Loop space

R. A. Brandt, A. Gocksch, and M.-A. Sato
Department of Physics, New York University, New York, New York 10003

F. Neri
Department of Physics, Rutgers University, New Brunswick, New Jersey 08903
(Received 30 March 1982)

Several topics in the loop-space formulation of non-Abelian gauge theories are considered. The basic objects dealt with are the unrenormalized dimensionally regularized gauge-invariant loop functions $W(C^i;g,\epsilon)$, where C^i is a set of loops, g is the unrenormalized coupling constant, and ϵ is the deviation from four space-time dimensions. The renormalization-group equations satisfied by the corresponding renormalized loop functions are derived and, using asymptotic freedom, used to determine the exact behavior of the functions when the length L of the loops approaches zero. The result is $(-\ln L\mu)^{a(\gamma)}$, where μ is the subtraction mass and γ represents the cusp and cross-point angles of the loops. The function $a(\gamma)$ is exactly computable and several examples are given. The equivalent result may be stated as the exact behavior of the renormalization-constant matrix $Z^{ij}(\gamma,g_R,\epsilon)$ for $\epsilon\to 0$ with fixed renormalized coupling constant g_R, or as the exact behavior of the unrenormalized loop function for $\epsilon\to 0$ and g_R fixed. It is shown next that the $W(C^i;g,\epsilon)$ satisfy dimensionally regularized Makeenko-Migdal equations in all orders of perturbation theory. The proof makes detailed use of dimensional regularization, Becchi-Rouet-Stora symmetry, gauge-field combinatorics, and properties of the area functional derivative of path-ordered multiple line integrals. Doubt is cast on the existence of such useful equations when other regularizations are used or when renormalization is performed. The Mandelstam constraints are considered next. Among other things, it is shown that the loop-function renormalization may be performed such that the renormalized functions satisfy a constraint which has the same form as the unrenormalized constraint $\sum_{i=1}^{(N+1)!} a_i W(C^i)=0$, for the U($N$) gauge group. The paper concludes with illustrations of how observable matrix elements of physical (color singlet, quark bilinear) flavor currents may be expressed in terms of loop functions. Among other topics discussed in the paper are the $N\to\infty$ limit, two-dimensional QCD, and normalization conditions on the renormalized loop functions.

I. INTRODUCTION

The ultimate goal of the loop-space formulation[1] of non-Abelian gauge theories is to eliminate all reference to gauges, gauge transformations, gauge fixing, ghosts, etc. and instead deal only with gauge-invariant scalar functionals $W_n(C_1,\ldots,C_n)$ of loops $C_1,\ldots,C_n$. The Yang-Mills equations of motion would be replaced by functional equations such as the Makeenko-Migdal[2,3] equations (for simplicity without fermions)

$$\partial^\nu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W_1(C)=Ng^2 \oint_C dy^\mu \delta^4(x-y) \times W_2(C_{xy},C_{yx}) , \tag{1.1}$$

and the information about the local gauge group G would be expressed as constraints such as the Mandelstam[4,5] constraints

$$\sum_{i=1}^{I} a^i W_{n_i}(C_1^i,\ldots,C_{n_i}^i)=0 . \tag{1.2}$$

The program would then be to solve (1.1), or something equivalent, subject to (1.2), or something equivalent, and to calculate observables such as the appropriate functional integrals

$$\mathscr{F}=\int dC_1 \cdots dC_n\, W_n(C_1,\ldots,C_n) f(C_1,\ldots,C_n) . \tag{1.3}$$

In all of this there is no mention of gauge transformation, quantum field operators, Hilbert spaces, etc.

