Semiclassical representation in quantum theory of a strong electromagnetic field

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A semiclassical representation is introduced to the quantum theory of a strong field, based on a possibility of defining the semiclassical evolution operator in quantum theory, whose effect on the operators of a field and a quantum system transforms them into the solutions of an appropriate semiclassical problem. This representation enables one to go, in a simple way, from a quantum problem to a corresponding semiclassical one, and to use equations of the semiclassical theory as a calculational apparatus in quantum theory and to determine the applicability limits of the semiclassical theory. The possibilities of using the semiclassical representation introduced are illustrated by its use in nonrelativistic quantum electrodynamics.

I. INTRODUCTION

The semiclassical representation method formulated in Refs. 1–3, which is based on the apparatus of operators of a classical field and their eigenstates, enables one to study the transition from a quantum description of an electromagnetic field to a semiclassical one. In this paper the semiclassical evolution operator, based on this approach, is constructed. It converts the free classical field operators and medium operators to the operator solutions of corresponding semiclassical equations. Its existence in the quantum theory is proved and its properties are investigated. The semiclassical representation is formulated, using the semiclassical evolution operator. It is shown that this representation simplifies the transition from quantum theory to semiclassical theory and the determination of the applicability limits of the latter. The transition from nonrelativistic quantum electrodynamics to a semiclassical approximation is considered as an illustration of the use of the semiclassical evolution operator and the semiclassical representation. The physical meaning of quantum correction is discussed using the semiclassical representation. In Sec. II the semiclassical evolution operator Ω and the semiclassical representation of the quantum evolution operator are obtained. The operator Ω operates both on the quantum system operators and the classical amplitude operators and converts them to the operator solutions of corresponding equations of semiclassical approximation, which shows that this representation is similar to a conventional representation in quantum theory.

Section III illustrates the use of the representation suggested by the example of nonrelativistic quantum electrodynamics. In particular, the transition to equations of semiclassical electrodynamics is studied, and the problems of the determination of their applicability limits and physical meaning of quantum corrections for the semiclassical approximation are discussed.

In Sec. IV the method of eliminating the consequences of heuristic elements of the semiclassical theories is indicated. Also, the representations of the quantum evolution operator, which have previously been discussed,[2,3] are compared with the semiclassical representation.

II. THE SEMICLASSICAL REPRESENTATION

Different variants of the semiclassical representation, considered in previous papers,[2,3] have one distinguishing feature: there is no possibility to determine one evolution operator in it whose operation on the variables of the quantum system and strong field would yield the time dependence of these values in accordance with the equations of semiclassical approximation. Now we show in what way such an evolution operator Ω can be constructed and we thereby show that the semiclassical representation based on the existence of Ω has all the properties of the conventional representation in quantum theory. This semiclassical representation enables one to simplify significantly the derivation of the semiclassical approximation from the quantum theory and to clarify the physical meaning of its applicability limits.

The Hamiltonian of our problem has the usual form

$$H = H_R + H_{12}(\hat{a}, \hat{a}^\dagger, x) + H_2(x) . \qquad (1)$$

Here H_R is the Hamiltonian for the radiation field, H_2 for a quantum system and H_{12} for the interaction, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the field, and x stands for an appropriate set of operators for the quantum system.

The field is considered to be strong,[4] hence, according to Refs. 2 and 3, it will be convenient for description of the field to go to the extended Hilbert space $W = \mathcal{H} \otimes \mathcal{H}$ and to introduce in W the operators of classical field amplitudes a_0 and $a_0^\dagger$ as follows: $a_0 = \hat{a} \otimes I + I \otimes \hat{a}^\dagger$, $a_0^\dagger = \hat{a}^\dagger \otimes I + I \otimes \hat{a}$.

Here $\mathcal{H}$ is the conventional Hilbert space for the field and I is the unit operator in $\mathcal{H}$. There is a complete and orthonormal system of eigenstates $|\psi_a\rangle$ of operators a_0 and $a_0^\dagger$ in W, which are defined from the equations $a_0|\psi_a\rangle = a|\psi_a\rangle$ and $a_0^\dagger|\psi_a\rangle = \bar{a}|\psi_a\rangle$. The eigenvalues of a_0 and $a_0^\dagger$ are the amplitudes of the classical field a and $\bar{a}$, which are determined as $a = \langle a|\hat{a}|a\rangle$, $\bar{a} = \langle a|\hat{a}^\dagger|a\rangle$, where $|a\rangle$ is the coherent state in $\mathcal{H}$. The operators $\hat{a}$ and $\hat{a}^\dagger$ are defined in W as $a_W = \hat{a}\otimes I$ and $a_W^\dagger = \hat{a}^\dagger\otimes I$ and we have $a_W = a_0 + \Delta a$, $a_W^\dagger = a_0^\dagger + \Delta a^\dagger$. If ρ is the field density matrix, the state R in W has to be defined as follows: $R = \rho\otimes|0\rangle_r\langle 0|_r$, where $|0\rangle_r$ denotes the vacuum state from the right-hand $\mathcal{H}$ in W.

