Unitary spin as a necessary concomitant of the structure of space-time

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It is shown that a unique Hamiltonian description of the geodesics in spaces of constant curvature leads, necessarily, to spins that are direct products of Dirac spin with generators of SU(3) or higher SU groups.

Since its introduction twenty years ago, unitary spin has stood as a system of cogent yet *ad hoc* internal degrees of freedom. But the query—where may it come from? — has, so far as can be seen, received no clear answer. The issue here is not whether unitary spin *may* be fitted into place as observation seems to require, but whether, from some underlying consideration, it must *necessarily* enter into the description of particles. In the present note it will be shown that unitary spin is not optional, but rather compulsory when quantum theory is consistently built, on a Hamiltonian base, within that simplest extension of flat Minkowski space, the space-time of constant curvature (de Sitter space). This space has in recent times been otherwise remarked as a background space for supersymmetry,[1] and for its possible bearing on quark confinement.[2] The de Sitter space in essential respects may be considered to be prototypical of other curved spaces (see concluding remark below).

While spinors in curved space have been extensively discussed,[3] the outcome has been largely unremarkable in that Dirac spinors alone appear to be sufficient. These treatments have relied on formal devices that produce formally covariant spinor wave equations, as from factorizations like $[\gamma^{\mu}(x)\nabla_{\mu}+m]\psi=0$ of Klein-Gordon equations $(\nabla^{\mu}\nabla_{\mu}+m^2)\psi=0$ (with ∇_{μ} being covariant derivative), without, however, addressing the more primitive question of how to fix the identity of those coordinates, of a particle going along a geodesic, which are to be used in a Hamiltonian treatment with canonical commutation rules. In the present discussion this question is placed in the foreground, where a unique answer is set forth telling, upon analysis, that Dirac spinors are not adequate after all, but must be fused with unitary spinors. Only first quantization need be studied initially in order to set the character of the Hilbert space for the later development of the full second-quantized field theory.

Since principles of general covariance prescribe that arbitrary coordinate transformations are to be allowed at the outset, the coordinates themselves apparently become immediately ambiguous. In the case of constant curvature, however, the ambiguity is clearly resolvable upon viewing the space from a simpler geometric standpoint than the usual differential-geometric one, namely the standpoint of projective geometry, as has been elaborated in an earlier discussion[4] (hereafter called III). Very briefly, the point is that the projective transformations $x_i'=\Lambda_i(x)/\Delta(x)$ (with Λ_i and Δ inhomogeneous linear functions of Cartesian space coordinates $x_1,x_2,x_3=\vec{r}$ and time $x_0=t$) send uniform straight-line motion into uniform straight-line motion (owing to the common denominator Δ), thereby defining a set of extended inertial frames, and having, as such, a distinguished role akin to the ordinary inertial frames linked by linear transformations. When the projective group is now specialized to recover the Poincaré group as a limiting case for $\Delta(x)\to 1$, this specialized projective group (containing a fundamental length scale a) comes to be isomorphic to the de Sitter group [of which only O(3,2) is here considered]. The characteristic differential invariant of the specialized projective group describes a space of constant curvature $1/a^2$, and yet the geodesics are, globally, the straight lines $d^2\vec{r}/dt^2=0$. In thus rewriting curvature as projection, the gain is that the simple setting, of distinguished coordinates of (extended) inertial frames with free-particle geodesics, is held central; and, just as in flat space, this clearly obviates any consideration of general covariance with its coordinate ambiguities.

We may now take it as a fundamental physical hypothesis, buttressed by all experience, that the coordinates of a free particle, seen as such in a family of equivalent inertial frames, are the distinctive ones for stating commutation rules of usual type and proceeding to quantum theory along Hamiltonian lines. There is, *now*, no problem in going to more useful coordinates, e.g., $\tau(t),\vec{\rho}(\vec{r},t)$ used in III, rephrasing the geodesics as harmonic-oscillator motions and giving a purely discrete spectrum to energy squared or, further, to the convenient coordinates $\vec{R}(\vec{\rho})$, which bring a Hamiltonian with square

$$H^2=\vec{P}^2+\vec{L}^2+1+\kappa^2(1+\vec{R}^2)$$
$$\equiv H_1{}^2+\kappa^2 H_2{}^2 ,$$

as derived in III and here written with $\hbar=c=a=1$ with $\vec{P}$ being $\frac{1}{2}(I+\vec{R}\vec{R})\cdot\vec{P}_c+\text{H.c.}$ and $\vec{P}_c$ being the canonical mate $-i\nabla_R$ to $\vec{R}$, while $\vec{L}$ is $\vec{R}\times\vec{P}$ and κ^2 is $m^2-\frac{1}{4}$.

