Horizon-free universe

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(Received 30 June 1981)

We have studied a theory of gravity, consistent with all astronomical observations, which is an extension of general relativity generated by a gravitational field Lagrangian proportional to $R + AR^2$ (where R is the scalar curvature and A is a new dimensional constant). We find cosmological solutions without a particle horizon for any value of A, compatible with cosmological observations if A is negative and small enough. It is pointed out that while alleviating the problem of the large-scale physical uniformity of the universe, the theory may also alleviate the problem of the development of smaller-scale structures from an initially isotropic state, and possibly the problem of "hidden mass."

Two of the most perplexing problems within the framework of the standard relativistic cosmological models are the origin of the large-scale homogeneity and isotropy of the universe, and of the small-scale structures within it. To date, the global smoothness, which is deduced from an array of observations from the distribution of galaxies to the smoothness of the 3-K microwave radiation (at least at the 0.1% level), has defied all attempts to be derived as a *consequence* of the big-bang model. The cause of this problem is directly connected with the existence of a particle horizon in the general-relativistic cosmological models which prevents communication between distant parts of the universe.

It is natural, therefore, to look for alternative theories without a particle horizon or without singularities. In other branches of physics (such as fluid mechanics), singularities are often removed by the introduction of higher-derivative terms (with suitable scale length). We are thus led to explore higher-order gravitational theories.

In the last few years a number of articles[1–5] have been published on "general relativity with higher derivatives" (GRHD). Field equations of GRHD, analogous to Einstein's equations of general relativity (GR) contain derivatives of higher order than the second. The equations can be derived by using an extended Lagrangian which contains, in addition to the Hilbert term linear in the scalar curvature R and generating GR, some additional invariants, nonlinear in the curvature tensor. It is most natural to first consider quadratic additions in the Lagrangian.

The resulting equations will have many "good" properties of Einstein's equations (except for the relative mathematical simplicity of GR). For example, they contain matter motion equations $T^i_{k;i}=0$ where T^i_k is the matter energy-momentum density tensor. Also, any vacuum solution of Einstein's equations is also a GRHD vacuum solution. Static spherically symmetric solutions, coupling to not too large amounts of matter, differ considerably from the Schwarzschild field only at arbitrarily small distances from the center (depending on the strength of the additional term[5]).

It is interesting to note that quantum gravity also leads to theories with higher derivatives (e.g., Refs. 1, 6, and 7).

The theory has the correct Newtonian limit. Furthermore, the classical solar system tests set only a weak restriction[5] on the magnitude of additional quadratic terms[8] in the Lagrangian. (One can see from simple dimensional considerations that the existence of neutron stars with properties more or less consistent with predictions of general relativity further restricts the allowed value of A to $|A| < 10^{13}$ cm^2.)

In cosmology, investigations in GRHD have been motivated by hopes to get "bounce models," without singularities. These hopes have not been borne out (see, however, Refs. 2 and 9).

Still, as far as we know no one looked at the GRHD theories in order to see if they admit solutions without a particle horizon, in advantageous distinction from GR (see, however, Ref. 1). We find this is the case. To point out this and some other advantages of such theories is the main purpose of this paper.

While flat-space cosmological models with $p=\epsilon/3$ have mainly been considered, we investigate complementary and more various cases of nonflat (e.g., closed or open) isotropic world models with the equation of state $p=\gamma\epsilon$, where p is the pressure and ϵ is the energy density of the "ideal fluid," and γ=const,$0\leq\gamma\leq\frac{1}{3}$.

Metrics can be expressed in the general form

$$ds^2=-dt^2+a^2(t)\left[d\chi^2+\left[\frac{\sin\sqrt{\sigma}\chi}{\sqrt{\sigma}}\right]^2\times(d\theta^2+\sin^2\theta d\phi^2)\right], \quad (1)$$

where $\sigma=1$, 0, or -1 for the closed, flat, or open models, respectively.

There are three quadratic scalar invariants that could lead to corrections to Einstein's equations, if added in the Lagrangian. These are $Q_1\equiv R^2$; $Q_2\equiv R^{ik}R_{ik}$; $Q_3\equiv R^{iklm}R_{iklm}$, where R is the scalar curvature, R_{ik} is the Ricci tensor, and R_{iklm} is the curvature tensor. However, one can show that identically (e.g., see Ref. 5) $\delta\int(Q_1-4Q_2+Q_3)\times\sqrt{-g}d^4x=0$. Moreover, for isotropic metrics (1) there is an identity (e.g., Ref. 10) $Q_1/3-2Q_2+Q_3=0$.

