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Homogeneous isotropic cosmological model in a new scalar-tensor theory

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A Friedmann-Robertson-Walker (FRW) model is studied in the new scalar-tensor theory of gravitation proposed by Schmidt *et al.* There is an assumption of a particular functional relationship between the cosmological expansion factor of the FRW model and the scalar field. Exact solutions are given, and it is observed that at least in one case $k=-1$ it is possible to obtain a model with oscillation between finite limits.

I. INTRODUCTION

In the theory presented by Schmidt *et al.*[1] the effective gravitational constant is given by the inverse of $\gamma_{\text{eff}}=\gamma-(\beta/12)\phi^2$. In order that the effective gravitational constant decrease with an increase of the scalar field ϕ, γ_{eff} has to increase and therefore β has to be negative. Again it is reasonable to assume that ϕ increases when matter density ρ increases, so that as the collapse proceeds density increases (which is shown later), and correspondingly the scalar field ϕ increases. This was mainly the idea of Schmidt *et al.* to prevent the collapse of sufficiently massive dense objects. But their theory was not adequate for this purpose because, even on varying the coupling constant between scalar and gravitational fields over the entire allowed range, one can vary the effective gravitational constant only within narrow bands.

The present theory may be applied to the case of a Friedmann-Robertson-Walker[2] cosmology. In fact, we have done so and have studied the flat $k=0$ case, that is, for zero spatial curvature. It is well known that in Einstein's theory as long as $(\rho+p)>0$, one cannot get a minimum of the proper volume of the universe so that it is not possible to avoid collapse to a point singularity. By introducing a creation field[3] in the field equations by the introduction of negative pressure[4] one can avoid such a singularity. The deviation from Einstein's theory such as Einstein-Cartan[5] theory, where the spin and torsion have been included in the field equations, can also formally avoid the appearance of a point singularity.

Here we also intend to examine whether in the theory proposed by Schmidt *et al.* one can avoid the problem of a singularity. With the flat open model ($k=0$) there is only one case where the universe oscillates between a finite upper limit and the lower limit of a point singularity itself. In no case, however, can the singularity be avoided. On the other hand, for nonzero values of k (for example, $k=\pm 1$) there are cases where it appears that oscillations are possible between finite limits.

II. FRW MODEL

According to the new scalar-tensor theory proposed by Schmidt *et al.* the field equations are

$$\left(\gamma-\frac{\beta}{12}\phi^2\right)(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)=-\tfrac{1}{2}T_{\mu\nu}-\tfrac{1}{2}[\phi_{,\mu}\phi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}(\phi_{,\alpha}\phi^{,\alpha}-\mu^2\phi^2)]+\frac{\beta}{12}[(\phi^2)_{;\mu\nu}-g_{\mu\nu}(\phi^2)^{;\alpha}{}_{;\alpha}]\ , \quad (2.1)$$

and the wave equation for the massive scalar field ϕ is given by

$$\Box\phi+\left(\mu^2+\frac{\beta}{6}R\right)\phi=0\ . \quad (2.2)$$

Here μ is the mass of the scalar field, β is an arbitrary coupling constant, $\gamma=c^2/16\pi G$ is half of the inverse gravitational constant. The effective inverse gravitational coupling in this theory becomes

$$\gamma_{\text{eff}}=\gamma-\frac{\beta}{12}\phi^2\ ,$$

and the effective mass of the scalar field now is

$$\mu_{\text{eff}} = \left(\mu^2 + \frac{\beta}{6} R \right)^{1/2} .$$

We consider the cosmological model with Robertson-Walker metric

$$ds^2 = -dt^2 + R^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) , \tag{2.3}$$

where $k = -1, 0, +1$. The matter content is assumed to be a perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) v_\mu v_\nu + p g_{\mu\nu} , \tag{2.4}$$

where ρ and p are the mean cosmic mass density and pressure, respectively, and v_μ is the four-velocity satisfying $v_\mu v^\mu = -1$. We choose a comoving coordinate system $v_\mu = \delta^0_\mu$. The nonvanishing components of the field equations (2.1) for the metric (2.3) and matter constant (2.4) are