It may readily be seen, first, that here *no Dirac factorization of* H^2 *is possible* (except for $\kappa=0$). For going to even the one-dimensional case (when the terms $\vec{L}^2$ and 1 in $H_1{}^2$ are both to be dropped), H^2 is simply $P^2+\kappa^2(1+X^2)$, and taking H as $F(X)P+G(X)$ (nothing else will do if H is to be a constant of motion) requires

$$F^2=1,\quad FG+GF=iFF',$$
$$G^2-iFG'=\kappa^2(1+X^2)$$

to be identically satisfied in the coordinate x [here F' means $(1+X^2)dF/dx$ and similarly for G']. Multiplying the second, right and left, by F tells $FF'=F'F$ while the first says $FF'+F'F=0$, so that $FF'=0=F'F$, i.e., $F'=0$ and $FG+GF=0$. Then, in the third again multiplying right and left by F brings $FG'=G'F$; but $(FG+GF)'=FG'+G'F=0$, so that $G'F=0=FG'$ or $G'=0$. In short, G is required to be constant, and the third above cannot be satisfied unless $\kappa=0$. The situation in the general case is somewhat similar, again admitting only $\kappa=0$ with H then being only $H_1=\vec{\alpha}\cdot\vec{P}+\vec{\nu}\cdot\vec{L}+\omega$, as shown in III, where $\vec{\nu}$ is Dirac $\vec{\alpha}$ or $-\vec{\sigma}$ and ω is $-i\gamma_5$ or $-I$, respectively. Thus, while $H_1{}^2$ and $H_2{}^2$ are separately Dirac linearizable (the latter, e.g., as $H_2=\beta+\vec{\lambda}\cdot\vec{R}$ with $\vec{\lambda}$ being $\vec{\alpha}$ or $i\beta\vec{\alpha}$, or in still other ways), the pieces H_1 and H_2 cannot be conjoined to produce one overall linear Hamiltonian: H_1 and H_2 are "incompatible" as regards linearization in total.

To meet this difficulty while staying within the structural framework of H^2, it is evidently necessary to relax the presumption that the wave equation $P_\tau{}^2\psi=H^2\psi$ (with $P_\tau\equiv i\partial/\partial\tau$) is to be obtained in the usual way from iteration of some supposed $P_\tau\psi=H\psi$. It is not necessary, after all, that $IP_\tau{}^2\psi=IH^2\psi$ (I being a unit matrix) result from iterating a supposed linear wave equation. It is enough to require only the relaxed linearization $\Gamma P_\tau=\mathscr{H}\psi$ such that upon iteration $\Gamma^2P_\tau{}^2\psi=\Gamma^2H^2\psi$ with Γ^2 a common (constant) matrix multiplier of both sides; but it is essential then that Γ be *singular*, as otherwise $P_\tau\psi=\Gamma^{-1}\mathscr{H}\psi$ must fail, as above demonstrated. It is clear that, in order to keep a grip on Dirac spins in a basic way, they must enter intact as sub-blocks in a larger matrix. The relaxed linearization is then conducted, minimally, by taking $\Gamma=I\times N_0$ (direct product of 4×4 I with singular N_0), and by taking $\mathscr{H}$ as $H_1\times N_1+\kappa H_2\times N_2$ with N_1,N_2 now so selected as to avoid the incompatibility of H_1,H_2 by themselves. In short the linearization

$$I\times N_0P_\tau\psi=(H_1\times N_1+\kappa H_2\times N_2)\psi$$

is to be invoked. The iteration producing $\Gamma^2P_\tau{}^2\psi=\Gamma^2H^2\psi$ now requires

$$(N_i,N_0)=0,\quad N_i{}^2=N_0{}^2\neq0,$$
$$N_1N_2=0=N_2N_1$$

($i=1,2$), where the last of this triplet is what circumvents the H_1,H_2 incompatibility.

