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Example of a gauge theory renormalizable even at infinite coupling constant

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A local SU(2) gauge theory with one multiplet of scalars in the adjoint representation is considered. In the limit of infinite gauge coupling constant ($g=\infty$), Yang-Mills fields become auxiliary and the action possesses a larger invariance than the usual gauge invariance; hence, the system develops a richer structure of constraints. The constraint analysis is carried out, the Faddeev-Popov-Senjanović determinant is evaluated, and all the functional integrals in the generating functional $W[J]$ over all canonical momenta, as well as over the gauge fields A_μ^a and two of three components of the scalar field ϕ_a, are evaluated, yielding the extremely curious result that, in this limit ($g=\infty$), the original theory is equivalent to a one-component self-interacting real scalar theory. This is an exact nonperturbative result. Some possible implications of this result are discussed.

The study of the gauge theories, broken or unbroken, have always been very difficult, if not outright impossible, in the strong-gauge-coupling limit. Most of the advances in this area were achieved by the methods of lattice gauge theories and Monte Carlo calculations, since the standard perturbative calculations become completely unreliable as the gauge coupling constant becomes strong.

In this paper, I would like to present a calculation in which I study the behavior of a local SU(2) gauge theory in the limit of infinite coupling constant. The treatment of the problem here is fundamentally different from the lattice calculations in two respects: (i) The theory is studied in the limit of strictly infinite gauge coupling constant, namely, $g=\infty$. For large, but finite values of the gauge coupling constant g, we have nothing to say. (ii) The method used here is not a lattice calculation. It is based on the manipulation of the functional integrals using the extra symmetries the theory possesses in this limit of infinite coupling constant. The correct handling of these extra symmetries enables us to perform most of the functional integrations exactly. At the end, the original theory is reduced to the relatively trivial theory of a one-component self-interacting real scalar field, and all references to Yang-Mills[1] fields disappear completely.

The example I present is an SU(2) gauge theory with a single scalar multiplet ϕ_a, $a=1,2,3$ in the *adjoint representation.* The Lagrangian is

$$\mathcal{L}^{(1)}=-\frac{1}{4g^2}F_{\mu\nu}^aF_a^{\mu\nu}+\tfrac{1}{2}(\mathfrak{D}_\mu\phi)_a(\mathfrak{D}^\mu\phi)_a-V(\phi) \quad , \tag{1}$$

where

$$(\mathfrak{D}_\mu\phi)_a=\partial_\mu\phi_a+f_{abc}\phi_bA_\mu^c \quad ,$$
$$V(\phi)=\tfrac{1}{2}m^2\phi_a\phi_a+\frac{\lambda}{4}(\phi_a\phi_a)^2 \quad ,$$

and f_{abc} are the totally antisymmetric SU(2) structure constants.

Now take the $g=\infty$ limit and drop the pure Yang-Mills part:

$$\mathcal{L}\equiv\lim_{g\to\infty}\mathcal{L}^{(1)}=\tfrac{1}{2}(\mathfrak{D}_\mu\phi)_a(\mathfrak{D}^\mu\phi)_a-V(\phi) \quad . \tag{2}$$

This is the Lagrangian we work with in the rest of this paper. In general, however, one cannot arbitrarily drop some of the terms in a renormalizable Lagrangian and maintain renormalizability, unless of course the new Lagrangian possesses a larger symmetry than the original one due to nonappearance of the terms dropped. The Lagrangian $\mathcal{L}^{(1)}$ is known to be renormalizable[2]; however, the renormalizability of $\mathcal{L}$ is suspect at this point, because the kinetic term for the Yang-Mills fields is dropped by taking the infinite-gauge-coupling limit. Fortunately this actually increases the symmetry. Now, the Lagrangian $\mathcal{L}$ is not only invariant under the ordinary local gauge transformations, but also under the following new exotic transformation:

$$A^a_\mu(x) \to A'^a_\mu(x) \equiv A^a_\mu(x) + \Lambda_\mu(x, \phi(x))\phi^a(x) \quad , (3)$$

where $\Lambda_\mu(x, \phi(x))$ are arbitrary functions of space-time and/or the scalar fields $\phi_a(x)$.