Unfortunately, all of the steps in this program are currently problematic. Some of the problems are the following: (1) The loop functions W are divergent and must be renormalized. (2) The area functional derivative $\delta/\delta\sigma_{\mu\nu}$ is not so well defined. (3) It is not clear what are a complete and effective set of constraints. (4) It is not known how to evaluate the functional integrals for observables. (5) It is not known how to solve the functional equations for the loop functions in four dimensions. The object of this paper is to investigate these problems in perturbation theory which, of course, brings back all the gauges, ghosts, etc. But in this context, we can solve the above problems as follows. (1) The loop functions are multiplicatively renormalizable. This is proved in Ref. 6 for nonintersecting differentiable loops and in Ref. 7 for loops with cusps and self-intersections. This work is discussed here and the corresponding renormalization-group equations are derived. Note that it is insufficient to consider only nonintersecting loops since intersecting loops are necessary for the nontriviality of (1.1), the statement (1.2), the integration in (1.3), etc. (2) We show that the Makeenko-Migdal[8] equations are valid in all orders of (dimensionally regularized, unrenormalized) perturbation theory. We also briefly treat the renormalization of the equations. (3) We derive the renormalized version of the Mandelstam constraints. (4) We illustrate how observables can be expressed as functional integrals over loop functions. (5) In view of (2) above, the perturbative solution of the MM equations must give expressions for the loop functions which agree with the perturbative expansions for the loop functions which follow from their definitions. We unfortunately have nothing to say about nonperturbative solutions in four dimensions.

The connection between the loop-space formalism (1.1)–(1.3) and the conventional local field-theoretic formalism is provided by the definition

$$W_n(C_1,\ldots,C_n) = \left\langle 0 \left| T^* \frac{1}{N}\,\mathrm{tr}\underline{\Phi}(C_1)\cdots\frac{1}{N}\,\mathrm{tr}\underline{\Phi}(C_n)\right| 0\right\rangle \tag{1.4}$$

of the loop functions in terms of the path-dependent phase-factor matrices

$$\underline{\Phi}(\Gamma_{xy}) = P\exp\left[ig\int_{\Gamma_{xy}} dz^\mu \underline{A}_\mu(z)\right]. \tag{1.5}$$

Here the path Γ_{xy} from x to y is defined as a map

$$\begin{aligned}&\Gamma_{xy}:\tau\to z^\mu(\tau),\quad 0\le\tau\le 1\,,\\ &z(0)=x,\quad z(1)=y\,,\\ &dz^\mu=\dot z_\mu(\tau)d\tau,\quad \dot z_\mu\equiv dz_\mu/d\tau\,.\end{aligned} \tag{1.6}$$

$\underline{A}_\mu(z)$ is the gauge-field $N\times N$ matrix, and g is the (unrenormalized) gauge coupling constant. The loops C_i are closed paths Γ_{xx} and (1.4) is independent of the starting points x because of the traces.

The field equations of motion and commutation relations satisfied by $\underline{A}_\mu$ imply the MM equations and constraints satisfied by the loop functionals, and conversely. The loop-space approach has the advantage that one deals with scalar gauge-invariant functions and the disadvantage that they are functions of all loops and there are vastly many more loops C than space-time points x. This is what gives rise to the constraints in loop space. The simplest constraints for the phase factors are invariance under reparametrization,

$$\underline{\Phi}(C_1)=\underline{\Phi}(C_2)\quad\text{when}\quad x_1(\tau)=x_2(f(\tau)) \tag{1.7}$$

for smooth functions f which satisfy $f(0)=f(1)-1=0$, and the inversion relation

$$\underline{\Phi}(C_1)=[\underline{\Phi}(C_2)]^{-1}\quad\text{when}\quad x_1(\tau)=x_2(1-\tau)\,. \tag{1.8}$$

The set of loops [with suitable identifications corresponding to (1.7) and (1.8)] with a fixed base point $x=x(0)=x(1)$ forms a group $\mathscr{G}$ (independent of x) with the composition multiplication law

$$C_1\cdot C_2=C_3\quad\text{when}\quad x_3(\tau)=\begin{cases}x_1(2\tau), & 0\le\tau\le\frac{1}{2}\\ x_2(2\tau-1), & \frac{1}{2}\le\tau\le 1\,.\end{cases} \tag{1.9}$$

Then, for each classical field $\underline{A}_\mu(x)$, the phase factors (1.5) give a representation of $\mathscr{G}$ on the gauge group G regarded as an N by N matrix subgroup of GL(N):

$$\underline{\Phi}(C_1)\underline{\Phi}(C_2)=\underline{\Phi}(C_1\cdot C_2)\,. \tag{1.10}$$

The Wilson loop functions

$$W(C)=\frac{1}{N}\,\mathrm{tr}\underline{\Phi}(C) \tag{1.11}$$

are then the characters of this representation, and (1.10) implies the constraint for loop functions[4]

$$W(C_1 \cdot C_2) = W(C_2 \cdot C_1) \ . \qquad (1.12)$$

The other Mandelstam constraints are discussed in Sec. V.