The main rules of using the classical amplitude operators a_0 and $a_0^\dagger$ to proceed to the case of the strong field can be formulated briefly as follows.

(i) One has to convert the initial state ρ and the operators $\hat{a}$ and $\hat{a}^\dagger$ into the Hilbert space W, to expand all the operators $f(\hat{a}_W, \hat{a}_W^\dagger)$ corresponding to the physical values and the Hamiltonian of the problem $H(a_W, a_W^\dagger)$ over the powers of Δa and $\Delta a^\dagger$, and then truncate the expansion of f and H by the terms $f(a_0, a_0^\dagger, x)$ and $H_W = H(a_0, a_0^\dagger, x) + [\partial H(a_0, a_0^\dagger, x)/\partial a_0]\Delta a + [\partial H(a_0, a_0^\dagger, x)/\partial a_0^\dagger]\Delta a^\dagger$.

(ii) One has to find the operators a_0 and $a_0^\dagger$ or the state R in the Heisenberg representation with the Hamiltonian H_W and to calculate the average values of f in accordance with the following formula:

$$\begin{aligned}\langle f\rangle &= \mathrm{Tr}\,S(t)RS^{-1}(t)f(a_0, a_0^\dagger, x)\\ &= \mathrm{Tr}\,RS^{-1}(t)f(a_0, a_0^\dagger, x)S(t)\\ &= \mathrm{Tr}_x\int da\langle\psi_a|SRS^{-1}|\psi_a\rangle f(a, \bar{a}, x)\,,\\ &\qquad da \equiv d\,\mathrm{Re}\,a\,d\,\mathrm{Im}\,a\,.\end{aligned}$$

Here Tr_x denotes the trace over the states of the quantum system.

It is necessary to add that in the interaction picture one has to use $a_0(t) = a_0 e^{-i\omega t}$, $a(t) = ae^{-i\omega t}$, and $\Delta a(t) = \Delta a e^{-i\omega t}$ instead of a_0, a, and Δa.

Now we start from our Hamiltonian H and write down the Schrödinger equation for the revolution operator in the interaction picture $U(t)$ as follows[5]:

$$\begin{aligned}i\hbar\frac{\partial U}{\partial t} = \Big(&H_2(x) + H_{12}(a_0(t), a_0^\dagger(t), x)\\ &+\frac{\partial H_{12}(a_0(t), a_0^\dagger(t), x)}{\partial a_0(t)}\Delta a(t)\\ &+\frac{\partial H_{12}(a_0(t), a_0^\dagger(t), x)}{\partial a_0^\dagger(t)}\Delta a^\dagger(t)\Big)U,\quad U(t_0) = 1\,.\end{aligned} \tag{2}$$

Let us write the evolution operator U in the form

$$U = QQ_1 \tag{3}$$

and require Q to obey the Schrödinger equation

$$\begin{aligned}i\hbar\frac{\partial Q}{\partial t} = \Bigg\{&\frac{\partial\Phi(a_0(t), a_0^\dagger(t)\,|\,\hat{t})}{\partial a_0(t)}\Delta a(t) + \frac{\partial\Phi(a_0(t), a_0^\dagger(t)\,|\,\hat{t})}{\partial a_0^\dagger(t)}\Delta a^\dagger(t)\\ &+\frac{1}{2}\left(\left[\Delta a(t), \frac{\partial\Phi}{\partial a_0(t)}\right] + \left[\Delta a^\dagger(t), \frac{\partial\Phi}{\partial a_0^\dagger(t)}\right]\right)\Bigg\}Q = H_Q Q,\quad Q(t_0) = 1\,.\end{aligned} \tag{4}$$

Here the designations $\partial\Phi(a_0(t), a_0^\dagger(t)|\hat{t})/\partial a_0(t)$ and $\partial\Phi(a_0(t), a_0^\dagger(t)|\hat{t})/\partial a_0^\dagger(t)$ mean that Φ is differentiated only with respect to a_0 and $a_0^\dagger$ written to the left of the vertical line, $\hat{t} = \hat{t}\{a_0, a_0^\dagger\}$. The operator Φ is not yet defined. The need to take into account the dependence of Φ on a_0 and $a_0^\dagger$ through $\hat{t}$ will become obvious later when we define Φ. The commutator terms in the right-hand side of Eq. (4), i.e., the functions of $a_0^\dagger$ and a_0, are added for H_Q to be Hermitian only.