Three points must be emphasized here. The first one is that the Lagrangian $\mathcal{L}^{(1)}$ is not invariant under the transformation given by Eq. (3), but $\mathcal{L}$ is. Therefore we expect the theory with the Lagrangian $\mathcal{L}$ still to be renormalizable since there is this new invariance corresponding to dropping the kinetic term.[3] The second point is that the invariance expressed by Eq. (3) is peculiar to the adjoint representation. If the scalar multiplet transformed by a different (say fundamental) representation of the gauge group we would not have this new invariance. The final point I make note of is that nowhere in this discussion is it essential that the gauge group in question be SU(2). All of the arguments above are equally valid for any semisimple compact Lie group. The group SU(2) is chosen to keep the calculations as simple as possible.

Taking the Lagrangian density $\mathcal{L}$ as our starting point we first compute the canonical momenta. The momenta canonical to the gauge fields A^a_μ vanish: $E^\mu_a \equiv \partial\mathcal{L}/\partial(\partial_0 A^a_\mu) \approx 0$. These are the primary constraints.[4] The momenta canonical to the scalar fields are $\xi_a \equiv \partial\mathcal{L}/\partial(\partial_0\phi_a) = (\mathfrak{D}^0\phi)_a$. The canonical Hamiltonian density $\mathcal{H}_c$ is then computed to be

$$\mathcal{H}_c = \tfrac{1}{2}\xi_a\xi_a - f_{abc}\xi_a\phi_b A^0_c - \tfrac{1}{2}(\mathfrak{D}_i\phi)_a(\mathfrak{D}^i\phi)_a + V(\phi) \quad ,$$

and the primary[5,6] Hamiltonian density is obtained by adjoining the primary constraints: $\mathcal{H}_p = \mathcal{H}_c + E^a_\mu K^\mu_a$. The primary constraints generate the secondary constraints χ_a and ψ^i_a:

$$\{E^0_a, H_p\} \approx f_{abc}\xi_b\phi_c \equiv -\chi_a \approx 0 \quad , \qquad (4)$$

$$\{E^i_a, H_p\} \approx -f_{abc}\phi_b(\mathfrak{D}^i\phi)_c \equiv -\psi^i_a \approx 0 \quad , \qquad (5)$$

where we use Dirac's symbol for weak equality,[4] and $H_p \equiv \int d^3x\,\mathcal{H}_p$. These secondary constraints generate no further constraints. The details of this calculation are too lengthy to be presented here and will be published elsewhere.[7] The secondary constraints given by Eqs. (4) and (5) are not completely independent, however.[8] In the case of SU(2), the 12 secondary constraints given above have only 8 independent ones. For example, χ_1 and ψ^i_1 can be solved in terms of the remaining 8. Therefore the total number of constraints is 20 for SU(2). Trivially, the primary constraints E^0_a are first class.[4] Other first-class combinations of the constraints can be found, such as $(\mathfrak{D}_i E^i)_a - \chi_a$, $\phi_a E^i_a$, $\xi_a E^i_a$, $(\mathfrak{D}_i\phi)_a E^i_a$, $(\mathfrak{D}_i\xi)_a E^i_a$, and $\xi_a\psi^i_a$. Of these, only 5 are independent, bringing the total number of first-class constraints to 8. (Therefore, there are 12 second-class constraints.)