Due to these two relations only one invariant is effectively independent, in the sense that an addition $q_1Q_1+q_2Q_2+q_3Q_3$ in the Lagrangian leads to the same equations as the term AR^2 does, with $A\equiv q_1+q_2/3+q_3/3$. So, A is the only new constant. (In general, nonisotropic case two invariants are independent, hence one needs to consider additional terms of the form $A_1Q_1+A_2Q_2$.) The T_0^0 equation yields

$$\frac{\dot a^2+\sigma}{a^2}+\frac{\alpha}{a^4}(2\dot a\dddot a a^2+2\ddot a\dot a^2a-\ddot a^2a^2-3\dot a^4-2\dot a^2\sigma+\sigma^2)=\frac{\epsilon_0}{a^{3(1+\gamma)}}, \quad (2)$$

where α for convenience in the further exposition is defined as $\alpha\equiv-6A$, ϵ_0 is constant; $\dot a\equiv da/dt$, and so on. The fact has been used that $T^{0k}{}_{;k}=0$ leads, as in GR, to the simple dependence of the energy density on the scale factor $a(t)$ on the right-hand side of (2), that is, $\epsilon=3\epsilon_0/a^{3(1+\gamma)}$.

We now investigate the possible asymptotic behavior of the scale factor a near $a=0$ and $a=\infty$.

Using the fact that coefficients of Eq. (2) do not depend on t, we can lower its order. Substitution of the variables

$$b\equiv(\dot a a)^{3/2},\ x\equiv[a^2/(2\sqrt{3})]^{3/2}$$

leads to the equation

$$b''-\frac{1}{\sqrt{3}}b^{-1/3}x^{-4/3}\sigma+b^{-5/3}x^{-2/3}\sigma^2+\alpha^{-1}(b^{-1/3}x^{-2/3}+2\sqrt{3}b^{-5/3}\sigma-\tilde\epsilon_0b^{-5/3}x^{-\gamma-1/3})=0, \quad (3)$$

where

$$b''\equiv d^2b/dx^2$$

and

$$\tilde\epsilon_0\equiv2^{(1-3\gamma)/2}3^{(1-3\gamma)/4}\epsilon_0 .$$

Now, $a\to0$ corresponds to $x\to0$. In this limit the fourth term (the first in parentheses) in Eq. (3) is small in comparison to the second by $x^{2/3}$, the fifth one is small in comparison with the third one in the same relation, the sixth term relates to the third one as $x^{1/3-\gamma}$ and cannot be neglected only if $\gamma=\frac{1}{3}$.

Hence, asymptotically one has the equation

$$b''-\frac{\sigma}{\sqrt{3}}b^{-1/3}x^{-4/3}+Bb^{-5/3}x^{-2/3}=0, \quad (3a)$$

where the constant $B=\sigma^2$ if $\gamma<\frac{1}{3}$ and $B=\sigma^2-\epsilon_0/\alpha$ if $\gamma=\frac{1}{3}$. (One can show that this equation is also applicable to the case $\sigma=0$, $\gamma=\frac{1}{3}$. Only the case $\sigma=0$, $\gamma<\frac{1}{3}$ demands special consideration, which we are not going to make here.) Now, the additional variable substitution $x\equiv\alpha^{3/2}e^w$, $b\equiv\alpha^{3/4}e^{w/2}v^{3/2}$ reduces Eq. (3a) to the form

$$\frac{d^2v}{dw^2}+\frac{1}{2v}\left[\frac{dv}{dw}\right]^2-\frac{2\sigma}{\sqrt{3}}v^{-1}+\tfrac{2}{3}Bv^{-3}=0 \quad (4)$$

not containing w in coefficients, so that the order can be lowered once more. This leads to an integrable equation. Using this procedure, one obtains an exact solution of (4):

$$w-w_0=\pm3\int_{v_0}^{v}\frac{vdv}{(v^4+4\sqrt{3}\sigma v^2+Cv+12B)^{1/2}}\equiv\pm3\int_{v_0}^{v}\frac{vdv}{[P(v)]^{1/2}} . \quad (5)$$

The solution of Eq. (4) is very similar to the solution corresponding to the case $\sigma=0$, $\gamma=\frac{1}{3}$ (in our notation) in the GRHD of Fischetti *et al.*[1] The solution family (5) is determined to be two-parameter with free parameters C and w_0. The limit $x\to0$ corresponds to $w\to-\infty$. The last relation can be reached in two ways.

(1) Take $v \to \infty$, choose the minus sign in integral (5). Then $w \sim -3\ln v$, $v \sim e^{-w/3}$. That corresponds to $b \to$const, and $a \sim \sqrt{t}$. This is a two-parameter family of asymptotic solutions.