In the representation determined by Q the operators of the problem vary with the time as follows:

$$\begin{aligned}&Q^{-1}xQ = x\,,\\ &Q^{-1}a_0(t)Q \equiv a_H(t) = a_0(t) - \frac{i}{\hbar}\int_{t_0}^{t}\frac{\partial\Phi(a_H(\tau), a_H^\dagger(\tau)\,|\,Q^{-1}\hat{\tau}Q)}{\partial a_H^\dagger(\tau)}e^{-i\omega(t-\tau)}d\tau\end{aligned} \tag{5}$$

which is equivalent after differentiation with respect to t to the conventional Hamiltonian equation

$$\dot{a}_H = -i\omega a_H - \frac{i}{\hbar}\frac{\partial\Phi(a_H, a_H^\dagger|Q^{-1}\hat{t}Q)}{\partial a_H^\dagger}\,,\quad a_H(t_0) = a_0(t_0)\,. \tag{6}$$

The equation for the operator Q_1 follows from Eqs. (2)–(4) and has the form

$$\begin{aligned}i\hbar\frac{\partial Q_1}{\partial t} = \Bigg\{&H_2(x) + H_{12}(a_H, a_H^\dagger, x) + \left(\left(\frac{\partial H_{12}(a_H, a_H^\dagger, x)}{\partial a_H} - \frac{\partial\Phi(a_H, a_H^\dagger|Q^{-1}\hat{t}Q)}{\partial a_H}\right)Q^{-1}\Delta a(t)Q + \mathrm{H.c.}\right.\\ &\left.+\tfrac{1}{2}Q^{-1}\left(\left[\Delta a(t), \frac{\partial\Phi}{\partial a_0(t)}\right] + \left[\Delta a^\dagger(t), \frac{\partial\Phi}{\partial a_0^\dagger(t)}\right]\right)Q\right)\Bigg\}Q_1\,.\end{aligned} \tag{7}$$

The structure of the Hamiltonian in Eq. (7) allows one to separate out from Q_1 the semiclassical evolution operator C of the quantum system, interacting with the classical field $a_H, a_H^\dagger$. This evolution operator obeys the following equation:

$$i\hbar\frac{\partial C}{\partial t}=[H_2(x)+H_{12}(a_H,a_H^\dagger,x)]C\,,\quad C(t_0)=1\,. \tag{8}$$

It can easily be seen that the operator C changes the time dependence only in the operators of the medium. The operators a_0, $a_0^\dagger$, a_H, $a_H^\dagger$ remain unchanged.

It follows from Eqs. (5), (6), and (8) that the operator Ω,

$$\Omega=QC\,, \tag{9}$$

changes both the field operators and the medium operators. The field amplitudes $\Omega^{-1}a_0(t)\Omega$ and $\Omega^{-1}a_0^\dagger(t)\Omega$ are the solutions of the Hamilton equation for motion in the potential $\Phi(a_H, a_H^\dagger\,|\,Q^{-1}\hat{t}Q)$; see Eqs. (5) and (6). The medium operators $\Omega^{-1}x\Omega=C^{-1}xC$; see Eqs. (5) and (9). Now we can say that Ω leads to the semiclassical theory which is based on Eqs. (6) and (8). These semiclassical equations are not yet defined because the potential Φ in Eq. (6) is not yet defined. In order to define Φ let us represent the quantum evolution operator U as follows:

$$U=\Omega\Theta \tag{10}$$

Now, using Eqs. (7) and (8) and the definition $Q_1=C\Theta$, which is an obvious consequence of Eq. (10), we can write down the equation for the operator Θ:

$$i\hbar\frac{\partial\Theta}{\partial t}=\left\{\left[\frac{\partial H_{12}(a_H,a_H^\dagger\,|\,x_C)}{\partial a_H}-\frac{\partial\Phi(a_H,a_H^\dagger\,|\,Q^{-1}\hat{t}Q)}{\partial a_H}\right]\Omega^{-1}\Delta a(t)\Omega+\text{H.c.}+\tfrac{1}{2}\Omega^{-1}\left(\left[\Delta a(t),\frac{\partial\Phi}{\partial a_0(t)}\right]+\left[\Delta a^\dagger(t),\frac{\partial\Phi}{\partial a_0^\dagger(t)}\right]\right)\Omega\right\}\Theta,\quad x_C\equiv C^{-1}xC\,. \tag{11}$$

The operator Θ describes the quantum and fluctuation corrections for the semiclassical scheme (5), (6), and (8). Equation (11) makes it possible to choose $\Phi(a_0, a_0^\dagger\,|\,\hat{t})$ in Eq. (4) in such a way that the Hamiltonian in Eq. (9) will have an evident fluctuation structure. For these purposes we define Φ as follows:

$$\Phi(a_0,a_0^\dagger\,|\,t)=\mathrm{Tr}_x TH_{12}(a_0,a_0^\dagger\,|\,QC^{-1}xCQ^{-1})\,. \tag{12}$$

Using the definition (12), the semiclassical theory equations, namely, Eqs. (6) and (8), can be written as follows:

$$\Omega^{-1}a_0(t)\Omega\equiv a_H,\quad \dot{a}_H=-i\omega a_H-\frac{i}{\hbar}\frac{\partial\langle H_{12}(a_H,a_H^\dagger\,|\,x_C)\rangle_x}{\partial a_H^\dagger}\,, \tag{13}$$

$$\Omega^{-1}x\Omega=C^{-1}xC\equiv x_C,\quad i\hbar\frac{\partial C}{\partial t}=[H_2(x)+H_{12}(a_H,a_H^\dagger,x)]C\,. \tag{14}$$