$$\left(\gamma - \frac{\beta}{12} \phi^2 \right) (k + 2R\ddot{R} + \dot{R}^2) = -R^2 \left\{ \tfrac{1}{2} p + \tfrac{1}{4} (\dot{\phi}^2 + \mu^2 \phi^2) - \frac{\beta}{12} \left[(\phi^2)^{\cdot\cdot} + \frac{2\dot{R}}{R} (\phi^2)^{\cdot} \right] \right\} , \tag{2.5}$$

$$3 \left(\gamma - \frac{\beta}{12} \phi^2 \right) (k + \dot{R}^2) = R^2 \left[\tfrac{1}{2} \rho + \tfrac{1}{4} (\dot{\phi}^2 - \mu^2 \phi^2) + \frac{\beta}{4} \frac{\dot{R}}{R} (\phi^2)^{\cdot} \right] . \tag{2.6}$$

The wave equation (2.2) with (2.3) reduces to

$$\ddot{\phi} + 3 \frac{\dot{R}}{R} \dot{\phi} + \left\{ -\mu^2 + \beta \left[\frac{k}{R^2} + \frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R} \right)^2 \right] \right\} \phi = 0 . \tag{2.7}$$

III. SOLUTION FOR FRW MODEL

The number of independent equations above are three, namely, (2.5), (2.6), and (2.7), whereas the number of unknown functions of time are R, ϕ, ρ, and p. We have therefore the freedom to assume one appropriate relation between them. Before making this assumption we observe that the field equations (2.1) satisfy the motion equation

$$T^{\mu\nu}{}_{;\nu} = 0 . \tag{3.1}$$

Substituting (2.3) into (3.1) one gets

$$\frac{\dot{\rho}}{\rho + p} = -3 \frac{\dot{R}}{R} . \tag{3.2}$$

It is clear from (3.2) that, since $(\rho + p) > 0$, the density ρ increases when the scale factor R decreases and, further, it is reasonable to expect that the scalar field ϕ increases when ρ increases. Thus the scalar field is expected to increase with the decrease in the volume of the universe. We assume therefore a relation

$$\phi = \phi_0 R^n , \tag{3.3}$$

with $n < 0$ and ϕ_0 being a constant. Substituting the relation (3.3) into the wave equation (2.7) we obtain an equation

$$aR\ddot{R} + b\dot{R}^2 + cR^2 + d = 0 , \tag{3.4}$$

with

$$a = n + \beta, \quad b = n^2 + 2n + \beta, \quad c = -\mu^2, \quad d = k\beta. \tag{3.5}$$

To solve the differential equation (3.4) we define a new function $f(R)$ such that $\dot{R} = f(R)$ and call $f^2 = u$, then (3.4) becomes

$$\tfrac{1}{2} aR \frac{du}{dR} + bu + cR^2 + d = 0 , \tag{3.6}$$

$$u = \dot{R}^2 = c_1 R^{-2b/a} - \frac{c}{a+b} R^2 - \frac{d}{b} , \quad \text{for } \frac{b}{a} \neq -1 , \quad b \neq 0 ; \tag{3.7}$$

$$u = \dot{R}^2 = c_1 R^2 - \frac{2c}{a} R^2 \ln R - \frac{d}{b} , \quad \text{for } \frac{b}{a} = -1 ; \tag{3.8}$$

$$u = \dot{R}^2 = c_1 - \frac{c}{a} R^2 - 2d \ln R , \quad \text{for } b = 0 \text{ or } n = -1 - (1 - \beta^2)^{1/2} . \tag{3.9}$$

We observe that $a < 0$ since $n < 0$ and $\beta < 0$ by the arguments of Schmidt *et al.* Now we analyze the properties of the solutions (3.7) to (3.9).

We consider first $k = 0$ and, consequently, $d = 0$. Here the universe is spatially flat.