The reader will remember that the original gauge theory with the Lagrangian $\mathcal{L}^{(1)}$ before we took the $g \to \infty$ limit and dropped the kinetic term has exactly 6 constraints: $E^0_\alpha \approx 0$, and $(\mathfrak{D}_i E^i)_a - \chi_a \approx 0$; all 6 of them are *first class.* These are, in fact, the infinitesimal generators of gauge transformations. For the Lagrangian $\mathcal{L}$, however, we have many more constraints: 12 second-class ones, but more importantly, 2 extra first-class constraints. *These 2 new first-class constraints are the generators of the exotic invariance the Lagrangian $\mathcal{L}$ possesses given by Eq. (3).*

As usual, each first-class constraint requires a gauge-fixing constraint to be introduced for it, in this case, a total of 8. The gauge-fixing constraints must be introduced such that the determinant of the matrix of all Poisson brackets must be nonzero.[9] It is also convenient, though in this case not really necessary,[10] to choose the gauge-fixing constraints to have vanishing Poisson brackets with each other. Then, the generating functional W can be written[9,11]

$$W \sim \int \prod_{a,i,\mu} [dA^a_\mu][d\phi_a][dE^a_\mu][d\xi_a]\delta(E^a_\mu)\delta(\chi_2)\delta(\chi_3)\delta(\psi^i_2)\delta(\psi^i_3) \times \prod_{s=1}^{8}\delta(\zeta_s)\,(\det M)^{1/2}\exp\left[i\int d^4x(\dot{A}^a_\mu E^\mu_a + \dot{\phi}_a\xi_a - \mathcal{L})\right] \quad , \qquad (6)$$

where $\zeta_1, \ldots, \zeta_8$ are the 8 gauge-fixing constraints, and M is the 28×28 matrix[12] of the Poisson brackets of all the constraints and the gauge-fixing conditions. The rest of the calculation involves three steps: (i) Choose ζ_s so that $\det M \neq 0$, and $\{\zeta_s, \zeta_{s'}\} \approx 0$; (ii) evaluate the Faddeev-Popov-Senjanović[11] determinant M; (iii) perform the functional integrations wherever possible.

We will choose to work in the axial gauge defined by $\eta_i A^i_a = 0$, where $\eta_\mu = (0, \eta_i)$ is a constant spacelike vector.[13] This could be achieved by a gauge transformation $A^a_\mu \to A'^a_\mu$ such that $\eta_i A'^i_a = 0$.[14] In this case, however, we can do more than just going to the axial gauge. Remembering the new invariance of the action expressed by Eq. (3), we can make a transformation $A'^a_i \to A''^a_i = A'^a_i + \Lambda_i\phi_a$, and choose Λ_i such that $\Lambda_i = -\alpha\cdot A'_i/(\alpha\cdot\phi)$, where $\alpha\cdot\phi$ is short for $\alpha_a\phi_a$ and $\alpha_b(x,\phi)$ are arbitrary functions of space-time and/or scalar fields. Then A''^a_μ satisfies $\alpha_a A''^a_i = 0$, as well as $\eta_i A''^i = 0$. We call this gauge the *doubly axial gauge.* It appears that we have specified six gauge-fixing conditions by $\alpha_a A^a_i = 0$ and $\eta_i A^i_a = 0$, but only five of these are independent: If we denote $\nu_a \equiv \eta_i A^i_a$, then clearly $\alpha_a\nu_a = 0$ on account of $\alpha_a A^i_a = 0$, thus proving that one of the ν_a can be solved in terms of others, and α_a.

To fix the gauge completely, then, we need three more constraints and a convenient choice is $A_a^0=0$, exactly as in the case of the quantization of the original Lagrangian $\mathcal{L}^{(1)}$.[15] Therefore, our gauge-fixing constraints are $\zeta_a=A_a^0$, $\zeta_{i+3}=A_i^a\alpha_a\equiv\mu_i$, and $\zeta_7=\eta_i A_2^i$, $\zeta_8=\eta_i A_3^i$. Clearly, we have $\{\zeta_s,\zeta_{s'}\}=0$ for $s,s'=1,\ldots,8$.