Now the set of equations (13) and (14) is the set of self-consistent Hamilton equations. The evolution operator Ω changes the operators a_0, $a_0^\dagger$, and x according to the semiclassical theory equations (13) and (14), so it can be called the semiclassical evolution operator. The equation for Ω follows from its determination [Eq. (9), (4), and (8)], and from the definition (12). We have

$$i\hbar\frac{\partial\Omega}{\partial t}=\left\{H_2(x)+H_{12}(a_0(t),a_0^\dagger,x)+\left(\frac{\partial\langle H_{12}(a_0,a_0^\dagger\,|\,Qx_CQ^{-1})\rangle_x}{\partial a_0(t)}\Delta a(t)+\frac{1}{2}\left[\Delta a(t),\frac{\partial\langle H_{12}(a_0,a_0^\dagger\,|\,Qx_CQ^{-1})\rangle_x}{\partial a_0(t)}\right]+\text{H.c.}\right)\right\}Q\,. \tag{15}$$

Thus, we formally constructed the semiclassical evolution operator, assuming the evident form of Φ, Eq. (10). Now we have to prove that the operator Ω exists in quantum theory, i.e., that our determination of Φ and Eqs. (4), (8), (9), and (13)–(15) are consistent. To check the consistency of these equations it is necessary to prove that the operator $QC^{-1}xCQ^{-1}$ is Hermitian and depends only on the operators x of the quantum system and the operators of classical amplitudes a_0 and $a_0^\dagger$. The Hermitian character of the operator Qx_CQ^{-1} follows immediately from the unitarity of Q and C. To show that the operator $QC^{-1}xCQ^{-1}$ is a functional of a_0, $a_0^\dagger$, and x only, let us consider the unitary operator $M=QCQ^{-1}$ in more detail. The equation for M follows from Eqs. (4) and (8):

$$i\hbar\frac{\partial M}{\partial t}=[H_Q,M]+[H_2(x)+H_{12}(a_0,a_0^\dagger,x)]M\,.$$

The equation for M enables one to write M as $M=C_0L$, where the operator C_0 is determined by the following Schrödinger equation:

$$i\hbar\frac{\partial C_0}{\partial t}=[H_2(x)+H_{12}(a_0,a_0^\dagger,x)]C_0\,,$$

and L satisfies the equation

$$i\hbar\frac{\partial L}{\partial t}=C_0^{-1}[H_Q,C_0]L+[H_Q,L].$$

Having written the solution of this equation as an iteration series with the zero-order approximation L^0, satisfying the equation

$$i\hbar\frac{\partial L}{\partial t}=C_0^{-1}[H_Q,C_0]L^0,$$

one can see that the operator L, and, therefore, the operator M are the functionals of the operators a_0, $a_0^\dagger$, and x only. Hence, our preceding equations are consistent. Therefore, the operator Ω exists in quantum theory and can be constructed according to the above procedure.

The equation for the operator Θ, determining the quantum and fluctuation corrections for semiclassical theory, based on Eqs. (13) and (14), can be rewritten as

$$i\hbar\frac{\partial\Theta}{\partial t}=\left\{\left(\frac{\partial H_{12}(a_H,a_H^\dagger\,|\,x_C)}{\partial a_H}-\frac{\partial\langle H_{12}(a_H,a_H^\dagger\,|\,x_C)\rangle_x}{\partial a_H}\right)\Omega^{-1}\Delta a(t)\Omega+\tfrac{1}{2}\Omega^{-1}\left[\Delta a(t),\frac{\partial\langle H_{12}(a_0,a_0^\dagger\,|\,QxcQ^{-1})\rangle_x}{\partial a_0}\right]\Omega+\text{H.c.}\right\}\Theta. \tag{16}$$

Rewriting Eq. (10),

$$U=\Omega\Theta, \tag{10'}$$

where the operators Ω and Θ obey Eqs. (15) and (16), respectively, one can see that Eq. (10′) enables one to introduce the representation which can be called a semiclassical one. In this representation the operators depend on the time according to Eqs. (13) and (14) and the time dependence of the state is described by the operator Θ. Contrary to the representations discussed previously,[2,3] here we introduce the semiclassical operator representation simultaneously for both groups of operators of the problem, the field operators and the operators of the quantum system.

The convenience of using the above representation for the transition from the quantum theory to the semiclassical theory and for defining the applicability limits of the semiclassical theory will be discussed in the next section, when we illustrate the application of the semiclassical representation method taking the nonrelativistic quantum electrodynamics as an example.

