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New two-soliton solution to the Einstein equations

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Using the inverse scattering method a new two-soliton solution to the Einstein equations for an axially symmetric space-time is obtained.

One of the most appealing methods used to solve the Einstein equations for stationary axially symmetric space-times is the inverse scattering method.[1,2] The actual application of this method requires the explicit integration of an overdetermined system of linear partial differential equations. To find integrals that can be expressed in closed form for the above-mentioned system of equations is not an easy task. Thus few new exact solutions to the Einstein equations have been obtained using the inverse scattering methods,[1–3] although some very interesting properties of known solutions have been discovered, e.g., the interpretation of the Tomimatsu-Sato solution as the "superposition" of n Kerr solutions.[4] The relation between the inverse scattering method with other solution-generating techniques, e.g., Bäcklund transformations, has been studied in some detail.[5–7]

The purpose of this paper is to exhibit a new two-soliton solution to the Einstein equations using the Belinsky-Zakharov solution-generating technique, i.e., the inverse scattering method.[1]

The stationary axially symmetric metric can be written as

$$ds^2 = g_{ab}\, dx^a dx^b + f(dr^2+dz^2) \quad , \tag{1}$$

where the indices a and b take the values 0 and 1. g_{ab} and f are functions of z and r only, and $(t,\theta) \equiv (x^0, x^1)$. Also,

$$\det g = -r^2 \quad , \tag{2}$$

where g is the 2×2 matrix associated to g_{ab}.

The Einstein equations for the metric (1) can be written as[1]

$$(rg_rg^{-1})_{,r} + (rg_zg^{-1})_{,z} = 0 \quad , \tag{3}$$

$$[\ln(rf)]_{,r} = (4r)^{-1}\operatorname{Tr}(U^2-V^2) \quad , \tag{4}$$

$$[\ln(rf)]_{,z} = (2r)^{-1}\operatorname{Tr}(UV) \quad , \tag{5}$$

$$U \equiv rg_rg^{-1}, \quad V \equiv rg_zg^{-1} \quad , \tag{6}$$

where the subscripts r and z denote partial differentiation. The condition of integrability of f, i.e., $f_{rz} = f_{zr}$, is exactly Eq. (3), thus any solution to (3) will give us an f that can be obtained as a simple quadrature of (4) and (5).

Soliton solutions to (3) are obtained solving the "Schrödinger" equations

$$D_r\psi_0 = \frac{rU_0 + \lambda V_0}{\lambda^2 + r^2}\psi_0 \quad , \tag{7}$$

$$D_z\psi_0 = \frac{rV_0 - \lambda U_0}{\lambda^2 + r^2}\psi_0 \quad , \tag{8}$$

$$D_r \equiv \partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\partial_\lambda \quad , \tag{9}$$

$$D_z = \partial_z - \frac{2\lambda^2}{\lambda^2 + r^2}\partial_\lambda \quad , \tag{10}$$

for the wave function ψ_0. This wave function is a 2×2 complex matrix function of r, z, and the spectral parameter λ. U_0 and V_0 are obtained replacing g in (6) by a known solution to (3), g_0. The solution g_0 is called the "seed" or "background" solution. The knowledge of ψ_0 allows us to find the new solution g to Eq. (3), given by

$$g_{ab} = (g_0)_{ab} - \sum_{k,l} \frac{N_a^{(l)}(\Gamma^{-1})_{lk}N_b^{(k)}}{\mu_k\mu_l} \quad , \tag{11}$$

$$\Gamma_{kl} \equiv \frac{m^{(k)}\cdot m^{(l)}}{r^2 + \mu_k\mu_l} \quad , \tag{12}$$

$$m^{(k)}\cdot m^{(l)} \equiv m_a^{(k)}(g_0)_{ab}m_b^{(l)} \quad , \tag{13}$$

$$N_a^{(k)} \equiv m_b^{(k)}(g_0)_{ba} \quad , \tag{14}$$

$$m_a^{(k)} \equiv m_{0b}^{(k)}M_{ba}^{(k)} \quad , \tag{15}$$

$$M^{(k)} \equiv \psi_0^{-1}|_{\lambda=\mu_k} \quad , \tag{16}$$

$$\mu_k \equiv a_k - z \pm [(\alpha_k - z)^2 + r^2]^{1/2} \quad , \tag{17}$$

where the summation convention on the indices a and b has been adopted. The indices k and l run from 1 to n, n being the number of solitons. $m_{0b}^{(k)}$ and α_k are sets of arbitrary constants. The matrix associated with (11) is symmetric and is a solution to (3), but, in general, it will not satisfy the condition (2). To remedy this problem we can define a new matrix,

