cdot \vec{\pi})^2 + a^2{}_3 \vec{\mathrm{n}} \cdot \vec{\pi}^3 . \tag{122c}$$

Now, it is clear that, since the f^α's are canonical variables,

$$\{f_\alpha(x), f_\beta(y)\}_{x_0 = y_0} = 0 . \tag{123}$$

This leads to

$$a^3{}_3 = a^2{}_3 = a^1{}_3 = 0 .$$

By this simple method, we rederive, in the SU(2) case, the gauge condition first obtained, for SU(N), by Izergin *et al.*[4] It consists in assuming $\partial_k A^\alpha_k = 0$ only for values of α corresponding to group generators which can be diagonalized. For the other indices, the class-I gauge conditions are $\vec{\mathrm{n}} \cdot \vec{\pi}^\alpha = 0$. Such a set of gauge conditions leads to an effective Lagrangian without Faddeev-Popov ghosts, which is easier to derive in the functional formalism. For details, see Refs. 4 and 12. We restrict ourselves to the derivation of the corresponding class-II gauge conditions

$$-g \vec{\mathrm{n}} \cdot \vec{\pi}^3 A^2_0 - D_j{}^1{}_\beta F^\beta_{ji} n_i \approx 0 , \tag{124a}$$

$$g \mathrm{n} \cdot \pi^3 A^1_0 - D_j{}^2{}_\beta F^\beta_{ji} n_i \approx 0 , \tag{124b}$$

$$\Delta A^3_0 - g(\partial_i A^2_i + A^2_i \partial_i) A^1_0 + g(\partial_i A^1_i + A^1_i \partial_i) A^2_0 - \partial_i \pi^3_i \approx 0 , \tag{124c}$$

$$\mathrm{Det}\kappa_I = (g \vec{\mathrm{n}} \cdot \vec{\pi}^3)^2 \Delta , \tag{125}$$

and Eqs. (124) can be solved with respect to A^α_0. The accidental vanishing of $\vec{\mathrm{n}} \cdot \vec{\pi}^3$ does not lead to any difficulty provided we first solve Eqs. (124) for nonvanishing values of $\vec{\mathrm{n}} \cdot \vec{\pi}^3$ and take later the limit for vanishing values.

C. Other generalization of the Coulomb gauge

In the derivation of the Izergin *et al.* gauge condition, we made the explicit assumption that each gauge condition has the same naive dimension. This may be too restrictive. In this section, we assume

$$\partial_k A^3_k = 0 \tag{126a}$$

while

$$f^\alpha = f^\alpha (A^\beta_k, \pi^\beta_k) = 0, \quad \alpha = 1, 2 , \tag{126b}$$

for which the simplest case is

$$f^\alpha = \mathrm{n} \cdot A^\alpha , \quad \alpha = 1, 2 . \tag{127}$$

We get, in this last case,

$$(\kappa^\alpha{}_\beta) = \begin{pmatrix} \vec{\mathrm{n}} \cdot \vec{\partial} & g \vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^3 & -g \vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^2 \\ -g \vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^3 & \vec{\mathrm{n}} \cdot \vec{\partial} & g \vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^1 \\ g \partial_i (A^2_i \cdot) & -g \partial_i (A^1_i \cdot) & \Delta \end{pmatrix}$$

$$\approx \begin{pmatrix} \vec{\mathrm{n}} \cdot \vec{\partial} & g \vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^3 & 0 \\ -g \vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^3 & \vec{\mathrm{n}} \cdot \vec{\partial} & 0 \\ g \partial_i (A^2_i \cdot) & -g \partial_i (A^1_i \cdot) & \Delta \end{pmatrix} , \tag{128}$$

where

$$\mathrm{Det}\kappa = [(\vec{\mathrm{n}} \cdot \vec{\partial})^2 + g^2 (\vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^3)^2] \Delta \tag{129}$$

is of constant sign. From the point of view of unicity, such a gauge condition is acceptable. Due to the A's dependence on detκ, it generates however, Faddeev-Popov ghosts. This will also be the case for more complicated gauge conditions of this type.

D. Axial gauge

Here, the set of class-I gauge conditions is simply

$$\vec{\mathrm{n}} \cdot \vec{\mathrm{A}}^\alpha = 0 . \tag{130}$$

We have

$$\kappa^\alpha_{A\beta} = n_i D_i{}^\alpha{}_\beta \approx \vec{\mathrm{n}} \cdot \vec{\partial} \delta^\alpha{}_\beta , \tag{131}$$

which is a constant diagonal matrix. There is therefore neither the problem of Faddeev-Popov ghosts nor of gauge ambiguity. The resulting class-II gauge condition is

$$\vec{\mathrm{n}} \cdot \vec{\partial} A^\alpha_0 - \vec{\mathrm{n}} \cdot \vec{\pi}^\alpha \approx 0 . \tag{132}$$

Quantization can be carried out according to the Dirac prescription and an effective Hamiltonian depending only on physical degrees of freedom can be written. We do not here reproduce the details which are given in Ref. 6. Owing to the fact that the left-sides of Eqs. (130) and (132) are not strictly variables canonically conjugate to the constraints, the axial gauge cannot, however, be considered as a natural gauge condition, as was the Coulomb gauge for Maxwell theory.