(2) The integral in (5) diverges at some finite value v_r of v, which value has obviously to be a root of the polynomial P in the denominator of the integrand in the solution (5). Moreover, it is a positive second-order root (otherwise the integral would converge in the singular point v_r): $P(v_r)=0$, $dP(v_r)/dv=0$. Such roots occur necessarily if $\gamma < \frac{1}{3}$. If $\gamma = \frac{1}{3}$, they occur if $\alpha^{-1}\epsilon_0 < 1$ and $\sigma = 1$; if $\alpha^{-1}\epsilon_0 < \frac{4}{3}$ and $\sigma = -1$; or if $\alpha < 0$, and $\sigma = 0$, in all cases at a special value of the constant C. The corresponding behavior of the scale factor a is

$$a = a_0 t + \cdots \;, \tag{6}$$

a one-parameter family. We thus obtain solutions (6) *without horizon.*

To clarify this, let us write down[8] the equation of light rays, coming into the coordinate origin (actually, any given point, by virtue of uniformity and isotropy of space) at the time t_0,

$$ds^2 = -dt^2 + a^2 d\chi^2 = 0 \;.$$

The solution is $\chi(t) = -\int_{t_0}^{t} [a(t)]^{-1} dt$ for the location of the point χ, where the light ray departs at time t. If this integral diverges at $t=0$, as in models (6), the χ can take arbitrarily large values, at t being sufficiently close to zero. *So, no matter how early a given part of the universe is considered, it could have exchanged signals with any other part. There is no particle horizon.*

Let us consider now a few other features of GRHD cosmologies.

Actually, in Ref. 1 for the case $\sigma=0$ and $p=\epsilon/3$, a similar one-parameter family of solutions without a particle horizon was found, but the authors do not discuss the solutions from the present point of view.

Note, by the way, that Fischetti *et al.*[1] construct their theory from quantum gravity as a theory with boundary conditions (rather than with initial ones) and point out that it does not manage to select single solutions by plausible boundary conditions; they impose the condition of Friedmannian behavior as $a \to \infty$, but still have a set of solutions: a one-parameter family with $a \sim \sqrt{t}$ and the solution corresponding to the asymptote $a \sim t$ as $t \to 0$. So, the addition of the boundary condition that the particle horizon is absent as $t \to 0$, together with the condition of Friedmannian behavior as $t \to \infty$, selects a single solution.

Now, we list the results of our investigation of the asymptotic behavior of the solutions of Eq. (3) as $t \to \infty$.

We sought growing asymptotes of the form

$$b \sim b_0 x^{\nu} (\ln x)^{\mu} \tag{7}$$

with $b_0 > 0$ and $\nu > 0$, or $\nu = 0$ and $\mu > 0$. Early research[10] in the case $\sigma=0$, $p=\epsilon/3$ revealed singularities of non-Einstein types, with unlimited growing four-curvature R due to runaway expansion of models as $t \to \infty$.

We have found that such asymptotes are present for $\alpha < 0$, but they are absent for $\alpha > 0$: in the last case the open model is only Friedmannian growing; and there are no monotonically growing asymptotes (7) at all in the closed models.

Note that, as Ref. 10 pointed out, the case $\alpha > 0$ is only possible[11] from the very beginning, if one demands minimal action rather than simply stationary action (although the demand of stationary action is sufficient to derive the field equations[10]).

It is interesting to note that there are no instabilities in the future, as $t \to \infty$, with respect to isotropic small perturbations (of the kind that Horowitz and Wald[4] had pointed out for other models) in our models with $\alpha > 0$: the equation of perturbations can be easily shown to have the form $D'' + Kx^{-4/3}D = 0$, with constant $K > 0$, and the general solution as $x \to \infty$ is $D = C_1\cos(\sqrt{K}x^{1/3}) + C_2\sin(\sqrt{K}x^{1/3})$. This means a damping of the form $\delta a/a \propto t^{-3/2}\cos(\omega t + \phi)$ in the original variables a and t (with some frequency ω and phase ϕ). We did not study nonisotropic perturbations in detail, but there are some arguments, too long to be presented here, that they do not cause any trouble.

We would like to point out some additional results for isotropic cosmology.

If $\alpha > 0$, there is a steady-state closed model, with "world radius"

$$a_0 = [\alpha(1-3\gamma)/(1+3\gamma)]^{1/2}, \quad \gamma < \tfrac{1}{3} \;,$$

where the parameter ϵ_0 is given by

$$\epsilon_0 = [6/(1-3\gamma)]a_0^{\,1+3\gamma} \;.$$

However, this is linearly unstable with respect to small perturbations.

A very interesting open problem is if there are oscillating solutions for closed models in GRHD with $\alpha > 0$ without any singularity, with Friedmannian behavior except within a small region of bounce from contraction to expansion, where the behavior is determined by the quadratic term in the Lagrangian. Related negative results for the flat models hardly mean anything for the nonflat ones: Their behavior can differ significantly, as our study indicates for open models at the late stages of expansion in distinction from that of flat models (see also Ref. 9).