$$g^{\rm ph} = rg/(-\det g)^{1/2} \quad , \tag{18}$$

that satisfies the condition (2) and is also a solution to (3). The determinant of (11) can be explicitly computed to give

$$\det g_n = (-1)^n r^{2n} \prod_{l=1}^{n} (\mu^l)^{-2} \det g_0 \ . \tag{19}$$

We shall take as our seed solution the solution to the vacuum Einstein equations,

$$ds^2 = (g_0)_{ab}\, dx^a dx^b + f_0(dr^2 + dz^2) \ , \tag{20}$$

$$(g_0)_{00} = -r^{1-b} \exp\{-[az + c(\tfrac{1}{2}r^2 - z^2)]\} \ , \tag{21a}$$

$$(g_0)_{11} = -r^2/(g_0)_{00};\ (g_0)_{01} = (g_0)_{10} = 0 \ , \tag{21b}$$

$$f_0 = r^{-(1-b^2)/2} \exp[abz + cb(\tfrac{1}{2}r^2 - z^2) - \tfrac{1}{4}a^2 r^2 + aczr^2 + c^2 r^2(\tfrac{1}{8}r^2 - z^2)] \ , \tag{22}$$

where a, b, and c are constants. It is a matter of simple computation to show that (20) satisfies Eqs. (3)–(5).

The function ψ_0 associated with the particular "potential" g_0 given by (21) can be found in the following way. First, we notice that ψ_0 must satisfy the initial condition

$$\psi_0|_{\lambda=0} = g_0 \ . \tag{23}$$

This condition is a consequence of Eqs. (7) and (8). Since the seed solution (21) is diagonal, i.e., $(g_0)_{12} = (g_0)_{21} = 0$, one may assume that ψ_0 is also a diagonal matrix. With this assumption, Eqs. (7), (8), (23), and (2) yield

$$\left(\partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\partial_\lambda\right) \det\psi_0 = \frac{2r}{\lambda^2 + r^2}\det\psi_0 \ , \tag{24}$$

$$(\lambda\partial_r + r\partial_z)\det\psi_0 = 0 \ , \tag{25}$$

$$\det\psi_0|_{\lambda=0} = -r^2 \ . \tag{26}$$

A solution to Eqs. (24)–(26) is

$$\det\psi_0 = -r^2 + \lambda^2 + 2\lambda z \ . \tag{27}$$

From Eqs. (7), (8), (21), and (23) we find that $(\psi_0)_{11}$ is determined by

$$\left(\partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\partial_\lambda\right)(\psi_0)_{11} = \frac{(1-b)r - \lambda r(a - 2cz) - cr^3}{\lambda^2 + r^2}(\psi_0)_{11} \ , \tag{28}$$

$$(\lambda\partial_r + r\partial_z)(\psi_0)_{11} = -r(a - 2cz)(\psi_0)_{11} \ , \tag{29}$$

$$(\psi_0)_{11}|_{\lambda=0} = -r^{1-b}\exp\{-[az + c(\tfrac{1}{2}r^2 - z^2)]\} \ . \tag{30}$$

Also, from (27) we find that $(\psi_0)_{22}$ is determined by

$$(\psi_0)_{22} = (-r^2 + \lambda^2 + 2\lambda z)/(\psi_0)_{11} \ . \tag{31}$$

A direct computation shows that

$$(\psi_0)_{11} = (-r^2 + \lambda^2 + 2\lambda z)^{(1-b)/2} \times \exp(-\{a(z + \tfrac{1}{2}\lambda) + c[\tfrac{1}{2}r^2 - (z + \tfrac{1}{2}\lambda)^2]\}) \tag{32}$$

is a solution to (28)–(30). Thus

$$(\psi_0)_{22} = (-r^2 + \lambda^2 + 2\lambda z)^{(1+b)/2} \times \exp\{a(z + \tfrac{1}{2}\lambda) + c[\tfrac{1}{2}r^2 - (z + \tfrac{1}{2}\lambda)^2]\} \ . \tag{33}$$