III. NONRELATIVISTIC QUANTUM ELECTRODYNAMICS

Let us consider the use of the semiclassical representation $U=\Omega\Theta$ for nonrelativistic quantum electrodynamics of a strong field. The multimode case will be considered. The Hamiltonian of the problem is of the form

$$H=H_0+H_R+\sum_j\vec{P}_j\vec{A}(\vec{r}_j),\qquad \vec{P}_j=-\sum_{i\in v_j}\frac{e_i}{m_ic}\vec{p}_i, \tag{17}$$

where H_0 is the Hamiltonian of a macroscopic medium, H_R is the Hamiltonian of the free field, j numbers the volumes v_j with the centers at r_j whose dimensions satisfy the conditions of applicability of the long-wave approximation, $\vec{A}(\vec{r}_j)$ is the operator of the transverse vector potential at the point $\vec{r}$, and the other designations are standard.

Now we proceed to the extended space W and write the operator of the vector potential in the interaction representation $\vec{A}(\vec{r},t)$ as follows:

$$\vec{A}(\vec{r},t)=\vec{A}_0(\vec{r},t)+\Delta\vec{A}(\vec{r},t), \tag{18}$$

where the operators $\vec{A}_0$ and $\Delta\vec{A}$ are defined by the following formulas:

$$\vec{A}_0=\sum_{k,\lambda}\left(\frac{2\pi\hbar C}{kL^3}\right)^{1/2}\vec{e}_\lambda(\vec{k})[a_{0\vec{k},\lambda}(t)e^{i\vec{k}\cdot\vec{r}}+\text{H.c.}],$$

$$\Delta\vec{A}=\sum_{k,\lambda}\left(\frac{2\pi\hbar C}{kL^3}\right)^{1/2}\vec{e}_\lambda(\vec{k})[\Delta a_{\vec{k},\lambda}(t)e^{i\vec{k}\cdot\vec{r}}+\text{H.c.}].$$

All designations are conventional and the operators $a_{0\vec{k},\lambda}$ and $\Delta a_{\vec{k}\lambda}$ are the operators a_0 and Δa for the mode with the momentum $\vec{k}$ and polarization λ.

In this case, using the results of Sec. II, the semiclassical representation can easily be determined.

Proceeding to the interaction representation, we have

$$U=\Omega\Theta,\quad \Omega=QC, \tag{19}$$

$$i\hbar\frac{\partial Q}{\partial t}=\sum_j\{\langle QP_{jC}Q^{-1}\rangle_x\Delta\vec{A}(\vec{r}_j,t)+\tfrac{1}{2}[\Delta\vec{A}(\vec{r}_j,t),\langle Q\vec{P}_{jC}Q^{-1}\rangle_x]\}Q, \tag{20}$$

$$i\hbar\frac{\partial C}{\partial t}=\sum_j\vec{P}_j(t)\vec{A}_M(\vec{r}_j,t)C, \tag{21}$$

$$\vec{A}_M(\vec{r},t)\equiv\Omega^{-1}\vec{A}_0(\vec{r},t)\Omega,\quad \Box\vec{A}_M(\vec{r},t)+\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\varphi)=\vec{J}_M, \tag{22}$$

$$\vec{J}_M=\frac{c}{4\pi}\left\langle\sum_j\vec{P}_{jC}\delta(\vec{r}_j-\vec{r})\right\rangle_x,$$

$$i\hbar\frac{\partial\Theta}{\partial t}=\sum_j\{(\vec{P}_{jC}-\langle\vec{P}_{jC}\rangle)\Omega^{-1}\Delta\vec{A}(\vec{r}_j,t)\Omega+\tfrac{1}{2}\Omega^{-1}[\Delta\vec{A}(\vec{r}_j,t),\langle Q\vec{P}_{jC}Q^{-1}\rangle_x]\Omega\}\Theta. \tag{23}$$

We see that the evolution operator Ω transforms the operators of the quantum problem $\vec{A}_0(\vec{r},t)$, $\vec{P}_j$ to the solutions of the self-consistent semiclassical electrodynamics equations, namely, the Schrödinger equations (21) and the Maxwell equations (22). The operator Θ describes the evolution of quantum corrections for semiclassical electrodynamics.

As above, it is not difficult to write the field operator in the Heisenberg representation, using the solutions of the problem (21) and (22), or to find the density matrix in the representation (19), thus showing that the equations of semiclassical electrodynamics can be used as a calculating apparatus of quantum electrodynamics. But we restrict ourselves by studying the physical meaning of quantum corrections to a semiclassical theory only.

In the representation defined by Eq. (18), the quantum average value, according to the density matrix $\rho_0=\rho T$ of arbitrary polynomial operator $f(\vec{A},x)$ at the moment of time t, is written as[6]

$$\langle f\rangle_t=\mathrm{Tr}R_\Theta(t)\left[f(\vec{A}_M,x_C)-\frac{1}{2}\frac{\partial^2 f(\vec{A}_M|x_C)}{\partial A_M^2}\langle 0|(\Delta A)^2|0\rangle+\cdots\right],\quad R_\Theta(t)=\Theta R\Theta^{-1}=\Theta\rho_0\otimes|0\rangle\langle 0|\Theta^{-1}. \tag{24}$$

Here, as in Refs. 1–3, ρ is the density matrix of the strong field and T is the density matrix of the quantum system.