The second step in our calculations involves the evaluation of the Faddeev-Popov-Senjanović determinant: $(\det M)^{1/2}$. At first, the evaluation of this 28×28 determinant might appear to be an insurmountable task, but it is quite straightforward, though admittedly it requires lengthy manipulations. To keep this paper as concise as possible, the calculation of the determinant will only be sketched here, and the details will be published elsewhere.[7]

We first note that the matrix M factors in two blocks, the smaller block generated by E_a^0, and A_a. The contribution of this block to the determinant is field independent; therefore, we can drop it. The remaining 22×22 determinant can be calculated in a number of ways. The easiest one is to exponentiate the matrix by Grassmann variables. Calling this 22×22 matrix $\overline{M}$, we can write

$$\det\overline{M}=\int dc_1^*dc_1\cdots dc_{22}^*dc_{22}\exp(-c^\dagger\overline{M}c)\ ,$$

where $c_1,\ldots,c_{22},c_1^*,\ldots,c_{22}^*$ are anticommuting Grassmann variables. The rest of the calculation proceeds by doing the integrations over the Grassmann variables be making use of the observation that the 9×9 submatrix $\{E_a^i,\rho_a^i\}$ is invertible and in block-diagonal form, where $\rho_1^i=\mu^i$, and $\rho_2^i=\psi_2^i$, $\rho_3^i=\psi_3^i$. At the end, we obtain the result

$$(\det M)^{1/2}=\alpha_1\phi_1{}^4(\alpha\cdot\phi)^2(\phi\cdot\phi)^3\int d\Omega^*d\Omega\,d\omega^*d\omega\exp\left[i\int d^4x\,(\Omega^*\partial^1\Omega+\omega^*\partial^1\omega)\right]\ .$$

Notice that the ghost contribution over the anticommuting ghost fields Ω^*, Ω, ω^*, and ω decouples from the fields and momenta of the Lagrangian $\mathcal{L}$. This is a nice property of the *doubly axial gauge.* Therefore, we can write

$$(\det M)^{1/2}\sim\alpha_1\phi_1{}^4(\alpha\cdot\phi)^2(\phi\cdot\phi)^3\ . \tag{7}$$

Some of the functional integrations in Eq. (6) can be performed trivially now, using the δ functions. These are the integrations over A_a^0 and E_μ^a. For convenience, choose $\eta^i=(1,0,0)$ and $\alpha_a=(\alpha,0,0)$. Then, using the δ functions $\delta(\alpha\cdot A^i)\delta(\eta_jA_2^j)\delta(\eta_jA_3^j)$, we can do the integrations over the fields A_a^1, A_1^2, and A_1^3 at once. To do the rest of the integration over the remaining gauge fields, we use the constraints ψ_a^i:

$$\prod_{i=1}^{3}\delta(\psi_2^i)\delta(\psi_3^i)=\delta(f_{2ab}\phi_a\partial^1\phi_b)\delta(f_{3ab}\phi_a\partial^1\phi_b)\delta^{(4)}(-Q\mathcal{A}+\mathcal{B})\ ,$$

where

$$\mathcal{A}=(A_2^2,A_3^2,A_2^3,A_3^3)\ ,\ \text{and}\ \mathcal{B}=(f_{2ab}\phi_a\partial^2\phi_b,f_{3ab}\phi_a\partial^2\phi_b,f_{2ab}\phi_a\partial^3\phi_b,f_{3ab}\phi_a\partial^3\phi_b)\ ,$$

and

$$Q=\begin{bmatrix}\overline{Q}&0\\0&\overline{Q}\end{bmatrix}\ \text{with}\ \overline{Q}=\begin{bmatrix}\phi_1{}^2+\phi_3{}^2&-\phi_2\phi_3\\-\phi_2\phi_3&\phi_1{}^2+\phi_2{}^2\end{bmatrix}\delta(\vec{x}-\vec{x}')\ .$$