E. Other class-I gauge conditions

To our knowledge, two other class-I sets of gauge conditions have been tried in order to find an ambiguity-free class-I gauge for non-Abelian Yang-Mills field theory. In the framework of SU(2), they read

$$\epsilon_{\alpha i\beta}\pi_i^\beta=0\,, \tag{133}$$

$$\pi_1^1=\pi_1^2=\pi_2^1=0 \tag{134}$$

and are, respectively, due to Goldstone and Jackiw[13] and Halpern and Koplik.[14] It is clear that, for the Goldstone-Jackiw[13] gauge

$$\kappa^\alpha_{GJ\beta}=g(\pi_\beta^\alpha-\pi_i^i\delta_{\alpha\beta})\,. \tag{135}$$

The class-II gauge condition

$$\kappa^\alpha_{GJ\beta}A_0^\beta-\epsilon^\alpha{}_{i\gamma}D^\gamma_{i\beta}F^\beta_{ji}\approx 0 \tag{136}$$

does not reduce to $A_0^\alpha\approx 0$. For this reason, the Goldstone-Jackiw gauge cannot be a natural gauge condition. Moreover, $\det\kappa_{GJ}$ is not of constant sign and no effective Lagrangian can be written.

The same difficulties occur with the so-called upper triangular gauge used by Halpern and Koplik,[14] for which

$$(\kappa^\alpha_{HK\beta})=\begin{pmatrix}0 & -g\pi_1^3 & 0\\ g\pi_1^3 & & 0\\ 0 & -g\pi_2^3 & g\pi_2^2\end{pmatrix}, \tag{137}$$

and the class-II gauge conditions are

$$\kappa^\alpha_{HK\beta}A_0^\beta-\delta_1^\alpha D^1_{j\beta}F^\beta_{ji}-\delta_2^\alpha D^2_{j\beta}F^\beta_{ji}-\delta_3^\alpha D^1_{j\beta}F^\beta_{ji}\approx 0\,. \tag{138}$$

F. $n^\mu A_\mu^\alpha=0$, $n_0\neq 0$ gauges

Up to now, we failed to find a natural class-I gauge for non-Abelian Yang-Mills theory, i.e., a gauge for which the class-I and the resulting class-II gauge conditions are strictly the variables canonically conjugate to the secondary and the primary constraints, respectively. Two gauges, however, admit an effective Lagrangian free from Faddeev-Popov ghosts. These ghosts are known to be absent for generalized axial gauge conditions

$$n^\mu A_\mu^\alpha=0\,, \tag{139}$$

which, for $n_0\neq 0$, are class II. In the Abelian case, we derived these from appropriate class-I conditions. Let us see if the same can be done here. In the case of a positive answer, the question of finding the natural gauge for non-Abelian Yang-Mills theory is solved by considering the case $\vec{n}=0$.

The weak identification of Eqs. (112) and (139) leads to

$$\frac{\partial f^\alpha}{\partial A_i^\gamma}\pi_i^\gamma+\frac{\partial f^\alpha}{\partial\pi_i^\gamma}D^\gamma_{j\beta}F^\beta_{ji}\approx\frac{1}{n_0}\kappa^\alpha{}_\beta\,\vec{n}\cdot\vec{A}^\beta \tag{140}$$

when $\kappa^\alpha{}_\beta$ is given by Eq. (113). Equations (140) are complicated nonlinear equations involving operators. For the temporal gauge $A_0^\alpha=0$, the solution of these equations should be a generalization of the Coulomb gauge to the non-Abelian case. Such a generalization has been discussed previously. The only suitable solution is the Izergin *et al.*[4] gauge whose resulting class-II condition does not reduce to $A_0^\alpha=0$. Therefore, there exists no class-I gauge condition equivalent to the temporal gauge or more generally to $n^\mu A_\mu^\alpha=0$.

We turn now to the indefinite-metric formulation. Since there are no Faddeev-Popov ghosts, the Lagrangian can be written as a simple generalization of the Abelian one

$$\mathfrak{L}=-\tfrac{1}{4}F^\alpha_{\mu\nu}F_\alpha^{\mu\nu}+\frac{1}{n_0}S_\alpha\partial_0(n^\mu A_\mu^\alpha)-\frac{a}{2}S_\alpha S^\alpha\,, \tag{141}$$

where a is an arbitrary constant and S_α is a ghost field canonically conjugate to A_0^α.