The two-soliton solution to Eq. (3) associated with the particular solution (21) is obtained letting k and l take the values 1 and 2 in (11)–(17), and from the ψ_0 given by (32) and (33). Thus

$$g_{ab} = (g_0)_{ab} - \hat{g}_{ab} \ , \tag{34}$$

$$\hat{g}_{ab}\Delta = \frac{m^{(2)}\cdot m^{(2)}}{(r^2+\mu_2^2)\mu_1^2} N_a^{(1)} N_b^{(1)} - \frac{m^{(1)}\cdot m^{(2)}}{(r^2+\mu_1\mu_2)\mu_1\mu_2} N_a^{(1)} N_b^{(2)} - \frac{m^{(1)}\cdot m^{(2)}}{(r^2+\mu_1\mu_2)\mu_1\mu_2} N_a^{(2)} N_b^{(1)} + \frac{m^{(1)}\cdot m^{(1)}}{(r^2+\mu_1^2)\mu_2^2} N_a^{(2)} N_b^{(2)} \ , \tag{35}$$

$$\Delta \equiv \frac{m^{(1)}\cdot m^{(1)}}{r^2+\mu_2^2}\,\frac{m^{(2)}\cdot m^{(2)}}{r^2+\mu_1^2} - \left(\frac{m^{(1)}\cdot m^{(2)}}{r^2+\mu_1\mu_2}\right)^2 \ , \tag{36}$$

$$m_0^{(k)} = m_{00}^{(k)}(2\alpha_k\mu_k)^{(-1+b)/2} \exp\left\{a(z + \tfrac{1}{2}\mu_k) + c\left[\frac{r^2}{2} - \left(z + \frac{\mu_k}{2}\right)^2\right]\right\} \ , \tag{37}$$

$$m_1^{(k)} = m_{01}^{(k)}(2\alpha_k\mu_k)^{-(1+b)/2} \exp\left[-\left\{a(z + \tfrac{1}{2}\mu_k) + c\left[\frac{r^2}{2} - \left(z + \frac{\mu_k}{2}\right)^2\right]\right\}\right] \ , \tag{38}$$

$$N_0^{(k)} = -m_{00}^{(k)}(2\alpha_k\mu_k/r^2)^{(-1+b)/2} \exp[\tfrac{1}{2}a\mu_k - c(z\mu_k + \tfrac{1}{4}\mu^2{}_k)] \ , \tag{39}$$

$$N_1^{(k)} = m_{01}^{(k)}(2\alpha_k\mu_k/r^2)^{-(1+b)/2} \exp\{-[\tfrac{1}{2}a\mu_k - c(z\mu_k + \tfrac{1}{4}\mu^2{}_k)]\} \ . \tag{40}$$

Let us first consider the general case, i.e., all $m_{0a}^{(k)} \neq 0$. In this case we can cast the different functions appearing in (35) as

$$\Delta = \frac{r^2B^2}{4\alpha_1\alpha_2\mu_1\mu_2}\left[\frac{\sinh(y_1+\delta_1)\sin(y_2+\delta_2)}{(r^2+\mu_1{}^2)(r^2+\mu_2{}^2)} - \frac{\sinh^2(x+\epsilon)}{(r^2+\mu_1\mu_2)^2}\right] , \tag{41}$$

$$m^{(k)}\cdot m^{(k)} = \frac{r}{2\alpha_k\mu_k}A_k\sinh(y_k+\delta_k) , \tag{42}$$

$$m^{(1)}\cdot m^{(2)} = \frac{r}{2(\alpha_1\alpha_2\mu_1\mu_2)^{1/2}}B\sinh(x+\epsilon) , \tag{43}$$

where the new constants A_k, δ_k, B, and ϵ are related to the old constants by

$$A_k = 2m_{00}^{(k)}m_{01}^{(k)} , \tag{44}$$

$$\tanh\delta_k = \frac{(m_{00}^{(k)})^2-(m_{01}^{(k)})^2}{(m_{00}^{(k)})^2+(m_{01}^{(k)})^2} , \tag{45}$$

$$B^2 = 4m_{00}^{(1)}m_{00}^{(2)}m_{01}^{(1)}m_{01}^{(2)} , \tag{46}$$

$$\tanh\epsilon = \frac{m_{00}^{(1)}m_{00}^{(2)} - m_{01}^{(1)}m_{01}^{(2)}}{m_{00}^{(1)}m_{00}^{(2)} + m_{01}^{(1)}m_{01}^{(2)}} . \tag{47}$$