We then obtain

$$W\sim\int\prod_a d\phi_a d\xi_a\delta(\chi_2)\delta(\chi_3)\delta(f_{2ab}\phi_a\partial^1\phi_b)\delta(f_{3ab}\phi_a\partial^1\phi_b)[\alpha^3\phi_1{}^6(\phi\cdot\phi)^3]\frac{1}{\alpha^3}(\det Q)^{-1}\exp\left[i\int d^4x\,L_{\text{eff}}\right]\ ,$$

where

$$\mathcal{L}_{\text{eff}}=\dot{\phi}_a\xi_a-\tfrac{1}{2}\xi_a\xi_a+\tfrac{1}{2}(\partial_i\phi)_a(\partial^i\phi)_a-V(\phi)+\tfrac{1}{2}\mathcal{B}^TQ^{-1}\mathcal{B}\ \text{(Ref. 16)}\ ,$$

and

$$\det Q=\phi_1{}^4(\phi\cdot\phi)^2\ ,$$

and

$$\mathcal{B}^TQ^{-1}\mathcal{B}=-(\partial_i\phi)\cdot(\partial^i\phi)+\frac{1}{\phi\cdot\phi}(\phi\cdot\partial_i\phi)(\phi\cdot\partial^i\phi)\ .$$

The integrations over ξ_2 and ξ_3 can now be performed using the δ functions $\delta(\chi_2)$ and $\delta(\chi_3)$, producing an extra factor of $\phi_1{}^{-2}$. Finally, the Gaussian integration over the momentum ξ_3 can be done, giving us another factor of $\phi_1(\phi\cdot\phi)^{-1/2}$, and we obtain

$$W\sim\int\prod_a d\phi_a\delta(f_{2ab}\phi_a\partial^1\phi_b)\delta(f_{3ab}\phi_a\partial^1\phi_b)\phi_1(\phi\cdot\phi)^{1/2}\exp\left[i\int d^4x\left(\frac{(\phi\cdot\partial_\mu\phi)(\phi\cdot\partial^\mu\phi)}{2\phi\cdot\phi}-V(\phi)\right)\right]\ .$$

Finally, observe that

$$\delta(f_{2ab}\phi_a\partial^1\phi_b)\delta(f_{3ab}\phi_a\partial^1\phi_b)\,d\phi_2 d\phi_3 = \delta(\partial^1 u)\delta(\partial^1 v)\,du\,dv \ ,$$

where $u=\ln(\phi_2/\phi_1)$, and $v=\ln(\phi_3/\phi_1)$, and the integrations over u and v can be performed yielding[17]

$$W \sim \int [d\phi] \exp\left(i \int d^4x [\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)] \right) \ , \quad (8)$$

where $\phi=\sqrt{3}\phi_1$, which is the generating functional for a single-component self-interacting real scalar field, the result I promised before. ϕ, of course, now is an SU(2) *(color) singlet,* since all reference to any group or gauge has completely disappeared.

I would like to conclude by making several comments on this result. First of all, it can be proven rigorously that when $g=\infty$ the theory becomes almost trivial; for large but finite values of g such is not the case in general. Therefore, we conclude that there has to be a phase transition at some value of g, perhaps at $g=\infty$, perhaps at a finite but a very large value.

Secondly, notice that the above treatment of the problem does not depend on whether or not the theory is broken, as long as we start from the Lagrangian $\mathcal{L}$, not $\mathcal{L}^{(1)}$. However, the connection between $\mathcal{L}$ and $\mathcal{L}^{(1)}$ becomes less physical if $m^2<0$, and thus the theory is spontaneously broken. To see that this is so, the reader has to remember that the SU(2) gauge theory with one multiplet of scalar fields in the adjoint representation has an asymptotically free gauge coupling constant g.[18,19] Therefore, we expect the theory with $\mathcal{L}^{(1)}$ to have a large gauge coupling constant at low energy scales, $g=\infty$ reached at $q^2=0$. This all hangs together very well if the theory is not broken spontaneously. However, if the theory is broken, the gauge coupling constant will not grow indefinitely as $q^2\to 0$. In other words, the Lagrangian $\mathcal{L}^{(1)}$ will never have $g=\infty$ for any q^2 and, therefore, it would not make sense to say that $\mathcal{L}$ is the limit of $\mathcal{L}^{(1)}$ as $g\to\infty$. For a broken asymptotically free theory, g is never infinite.