Case (i). $-b/a > 1$. This means $b > 0$ since $a < 0$, hence $-c/(a+b) > 0$. We define $-c/(a+b) = v^2$. There may be two situations:

If $c_1 > 0$, putting $c_1 = p^2$ we have from (3.7)

$$\dot{R}^2 = p^2 R^{-2b/a} + v^2 R^2 .$$

There is neither a maximum nor a minimum of R for its finite values.

If $c_1 < 0$, putting $c_1 = -p^2$, we have from (3.7)

$$\dot{R}^2 = -p^2 R^{-2b/a} + v^2 R^2 .$$

We have $\dot{R} = 0$ at $R = 0$ and subsequently, for an exploding model, $\dot{R} = 0$ again for $R = R_0$ where $v^2 R_0{}^2 = p^2 R_0{}^{-2b/a}$. It can be shown that $\ddot{R} < 0$ at $R = R_0$, which corresponds to the maximum of R. On the other hand, $\ddot{R} = 0$ at $R = 0$. So we get only a point of inflection at $R = 0$ and there is no further bounce from this point for a collapsing model.

Case (ii). $0 < -b/a < 1$. Since $a < 0$, in this case we have $b > 0$ and $-b > a$. Hence $c/(a+b) > 0$ and we define $c/(a+b) = \bar{v}^2$. Therefore again there are two situations. For $c_1 > 0$ we define $c_1 = \bar{p}^2$ and we have from (3.7)

$$\dot{R}^2 = \bar{p}^2 R^{-2b/a} - \bar{v}^2 R^2 .$$

In the same way as in case (i) one can show that here also there is an upper limit for R also and no lower finite limit, and so the singularity inevitably occurs.

If $c_1 < 0$, we define $c_1 = -\bar{p}^2$ and from (3.7) we get

$$\dot{R}^2 = -\bar{p}^2 R^{-2b/a} - \bar{v}^2 R^2 .$$

This is not an allowed case since the right-hand side is a negative quantity.

Case (iii). $-b/a < 0$. Since $a < 0$, we have in this case $b < 0$ and hence $c/(a+b) = v^{*2}$. Now assuming $c_1 > 0$, we write $c_1 = p^{*2}$. We have therefore, from (3.7),

$$\dot{R}^2 = p^{*2} R^{-2b/a} - v^{*2} R^2 .$$

Here $\dot{R} = 0$ at $R = R_0$, which satisfies the relation

$$p^{*2} R_0^{-2b/a} = v^{*2} R_0^2 .$$

So there is an upper bound for the model with an initial explosion at $R = 0$, where $\dot{R}^2$ approaches infinite magnitude. The behavior is similar to that of a dust Friedmann universe with $k = +1$.

The situation $c_1 < 0$ is not allowed for the same reason as in case (ii).

The other solutions given in (3.8) and (3.9) may be examined in a similar way. One should, however, carefully note that the models $k = +1$ or -1 may be particularly interesting for the oscillatory characters of some of them.

We study a particularly simple example. We put $-2b/a = 1$ and consider first the model with negative spatial curvature ($k = -1$). The constant d then turns out to be positive. Further, b is positive and we define $m^2 = d/b$ and $v^2 = c/(a+b)$. If we now choose $c_1 > 0$ and write $p^2 = c_1$, Eq. (3.7) leads to

$$\dot{R}^2 = p^2 R - v^2 R^2 - m^2 .$$

Here $\dot{R} = 0$ corresponds to an equation which has two real roots given by

$$R = \frac{1}{2v^2}[p^2 \pm (p^4 - 4m^2 v^2)^{1/2}] ,$$

provided $p^2 > 2mv$. The model oscillates between these two limits. $c_1 < 0$ is not an allowed case.

We next consider the model with $k = +1$. It is not difficult to analyze the behavior of the models with $c_1 > 0$ or $c_1 < 0$ in a similar fashion. In both cases it is observed that the model may start from the initial singularity $R = 0$, expand, and finally contract again after reaching an upper limit, which is the turning point for the model. There is, however, no case of oscillation between finite limits in this case.

In all the above cases we have not given exact expressions for the matter density and pressure. However, it is not difficult to see that one can restrict ρ and p to positive values only by putting restrictions on the constant parameters like n, β, and μ.

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