The N matrices are easily found. Clearly $N_i{}^3=0$, while $N_i{}^2$ is nonzero, i.e., the N_i are nilpotent of index 3. As follows from their Jordan canonical forms, all such matrices may be written as SuS^{-1}, where u is in the present case at least 4×4 of form (dots meaning 0)

$$u=\begin{bmatrix}\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{bmatrix}$$

(no 3×3 matrix will do). We may therefore take N_1 to be SuS^{-1}, leaving S unspecified, and place $N_2=TuT^{-1}$, or more simply SvS^{-1}, finding

$$v=\begin{bmatrix}\cdot&\cdot&e&f\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&g&\cdot\end{bmatrix},$$

e.g.,

$$v=\begin{bmatrix}\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot\end{bmatrix},$$

with e arbitrary and $fg=1$. The general N_0 is then also quickly found, but is suffices to take $N_0=(N_1+N_2)/\sqrt{2}$.

The appearance of singular non-Hermitian elements is a little disconcerting at first glance. This is answered by requiring S to be unitary and $Su\equiv n_1$ and $Sv\equiv n_2$ to be Hermitian, while introducing the harmless unitary transform $\varphi=I\times S^{-1}\psi$ of ψ. Simply typical S,n_1,n_2 (using the simplified v above) are

$$S=\begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}, \quad n_1=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$n_2=\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix},$$

bringing the final spinor wave equation in Hermitian operators $[n_0=(n_1+n_2)/\sqrt{2}]$

$$I\times n_0 P_\tau \varphi=(H_1\times n_1+\kappa H_2\times n_2)\varphi \ .$$

Here probability conservation comes to be expressed by the familiar $\partial\rho_0/\partial\tau+\nabla_\rho\cdot\vec{j}_0=0$ in the $\tau,\vec{\rho}$ coordinates mentioned earlier, where

$$\rho_0=\varphi^\dagger\{I\times n_0{}^2\}\varphi \ ,$$

$$\vec{j}_0=\varphi^\dagger\{[\vec{\alpha}(1-\rho^2)^{1/2}+\vec{v}\times\vec{\rho}]\times n_0 n_1\}\varphi \ .$$

It may be noted also that indexing φ first according to the rows of the n matrices as, say $\varphi_a,\varphi_b,\varphi_c,\varphi_d$ (with φ_a idle owing to the null first rows), the component φ_c has simultaneously to satisfy $i\varphi_c'=H_1\varphi_c$ and $i\varphi_c'=\kappa H_2\varphi_c$ (with φ_c being further fitted with ordinary spinor indices). This excludes stationary φ_c as examination of H_i shows, while it is demanded from φ_c as either $\exp(-iH_1\tau')\theta_c$ or $\exp(-i\kappa H_2\tau')\theta_c$ that $H_1\theta_c=\kappa H_2\theta_c$, placing the mass parameter κ in the position of eigenparameter rather than a simple assignable numeric (here τ' is $\tau\sqrt{2}$ and φ' is $\partial\varphi/\partial\tau'$). Considering all the alternate permissible forms for H_i and n_i, the final spinors φ make up a rather high spin-tower with a large number of components.

Above, one recognizes in the marked 3×3 subblocks of n_1 and n_2 the elements λ_1 and λ_6 of the conventional set of generators of SU(3) (other members of the set may occur under other choices of v and S), so that it is four-dimensional representations of SU(3) that are compounded with Dirac spin. It may be shown as well that a family of higher SU generators results from higher-dimensional nilpotent N_1,N_2,N_0, but nothing lower than SU(3) is possible.

This concludes the demonstration that *within a Hamiltonian framework, spin in curved space-time has naturally and necessarily to be a fusion of Dirac spin and unitary spin, minimally* SU(3).

As regards the situation in more generally (e.g., gravitationally) curved spaces, one notices that the usual elemental flatness and accompanying local Poincaré covariance may be significantly enlarged[5] to better-fitting local contact or osculating spaces of constant curvature, where then the projective or de Sitter covariance holds, the local recognition of curvature then implying Dirac $\times$ unitary spin locally and eventually globally.

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