Equation (24) shows in what way the known solutions of the problem of semiclassical electrodynamics enable one to find the unknown quantum averages. Besides, using (24), one can find the applicability limits of semiclassical electrodynamics and can physically interpret the quantum corrections for it. It should be noted that the applicability limits of semiclassical approximation depend on the value of field amplitudes and on the explicit form of f [a possibility of neglecting the term $\partial^2 f/\partial A_M^2$ in comparison with $f(A_M)$], and on the characteristics of the quantum system (the dependence of f on x_C), and also on the contribution of fluctuations to the process considered [the difference between $R_\Theta(t)$ and R]. Therefore, the applicability limits of semiclassical approximation should be determined in any particular cases.

An analysis of the derivation of Eq. (24) which is analogous to derivations of Eqs. (26) and (59) in Ref. 3 shows that the origin of the second term in the right-hand part of Eq. (24) is the noncommutativity of the operators $\hat{a}$ and $\hat{a}^\dagger$ in quantum theory. If f is the field intensity, this term gives the intensity of the usual spontaneous radiation. The appearance of corrections of this kind is obvious and usually is pointed out. As a matter of fact, the expansion in the brackets in Eq. (24) is the expansion of f over the powers of $1/I$ where I is the field intensity.

An evident fluctuation structure of the Hamiltonian in the Schrödinger equation (23) shows that the appearance in Eq. (24) of the time-dependent density matrix $R_\Theta(t)$ is a consequence of the dependence of solution of the Maxwell equation (21)—$\vec{A}_M$ on the current $\vec{J}_M$, but not on the corresponding operator of the quantum system. The corrections of this kind have been discussed previously,[7] but here we obtain these corrections from the quantum theory together with the appropriate mathematical procedure for their estimation.

To obtain a clearer physical interpretation of these corrections and to point out a possible way to take them into account, consider the time dependence of the density matrix $R_\Theta(t)$ in more detail. Assuming that the medium, interacting with the classical field $\vec{A}_M$, is the dissipative subsystem, one can show[8] that for calculating the averages it is sufficient to limit oneself to the density matrix of the form

$$R_\Theta(t)=R^F(t)T,\quad R^F(t)=\mathrm{Tr}R_\Theta(t). \tag{25}$$

Now, in the average value, Eq. (24), the trace can be written in the explicit form

$$\langle f\rangle_t=\int da\,R^F(a,t)\left(\mathrm{Tr}_x Tf(\vec{A}_M,x_C)-\tfrac{1}{2}\mathrm{Tr}_x T\frac{\partial^2 f(\vec{A}_M,x_C)}{\partial\vec{A}_M^2}\langle 0|(\Delta\vec{A})^2|0\rangle+\cdots\right),\quad R^F(a,t)=\langle\psi_a|R^F(t)|\psi_a\rangle. \tag{24$'$}$$

As in Refs. 1–3, it is easy to obtain the equation for $R^F(a,t)$ being the distribution function of the field amplitudes:

$$\frac{\partial R^F(a,t)}{\partial t}=\frac{1}{\hbar^2}\int_{t_0}^t\sum_{i,j}\frac{\delta}{\delta\vec{A}(\vec{r}_j,t)}\left(\langle\langle\vec{P}_{jC}(a,t);\vec{P}_{iC}(a,\tau)\rangle\rangle_x\frac{\delta}{\delta\vec{A}(\vec{r}_i,\tau)}+\left\langle\!\!\left\langle\vec{P}_{jC}(a,t);\frac{\delta\vec{P}_{iC}(a,\tau)}{\delta\vec{A}(\vec{r}_i,\tau)}\right\rangle\!\!\right\rangle_x\right)R^F(a,\tau)d\tau\,,$$

$$\frac{\delta}{\delta\vec{A}(\vec{r}_j,t)}\equiv\sum_{\vec{k},\lambda}\left(\frac{2\pi\hbar c}{kL^3}\right)^{1/2}\vec{e}_\lambda(\vec{k})\frac{\partial}{\partial a_{\vec{k},\lambda}}e^{i\vec{k}\cdot\vec{r}-ikct}+\mathrm{H.c.}\,, \tag{26}$$

$$\langle\langle A;B\rangle\rangle \equiv \tfrac{1}{2}\langle AB\rangle_x + \tfrac{1}{2}\langle BA\rangle_x - \langle A\rangle_x\langle B\rangle_x ,$$

$$\vec{P}_{jC}(a,t)\delta(a-b) \equiv \langle\psi_a|\vec{P}_{jC}(t)|\psi_B\rangle .$$

Here some terms, small in the case of a strong field, are omitted.

As previously, the equation for the distribution function $R^F(a,t)$ is the non-Markovian Fokker-Planck equation. The presence of a friction coefficient in it shows a possibility of making more precise the nonlinear Maxwell equations (22). This possibility was discussed previously[1-3] and here we do not dwell on it.

