Comment on the Green's function for the anharmonic oscillators

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Application of the analytic theory of continued fractions to anharmonic oscillators of the type $ax^2+bx^4+cx^6$ by Singh *et al.* is shown to give an unphysical result for negative coupling, $b<0$. For $b>0$, their construction of the Green's function $G(E)$ is rigorously proved and extended to the $b \leq 0$ cases.

Let us consider the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x)=E\psi(x) \ ,$$
$$V(x)=ax^2+bx^4+cx^6, \quad c=\alpha^2>0 \ . \tag{1}$$

In accord with Singh *et al.*,[1] we may write its general solutions $\psi_0(x)$, $\psi_1(x)$ in the standard power-series form

$$\psi_\nu(x)=x^\nu e^{-f(x)}\phi(x) \ ,$$
$$\phi(x)=\sum_{n=0}^{N} a_n x^{2n}, \quad \nu=0,1 \ , \tag{2}$$
$$f(x)=\tfrac{1}{4}\alpha x^4+\frac{1}{4\alpha}bx^2, \quad N\leq\infty \ ,$$

where the a_n satisfy recurrence relations of the type

$$\sum_{m=n-1}^{n+1} Q_{nm}a_m=Ea_n, \quad n=0,1,\ldots \tag{3}$$

(see Ref. 1 for further details). The authors of Ref. 1 succeeded in finding an explicit ($n\times n$-determinantal) form of the Taylor coefficients a_n. Requiring termination of the infinite series in Eq. (2), they also obtained an important class of exactly polynomial bound-state solutions $\psi(x)$ for some particular values of a, b, and c in Eq. (1). These results were successfully generalized recently.[2]

Our present aim is to make a few comments concerning the second half of Ref. 1—construction of the physical Green's function $G(E)$ for Eq. (1).

(i) The Hill-determinant method is not fully applicable to Eq. (1) in the formulation of Ref. 1: Indeed, the incorrect initial methodical statements[1] are the following:

(a) "The necessary and sufficient condition that nontrivial a_n's (for $n=0,1,\ldots$) exist which solve Eq. (3) is that the ... infinite determinant $\det(Q-E)$ vanish."

(b) "The difference equation [Eq. (3)] is an equivalent description of the original differential equation (1) for the eigenvalue problem."

Proof. Ad (a) Whenever $N=\infty$, we may choose the value of a_0 as an arbitrary normalization. Then, the recursive treatment of Eq. (3) gives the explicit determinantal formulas for any a_n [$a_1=-a_0Q_{00}/Q_{01}$, $a_2=a_0(Q_{00}Q_{11}-Q_{10}Q_{01})/Q_{01}Q_{12}$, etc.] as given by Eq. (11) in Ref. 1. In general, these a_n's are nonzero irrespective of the value of E. Even for $N<\infty$, we must guarantee that $a_{N+1}=0$ and $a_{N+2}=0$. The first condition coincides indeed with the Hill-determinant specification of energies while the second one [$Q_{N+1N}=0$, cf. Eq. (18) in Ref. 1] fixes one of the couplings as a function of N.

Ad (b) For $N=\infty$ and E not lying in the spectrum of Eq. (1) we still have[1] $a_{n+1}/a_n \sim O(1/\sqrt{n})$, $n>>1$ since $\phi(x)$ is convergent for any x (though not in the norm). Nevertheless, the recurrently defined vector $(a_0,a_1,\ldots)$ remains normalizable ($\sum a_n{}^2<\infty$) and must therefore be admitted as a valid solution to the eigenvalue problem Eq. (3) in principle. Q.E.D.

We may summarize as follows:

Ad (a) Instead of the "nontrivial" a_n's in (a), we have to speak about their N-dependent "approximants" $a_n(N)$, and to replace the "infinite Hill determinant" in Eq. (9) of Ref. 1 by its Nth minor with the N-dependent zeros $E(N)$, $N<\infty$.

Ad (b) When $N=\infty$ in Eq. (2), it is necessary to deliver the missing proof of coincidence of the physical eigenvalues of Eq. (1) with the intuitively chosen "approximants" $E(N)$ in the limit $N\to\infty$ (see below).

(ii) The algebraic manipulations of Ref. 1 only disguise the inapplicable Hill-determinant philosophy: Eq. (3) with $T_n=a_n/a_{n-1}$, $n=1,2,\ldots$ is rewritten in the form[1]

$$B_0+A_0T_1=0 \ ,$$
$$C_1/T_1+B_1+A_1T_2=0 \ ,$$
$$C_2/T_2+B_2+A_2T_3=0 \ , \tag{4}$$
$$\ldots \ ,$$

where (a) the first row is omitted, (b) the continued-fraction approximants $T_n^{(N)}$, $n=N-1, N-2,\ldots,2,1$ to the quantities T_n are defined by the initialization $T_N=T_N^{(N)}=0$ at some $N>>1$, and (c) the first row is "self-consistently" included again in the limit $N\to\infty$.

In this setting, the "self-consistent eigenvalues" $E=E_0$, i.e., the zeros of the auxiliary continued fraction

$$\frac{1}{T_0^{(\infty)}}=-(B_0+A_0T_1^{(\infty)}) \tag{5}$$

are not derived from the Schrödinger boundary condition

$$|\psi(x)|<\infty \tag{6}$$

but from the purely formal requirement (b). As a consequence, it is possible to prove the following:

(iii) For $b<0$, the Green's function of Singh *et al.*[1] defined as $G(E)=T_0^{(\infty)}(E)$ is unphysical:

(a) From the asymptotic form of Eq. (4)

$$T_n=\frac{\alpha}{-b/2\alpha+nT_{n+1}}\left[1+O\left(\frac{1}{n}\right)\right] \tag{7}$$

we find out that the repeated action of mapping $T_{n+1}^{(N)}\to T_n^{(N)}$ leads in general to the accumulation of the values $T_n^{(N)}$, $N>>n>>1$ near the N-independent fixed points,

$$T_n^{(N)}=Z_n\left[1+O\left(\frac{1}{n}\right)\right]\ ,\quad Z_n=-\left(\frac{\alpha}{n}\right)^{1/2}\mathrm{sgn}b+O\left(\frac{1}{n}\right)\ . \tag{8}$$

The proof follows from the geometrical interpretation of Eq. (7) as given in Fig. 1.