Finally, there are some heuristic indications that if we trace back in time modern density perturbations $\delta\rho/\rho$ and metric perturbations δg^i_k, we have in our models $\delta\rho/\rho \to 0$, $\delta g^i_k \to 0$. This is in favorable distinction to the perturbation behavior in the standard GR Friedmann models, where $\delta\rho/\rho \to 0$, but $\delta g^i_k \to \text{const} \neq 0$.

One can see this as follows (after Ref. 12): The well-known Jeans result is that $\delta\rho/\rho \sim \exp(\omega t)$ for perturbations on a stationary background. But in nonstationary models of GR, $\rho \neq \text{const}$. The natural generalization is $\delta\rho/\rho \sim \exp[\int \omega(t) dt]$, with varying $\omega(t) = [4\pi G\rho(t)]^{1/2}$. In the GR Friedmann models, $\rho \sim t^{-2}$, $\omega \sim t^{-1}$, so $\int \omega dt \sim \ln t$ and $\delta\rho/\rho \sim \exp(\int \omega dt) \sim t^r$ (with some $r > 0$). A power law is a consequence of the logarithmic behavior of $\int \omega dt$, which is in turn due to the t^{-2} behavior of ρ. But in our models by contrast, $\rho \sim t^{-3(1+\gamma)}$, and therefore $\omega \sim \sqrt{\rho} \sim t^{-3/2-3\gamma/2}$, so $\int \omega dt \sim t^{-1/2-3\gamma/2} \equiv t^{-q}$, with some $q \geq \frac{1}{2} > 0$. Therefore, $\delta\rho/\rho \sim \exp(\int \omega dt) \sim \exp[-(t_0/t)^q]$, where t_0 is a constant.

So, in horizon-free models the perturbations $\delta\rho/\rho$ tend to zero much more powerfully as $t \to 0$. Since all unperturbed quantities show only powerlike behavior, all coefficients in the perturbation equations are powerlike, while on the right-hand side we have $\delta\rho$ exponential in t. Therefore, the δg^i_k should exhibit exponential behavior,

$$\delta g^i_k \sim (t/t_0)^n \exp[-(t_0/t)^q]$$

and $g^i_k \to 0$ as $t \to 0$ for any n.

Hence in horizon-free models the modern galaxy size structures might develop in an initially isotropic world. It is remarkable that while alleviating the problem of the large-scale homogeneity of the universe, this theory *also* alleviates the problem of the development of small-scale structure from an initially isotropic state.

This work was supported in part by the National Science Foundation under Grant No. AST-8020756.

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[9]We have found bounce-type asymptotic solutions of Eq. (2) with $p = \epsilon/3$:

$$a \sim a_0 + a_1 t^2 + a_2 t^4 + \cdots + a_n t^{2n} + \cdots ,$$

with

$$\epsilon_0 = \alpha(1 - 4a_0^2 a_1^2) - a_0^2 ,$$

$$a_2 = [a_0(2a_0 a_1 - 1) - 4\alpha a_1(a_0 a_1 + 1)]/(4! \alpha a_0^2) ,$$

and so on for a_3, a_4, etc., in terms of a_0 and a_1. The solution has two parameters, e.g., a_0 and a_1. We hope to investigate this solution elsewhere.

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[11]According to Stelle, in GRHD with $R \to R + AR^2 (\equiv R - \alpha R^2/6)$ in the Lagrangian we will have a massive spin-0 gravitational radiation, with $m = hc^{-1}\alpha^{-1/2}$. We note that these massive gravitons could play the same cosmological role as that of massive neutrinos. If we assume that the gravitons were in thermodynamic equilibrium at some very early time (before becoming noninteractive at $t \sim \alpha^{1/2} c^{-1}$), so that now there is one graviton for $10-100$ 3-K photons, then a graviton mass of the order of $10-100$ eV will account for the "hidden mass" of the clusters of galaxies. Those values of the mass yield

$$l \sim 10^{-7} - 10^{-6} \text{ cm } (l^2 \equiv \alpha \sim 10^{-15} - 10^{-11} \text{ cm}^2) .$$

In this assumption of graviton equilibrium, taking $l < 10^{-10}$ cm would lead to a contradiction with the observed expansion and age ($t \gtrsim 10^{10}$ yr) of the universe. (Too large a mass density due to the too massive gravitons would force the universe to contract much earlier.) Thus, the "quantum-origin" value of l, of the order of 10^{-33} cm, turns out to be well prohibited by the assumption of primordial graviton equilibrium in GRHD. We will consider in detail elsewhere the possible role of massive gravitons in originating galactic structure.

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