Finally, I emphasize that the above result is not a peculiarity of the group SU(2). The same treatment can be applied to any semisimple compact Lie group.[20] This kind of an extra invariance in the limit of $g\to\infty$ appears only for the adjoint representation of scalars and not for other representations, and not for models containing fermions.

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[1]C. N. Yang and R. Mills, Phys. Rev. 96, 191 (1954).

[2]G. 't Hooft, Nucl. Phys. B35, 167 (1971); B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972).

[3]This of course is verified by our final result that the theory is equivalent to a single-component self-interacting real scalar field theory, which is, of course, renormalizable.

[4]For the notation and terminology, refer to P. A. M. Dirac, Can. J. Math. 2, 129 (1950); J. Anderson and P. Bergmann, Phys. Rev. 83, 1081 (1951); P. Bergmann, Rev. Mod. Phys. 33, 510 (1961); P. A. M. Dirac, *Lectures on Quantum Mechanics,* Belfer Graduate School of Science, Yeshiva University, New York (Academic, New York, 1964).

[5]Sometimes, this is referred to as the total Hamiltonian.

[6]In this paper the Latin indices $a,b,c,\ldots$ are used for the group index, and $i,j,k,\ldots$ for the spatial Lorentz index. The Greek letters $\mu,\nu,\alpha,\ldots$ refer to all four Lorentz indices. Summation is always implied over the repeated indices.

[7]S. Kaptanoglu, in preparation.

[8]For any semisimple Lie group, if we define $S_{ab}=f_{abc}\phi_c$, then $\det S=0$. This is why χ_a and ψ_a^i are not all independent.

[9]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967); Kiev Report No. ITP-67-36, 1967 (unpublished); L. D. Faddeev, Teor. Mat. Fiz. 1, 3 (1969) [Theor. Math. Phys. (USSR) 1, 1 (1970)]; V. N. Popov, CERN Report No. Th. 2424, 1977 (unpublished).

[10]Sinan Kaptanoglu, Phys. Lett. 98B, 77 (1981).

[11]P. Senjanović, Ann. Phys. (N.Y.) 100, 227 (1976).

[12]Actually M is a matrix of infinite size due to space-time indices. It is a bilocal matrix taking a different value at each different pair of space-time points.

[13]For convenience, later we will choose $\eta^\mu=(0,1,0,0)$.

[14]R. Arnowitt and S. Fickler, Phys. Rev. 127, 1821 (1962); R. N. Mohapatra, Phys. Rev. D 4, 2215 (1971).

[15]Once the gauge is partially fixed by $\eta_i A_a^i=0$, it is strictly not possible to make a further gauge transformation and also make A_a^0 vanish. However, the use of $A_a^0=\frac{1}{2}\int d\xi \times |x^1-\xi|(\eta_i E_a^i)_{x_1=\xi}$ instead of $A_a^0=0$ does not really affect the calculations in the functional integral. Therefore $A_a^0=0$ and $\eta_i A_a^i=0$ together can be used to fix the gauge completely, even though this does not coincide with any of the classical trajectories in the phase space. Also see Ref. 10.

[16]When taking inverses, one must not neglect the dependence on spatial indices; for example, $(Q)^{st}_{xx'}(Q^{-1})^{tu}_{x'x''} = \delta^{su}\delta(\vec{x}-\vec{x}'')$.

[17]In this computation we dropped the field-independent determinant $\{\det[\partial^1\delta(\vec{x}-\vec{x}')]\}^2$, and absorbed it into the normalization of the functional W.

[18]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 39, 1343 (1973); D. Politzer, *ibid.* 30, 1346 (1973); D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).

[19]The scalar self-coupling constant λ, however, is not asymptotically free in general.

[20]Like the SU(2) theory presented here, all gauge theories based on a semisimple compact Lie group with one multiplet of scalars in the adjoint representation are asymptotically free. See Ref. 18.