Now we see that the use of the semiclassical theory equations instead of equations of quantum theory will be possible when the correlators $\langle\langle P_{jC}(a,t);P_{iC}(a,\tau)\rangle\rangle_x$ and $\langle\langle P_{jC}(a,t);\delta P_{iC}(a,\tau)/\delta\vec{A}(\vec{r}_i,\tau)\rangle\rangle_x$ are negligible. In this case the quantum noise influence on the statistical properties of a strong field can be neglected. Equation (26) gives us the possibility of improving the semiclassical approximation and to take into account the change of statistical properties of a strong field interacting with a quantum medium, using the known solutions of an appropriate semiclassical problem.

IV. SOME REMARKS ON THE SEMICLASSICAL THEORIES AND COMPARISON OF THE EVOLUTION OPERATOR REPRESENTATIONS

It is well known that semiclassical theories are inconsistent and include heuristic elements. From our point of view all the semiclassical theories are certain approximations of quantum theory of interacting subsystems and the main problem is to construct a convenient method for the transition from quantum problems to semiclassical ones. We hope that the previous analysis completes the formulation of such a method, which can be called the semiclassical representation method. This method allows one not only to move to the semiclassical theory but also to use the semiclassical theory equations as a calculational apparatus of quantum theory.

Using this method, one can find the quantum meaning of consequences emanating from the presence of heuristic elements in semiclassical theories, and the way in which these consequences can be eliminated. It is clear that the heuristic elements lead to the change-of-motion equations of a field or of a medium only. As is seen from the analysis of the semiclassical representation given in Sec. II, this change leads to the redetermination of the semiclassical evolution operator Ω, Eq. (15). It can easily be seen from Eq. (10) that a new operator Θ will describe the time evolution of the quantum corrections to a chosen variant of the semiclassical theory. So the semiclassical representation method allows one to eliminate the influence of all the heuristic elements of the semiclassical theory.

Now we shall discuss briefly different semiclassical representations of the evolution operator and compare them with each other.

The results of Refs. 2 and 3 and the preceding analysis allow the following relationships for the evolution operator U to be written:

$$U = GQ = CQ = C_0QQ_1 = \Omega\Theta . \tag{27}$$

In the first case $U = GQ$ and the operators G and Q obey the following equations:

$$i\hbar\frac{\partial G}{\partial t} = [H_2(x) + H_{12}(a_H(t), a_H^\dagger(t), x)]\,G, \quad G(t_0) = 1 , \tag{28}$$

$$i\hbar\frac{\partial Q}{\partial t} = \left(\frac{\partial H_{12}(a_0(t), a_0^\dagger(t)\,|\,x_G)}{\partial a_0(t)}\,\Delta a(t) + \frac{\partial H_{12}(a_0(t), a_0^\dagger(t)\,|\,x_G)}{\partial a_0^\dagger(t)}\,\Delta a^\dagger(t) + \cdots\right)Q \quad Q(t_0) = 1 , \; x_G \equiv G^{-1}xG . \tag{29}$$

Here the operators a_H, $a_H^\dagger$, and $x_G = x_C$ are given by Eqs. (13) and (14). From the above equations it can easily be seen that G is the semiclassical evolution operator for the quantum system only, and for describing the semiclassical evolution of the field it is necessary to take into account, partially, the action of the operator Q on the field operators.

A similar situation occurs in the case $U = CQ$ when the equations for C and Q are similar to Eqs. (28) and (29) with other definitions of a_H, $a_H^\dagger$, and x_C only. The case of $U = C_0QQ_1$ is somewhat different from that mentioned above. These operators obey the following equations:

$$i\hbar\frac{\partial C_0}{\partial t} = [H_2(x) + H_{12}(a_0(t), a_0^\dagger(t), x)]\,C_0 , \quad C_0(t_0) = 1 , \tag{30}$$

$$\begin{aligned} i\hbar\frac{\partial Q}{\partial t} = \Bigg(&\left\langle\frac{\partial H_{12}(a_0(t), a_0^\dagger(t)\,|\,x_0)}{\partial a_0(t)}\right\rangle_x \Delta a(t) + \left\langle\frac{\partial H_{12}(a_0(t), a_0^\dagger(t)\,|\,x_0)}{\partial a_0^\dagger(t)}\right\rangle_x \Delta a^\dagger(t) \\ &+ \frac{1}{2}\left[\Delta a(t), \left\langle\frac{\partial H_{12}(a_0(t), a_0^\dagger(t)\,|\,x_0)}{\partial a_0(t)}\right\rangle_x\right] + \frac{1}{2}\left[\Delta a^\dagger(t), \left\langle\frac{\partial H_{12}(a_0(t), a_0^\dagger(t)\,|\,x_0)}{\partial a_0^\dagger(t)}\right\rangle_x\right]\Bigg)Q , \end{aligned} \tag{31}$$