(b) The above asymptotic estimate of T_n and of $a_n=a_{n-1}T_n^{(\infty)}=a_{n-2}T_{n-1}^{(\infty)}T_n^{(\infty)}=\cdots$ enables us to decompose

$$\phi(x)=\phi_o(x)+\phi_e(x)\ ,$$
$$\phi_{o/e}(x)=\sum_{n=\text{odd/even}}^{\infty}a_nx^{2n}=O(e^{2f(x)}),\quad x>>1 \tag{9}$$

which follows from the Stirling formula. As a consequence, the quantity $\mathrm{sgn}T_n^{(\infty)}$, $n>>1$, determines the relative sign of the two growing exponents ϕ_o and ϕ_e. The asymptotic cancellation of ϕ_o with ϕ_e (i.e., $Z_n<0$) is the necessary condition for the normalizability of $\psi(x)$—this completes the proof.

Due to the rather exceptional character of the $x\to\infty$ boundary condition Eq. (6), a part of the construction of Singh *et al.* is correct:

(iv) For $b>0$, the poles of $G(E)=T_0^{(\infty)}(E)$ coincide with the anharmonic binding energies: We may verify the algebraic identity

$$T_n=T_n^{(\infty)}+a_0\prod_{j=1}^{n}\frac{C_j}{T_{j-1}^{(\infty)}A_{j-1}}=\frac{a_n}{a_{n-1}} \tag{10}$$

which is valid for any parameter E and values $1/T_{j-1}^{(\infty)}\neq 0$, $j=1,2,\ldots,n$. At $E=E_0$, the difference $T_n-T_n^{(\infty)}$ [i.e., its $1/T_0^{(\infty)}(E)$ factor] changes sign. Since $T_n/T_n^{(\infty)}\sim 1+\text{const}\times(E-E_0)\sim\phi_o/\phi_e$ for $x>>1$, the dominant part of $\phi_o+\phi_e$ also vanishes for $x>>1$. This represents the missing mathematical foundation of the construction of Singh *et al.* Comparison with (iii) clarifies also the puzzling discontinuity[1] of $T_0^{(\infty)}(E)$ at $b=0$.

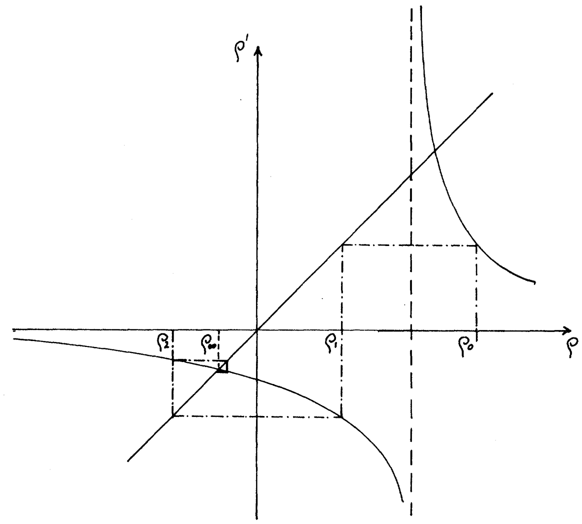

FIG. 1. The mapping $\rho=T_{n+1}^{(N)}\to\rho'=T_n^{(N)}$ with the fixed point $\rho_\infty=Z_n$.

(v) We have seen that the physical fixed point

$$T_n=-(\alpha/n)^{1/2}+O(1/n) \tag{11}$$

of the mappings $T_n\leftrightarrow T_{n+1}$, $n>>1$ in Eq. (4) is determined uniquely by the cancellation requirement. For $b<0$, it is stable if and only if the direction of the recurrences in Eq. (4) is reverted, $T_1\to T_2\to T_3\to\cdots$, $T_1=-B_0/A_0$. In full analogy with the construction of Singh *et al.*, the $b<0$ oscillator may therefore be assigned the Green's function

$$G(E)=\frac{1}{N^{1/2}T_{N+1}(E)} \tag{12}$$

in the limit $N\to\infty$. In this formulation, only the initialization and termination roles of the first and Nth $(N\to\infty)$ row of Eq. (4) are interchanged.

(vi) The most difficult $b=0$ case is characterized by the oscillatory divergence[1] of the recurrences Eq. (4) in both directions—the mappings $T_n\leftrightarrow T_{n+1}$ have no fixed points in the leading order. Hence, the necessary physical cancellation in Eq. (9) [or boundary conditions in infinity, Eq. (6)] and the corresponding smooth n dependence of T_n must be enforced by the "artificial" asymptotic requirement Eq. (11) compatible with the continuity of both the $b\to 0^{\pm}$ limits of the physical $G(E)$ and incompatible with the $b=0$ conjecture of Ref. 1. We omit the further details here.

[1] V. Singh, S. N. Biswas, and K. Datta, Phys. Rev. D 18, 1901 (1978).

[2] E. Magyari, Phys. Lett. 81A, 116 (1981).