G. Relativistic gauges

As in the Abelian case, relativistic class-III gauge conditions are written in the weak form

$$\partial^\mu A_\mu^\alpha\approx 0\,. \tag{142}$$

It is known they involve Faddeev-Popov ghosts in order to assure a complete cancellation between unphysical degrees of freedom. Since these ghosts are outside of the program fixed in this paper, we do not look further in relativistic gauges and send the reader to the existing literature.[9]

V. CONCLUSIONS

We have made a review of the various gauges used in Abelian and non-Abelian gauge theories in the light of a classification based on the number of unphysical degrees of freedom which are conserved. For class-I gauge conditions, the theory does not contain any unphysical variables at all. For class-II gauge conditions, the secondary constraints like the Gauss law are not decoupled from the physics, while, for class III all the unphysical degrees of freedom are kept. Relativistic gauge conditions are class III and cannot be derived from appropriate class-I or class-II conditions. This is the fundamental reason that any relativistic formulation of the Maxwell theory implies an indefinite-metric space.

For the free Maxwell theory, we succeeded in deriving the class-II generalized axial gauges from equivalent class-I gauge conditions. As a particular case, Coulomb and temporal gauges are equivalent from the physical point of view. Only the formalisms are different. A suitable formalism for the Abelian temporal gauge has been given by Creutz[15] and Willemsen[16] and confirms the equivalence.

For the non-Abelian case, there exists no class-I conditions equivalent to the generalized axial gauges. As a particular case of this result, there exists no natural gauge condition for Yang-Mills theory, i.e., a gauge condition representing a surface orthogonal to the orbits. This result has been proven by many authors[2,11,17] using geometrical arguments. Some class-I gauge conditions are, however, free from ambiguities and ghosts problems. They allow Yang-Mills theory to be described by a number of variables equal to the number of degrees of freedom.

We developed the arguments for gauge fields without interaction with matter fields. The last ones can be introduced into the formalism without any particular difficulty. Let us do it for the Coulomb gauge in the Abelian case. The secondary constraint is now

$$\partial_k \pi_k(x) = j_0(x) \tag{143}$$

and the class-II condition corresponding to the Coulomb gauge is

$$\Delta A_0 \approx j_0(x) \tag{144}$$

instead of $A_0 \approx 0$. Since the Poisson brackets between the charge density j_0 and any gauge field or conjugate momentum vanish, there is no essential difference between the quantization procedure for 0 and j_0.

The problem could be different if we look at the class-I condition corresponding to the temporal gauge. It is

$$\partial_k A_k(x) = \alpha(x) \tag{145}$$

with

$$\partial_0 \alpha = -j_0 \,. \tag{146}$$

Again, the Poisson brackets between α and any gauge quantity vanish, while, from naive dimensional arguments,

$$\{\alpha(x), j_0(y)\}_{x_0=y_0} = \kappa e^2 \Delta \delta^{(3)}(\vec{x}-\vec{y}) \,, \tag{147}$$

where e is the coupling constant and κ a constant depending on the chosen matter field. We have

$$\{\partial_k A_k(x) - \alpha(x), \partial_l \pi_l(y) - j_0(y)\}_{x_0=y_0} = \Delta(1+\kappa e^2)\delta^{(3)}(\vec{x}-\vec{y}) \tag{148}$$

so that there is again no essential difference between the pure gauge theory and the interacting case.

[1]L. D. Faddeev and V. Popov, Phys. Lett. 25B, 29 (1967).
[2]V. Gribov, Lecture at 12th Winter School, Leningrad 1977 (unpublished); Nucl. Phys. B139, 1 (1978).
[3]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
[4]A. Izergin, V. Korepin, M. Semenov-Tian-Shansky, and L. D. Faddeev, Theor. Math. Phys. (USSR) 38, 1 (1979).
[5]A. Burnel, Eur. J. Phys. (to be published).
[6]A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Academia Nazionale dei Lincei, Roma, 1976).
[7]B. Lautrup, Mat. Fys. Medd. Dan Vid. Selsk 35, No. 11 (1967).
[8]L. D. Faddeev, Theor. Math. Phys. 1, 1 (1970).
[9]E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973); L. D. Faddeev and A. A. Slavnov, *Gauge Fields* (Benjamin, Reading, Mass., 1980); R. Jackiw, Rev. Mod. Phys. 52, 661 (1980).
[10]N. H. Christ and T. D. Lee, Phys. Rev. D 22, 939 (1980); T. N. Tudron, Phys. Rev. D 21, 2348 (1980).
[11]M. Creutz, I. Muzinich, and T. Tudron, Phys. Rev. D 19, 531 (1979).
[12]V. Baluni and B. Grossman, Phys. Lett. 78B, 226 (1978).
[13]J. Goldstone and R. Jackiw, Phys. Lett. 74B, 81 (1978).
[14]M. B. Halpern and J. Koplik, Nucl. Phys. B132, 239 (1978).
[15]M. Creutz, Ann. Phys. (N. Y.) 117, 471 (1979).
[16]J. F. Willemsen, Phys. Rev. D 17, 574 (1978).
[17]I. M. Singer, Commun. Math. Phys. 60, 7 (1978); A. Chodos and V. Moncrief, J. Math. Phys. 21, 364 (1980).