$$i\hbar\frac{\partial Q_1}{\partial t}=\left[\left(\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\,\tilde{x}_0)}{\partial\tilde{a}}-\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\,\tilde{x}_0)}{\partial\tilde{a}}\right\rangle_x\right)Q^{-1}C_0{}^{-1}\Delta a(t)C_0Q\right.$$
$$\left.+\left(\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\,x_0)}{\partial\tilde{a}^\dagger}-\left\langle\frac{\partial H_{12}(\tilde{a}^\dagger,\tilde{a}\,|\,\tilde{x}_0)}{\partial\tilde{a}^\dagger}\right\rangle_x\right)Q^{-1}C_0{}^{-1}\Delta a^\dagger(t)C_0Q\right]Q_1, \tag{32}$$
$$\tilde{a}\equiv Q^{-1}C_0{}^{-1}a_0(t)C_0Q,\ \ x_0\equiv C_0{}^{-1}xC_0,\ \ \tilde{x}_0=Q^{-1}x_0Q.$$

One can see that in this case the operator C_0Q changes both the medium operators and the field operators. However, as we have shown previously,[2,3] in this case the description of a medium behavior in a classical field is inconsistent. This can easily be seen from the fact that due to Eq. (65) in Ref. 3 C_0 depends on the operators $a_0(\tau)$ and $a_0^\dagger(\tau)$, where τ is any time, and that there is no such operator $Q(t)$ which would convert all $a_0(\tau)$ and $a_0^\dagger(\tau)$ to the solutions of corresponding Hamiltonian equations simultaneously.

Only the representation $U=\Omega\Theta$ solves the problem of transition from quantum theory to semiclassical theory in the most convenient and mathematically complete way, since the semiclassical evolution operator Ω transforms both the medium operators x and the field operators $a_0(t)$ and $a_0^\dagger(t)$ to the solutions of a corresponding semiclassical problem.

It should be noted that all the representations lead finally to similar results, because they are the representations of the quantum evolution operator U. So the choice of one of them for the solution of a particular problem will depend on specific conditions of the problem. This choice is similar to that of a convenient variant of the semiclassical theory for the particular problem.

It should be noted that the possibility of writing Eqs. (27) for the quantum evolution operator U illustrates the above remarks on the heuristic elements in semiclassical theories and their elimination.

V. CONCLUSIONS

Thus we have shown that the semiclassical evolution operator can be constructed in a quantum theory which converts the classical field operators and the medium operators into the solutions of the semiclassical theory equations. The existence of this operator allows one to introduce the semiclassical representation into quantum theory, to obtain a simple method for proceeding to the semiclassical theory and to take into account quantum corrections to it.

This method, which we call the semiclassical representation method, allows one to find the quantum meanings of consequences of heuristic elements in the semiclassical theory. The application of the semiclassical representation method has been illustrated by its use in nonrelativistic quantum electrodynamics. In this case the method gives the possibility of using, for the calculations in nonrelativistic quantum electrodynamics, the nonlinear Maxwell equations (21) and (22) and the Fokker-Planck equation (26). It should be mentioned that the use of the fluctuation-dissipation theorems, both linear[9] and nonlinear,[10] enables one, in many cases, to rewrite the nonlinear correlations in Eq. (26) in terms of corresponding nonlinear susceptibilities of the medium. In this case, for calculating the quantum averages using the formula (24) it is sufficient to know the susceptibilities of a medium and the solutions of corresponding nonlinear Maxwell equations (22), and to solve the Fokker-Planck equation (26). It is clear that the above-suggested procedure is simpler than the quantum-electrodynamical calculations in the case of interaction of the strong field with the macroscopic medium. It should be noted that the Fokker-Planck equation (26) gives us a real and correct foundation, from the standpoint of quantum theory, for studying the statistical phenomena in nonlinear optics.

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[4]It is clear that the case of the strong field can be obtained when we use the appropriate initial field density matrix ρ, for which the condition $\langle n\rangle\gg 1$ is fulfilled.

[5]It can easily be seen that in spite of noncommutativity of the operators $\partial H_{12}/\partial a_0$ with Δa and $\partial H_{12}/\partial a_0^\dagger$ with $\Delta a^\dagger$, the Hamiltonian in Eq. (2) is Hermitian, because the operation of Hermitian conjugation results in the

appearance of two similar terms in it with different signs.

[6]It should be noted that Eq. (24) is the consequence of the following formula for the arbitrary normal-ordering operator $f(\hat{a}^\dagger, \hat{a})$:

$$\langle f \rangle = \mathrm{Tr}\, R \left[f(a_0^\dagger, a_0) - \frac{\partial^2 f(a_0^\dagger, a_0)}{\partial a_0 \partial a_0^\dagger} + \ldots \right] .$$

The expansion in the brackets is the result of the antinormal ordering of the operator f and the replacement of the operators a_W and $a_W^\dagger$ by a_0 and $a_0^\dagger$, respectively.

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