The functions y_k and x are related to r and z by

$$y_k = b\ln(2\alpha_k\mu_k/r) + a(z+\mu_k) + c(\tfrac{1}{2}r^2+z^2) - 2c(z+\tfrac{1}{2}\mu_k)^2 , \tag{48}$$

$$x = \frac{b}{2}\ln(4\alpha_1\alpha_2\mu_1\mu_2/r^2) + a[z+\tfrac{1}{2}(\mu_1+\mu_2)] + c[\tfrac{1}{2}r^2 - z^2 - \tfrac{1}{4}(\mu_1{}^2+\mu_2{}^2) - z(\mu_1+\mu_2)] . \tag{49}$$

The physical g is obtained from (34) as follows:

$$g^{\mathrm{ph}} = (\mu_1\mu_2/r^2)(g_0-\hat{g}) . \tag{50}$$

To complete the integration of (3)–(5) we need to compute f; this can be done using the method of Ref. 1. We find

$$f = \frac{Kf_0r^2\mu_1{}^2\mu_2{}^2}{(r^2+\mu_1{}^2)(r^2+\mu_2{}^2)}\Delta , \tag{51}$$

where f_0 and Δ are given by (22) and (36), respectively, and K is an arbitrary constant.

Some interesting particular cases of (34) are obtained, making the following restrictions on the constants $m_{0b}^{(k)}$: $m_{01}^{(1)}=m_{00}^{(2)}=0$, $m_{00}^{(1)}=m_{01}^{(2)}=0$, $m_{01}^{(1)}=m_{01}^{(2)}=0$, and $m_{00}^{(1)}=m_{00}^{(2)}=0$. For these particular cases, we get

$$g_{00}^{\mathrm{ph}} = \pm(\mu_1/r)^{\epsilon_1}(\mu_2/r)^{\epsilon_2}(g_0)_{11} , \tag{52}$$

$$g_{01}^{\mathrm{ph}} = g_{10}^{\mathrm{ph}} = 0 , \tag{53}$$

$$g_{11}^{\mathrm{ph}} = -r^2/g_{00}^{\mathrm{ph}} , \tag{54}$$

where ϵ_1 and ϵ_2 are constants that can take the values ±1. The form of the relations (52)–(54) suggests how to construct a diagonal $2n$-soliton solution[8,9] out of *any* diagonal seed solution g_0.

In order to give a physical interpretation to the solution obtained we shall first study the seed solution (f_0, g_0). This solution has an essential singularity at $r=0$, and at $r\to\infty$ the physical components of the Riemann-Christoffel tensor go to zero. By physical components we mean the components projected along the vierbeins associated with (20)–(22). One can prove the previous statements by computing the curvature tensor and noticing that when $r\to\infty$, $f_0\to\infty$ faster than any power of r, as long as $c\neq0$ and z remains finite. Thus the solution (f_0, g_0) represents a wire located on the z axis. In the same way, one can prove that the solutions (52)–(54) also describe a wire, but with different density distributions. We note that the asymptotically flat character of the solution at $r\to\infty$ is maintained after the "dressing" of the solution with two solitons.

The general case given by (50)–(51) might be interpreted as a rotating wire. The singular character of the solution at $r=0$ is apparent, but that this solution is asymptotically flat when $r\to\infty$ needs to be proved. Cosgrove[5] proved the following theorem: It is always possible to choose constants $m_{0b}^{(k)}$ in such a way that if we start with an asymptotically flat seed solution, after the dressing of this solution we end up with another asymptotically flat solution. Unhappily, the concept of asymptotical flatness used to show the theorem cannot be used in the present case.

The functions μ_1 and μ_2 can be either real or complex conjugated. In this last case it is always possible to choose constants $m_{0b}^{(k)}$ in such a way that the final solution is real.[1] Letting $\mu_1=\mu_2^*$, $b=1$, and $a=c=0$ in (34) and (35), we find that (34) and (51) reduces to the Kerr-NUT (Newman-Unti-Tamburino) solution. This can be proved directly using the transformations of coordinates given in Ref. 1, or by noticing that the particular choices $b=1$ and $a=c=0$ reduce the seed solution to the Minkowski metric in cylindrical coordinates, and that the Kerr-NUT solution is obtained[1] dressing this metric with two complex conjugated "poles" μ and μ^*.

Finally, we want to add that the solution (34) is closely related to a two-soliton solution for self-dual SU(2) gauge fields on Euclidean space. This relation was studied in Ref. 9.

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