Let us note here an interesting connection between the three main topics of this paper—the renormalization program, the MM equations, and the Mandelstam constraints. Consider a loop C which crosses itself one or more times. Then the loop function $W(C)$ is mixed with a set of other loop functions $W_i^{(1)}$ ($i=1,\ldots,N_1$) upon renormalization,[7] is mixed with a set of other loop functions $W_i^{(2)}$ ($i=1,\ldots,N_2$) by the MM equation,[3] and is mixed with a set of other loop functions $W_i^{(3)}$ ($i=1,\ldots,N_3$) by the Mandelstam constraints appropriate to a suitable gauge group.[4,5] It turns out that these three sets are identical: $W_i^{(1)}=W_i^{(2)}=W_i^{(3)}$ ($i=1,\ldots,N_1=N_2=N_3$). Although the technical reasons for this are clear, we feel that there may be a deeper significance to this remarkable fact.

In Sec. II, we summarize the conventional Becchi-Rouet-Stora[9] formalism for Yang-Mills theories and consequent Ward-Takahashi identities involving loop functions. The renormalization program is discussed in Sec. III. Part A summarizes the Feynman rules and dimensional regularization. In part B we derive the (trivial) renormalization-group equation for loop functions of smooth (i.e., differentiable) and simple (i.e., non-self-intersecting) loops. In part C we derive the renormalization-group equation for renormalized loop functions $W_R(C_\gamma)$ of simple loops C_γ with a cusp (i.e., a point where the tangent vector to the loop jumps through a finite angle γ). The exact[10] behavior of $W_R(C_\gamma)$ when the length L of C_γ becomes small is seen to be $(\ln L\mu)^{a(\gamma)}$ with $a(\gamma) \propto \gamma\cot\gamma - 1$. The exact behavior of the renormalization constant for dimension $D \to 4$ is also given. In part D we do the same thing for loops with cross points. Here the renormalization-group equation is a matrix equation.

Section IV deals with the MM equations. In part A we review the formal derivation of the equations and in part B we discuss some formal properties of the equations. In part C we derive an expression for the area derivative of loop functionals defined as path-ordered multiple line integrals over the loop. Part D contains the proof of the dimensionally regularized MM equation in all orders of perturbation theory. The proof is a diagrammatic one which makes detailed use of the dimensional regularization, a Ward-Takahashi identity derived in Sec. II, the combinatorics associated with the Yang-Mills action, and the results of part C. The renormalization of the MM equation is considered in part E and the solution in two dimensions is reviewed in part D.

In Section V we prove that the loop functions can be renormalized such that the renormalized functions satisfy the same Mandelstam constraints as do the unrenormalized functions. Section VI illustrates how observable matrix elements of physical (color singlet) flavor currents can be expressed as functional integrals over loop functions. Section VII contains a brief statement of our conclusions.

II. GAUGE INVARIANCE

We take the gauge group G to be a subgroup of GL(N). The vector potential A_μ^a and scalar ghosts $\bar{C}^a$ and C^a transform as adjoint representations of G with $a=1,2,\ldots,d=\dim G$. Other fields in other representations, covariantly coupled to the vector potentials, could be included but will not be considered in this paper for simplicity.

The $N\times N$ matrix generators $\underline{\lambda}^a$ of G satisfy the group algebra

$$[\underline{\lambda}^a,\underline{\lambda}^b]=if^{abc}\underline{\lambda}^c \qquad (2.1)$$

and may be normalized by

$$\mathrm{Tr}\underline{\lambda}^a\underline{\lambda}^b=\delta^{ab} \ . \qquad (2.2)$$

In terms of the field operator matrices

$$\begin{aligned} &\underline{A}_\mu=A_\mu^a\underline{\lambda}^a, \quad \underline{C}=C^a\underline{\lambda}^a, \quad \underline{\bar{C}}=\bar{C}^a\underline{\lambda}^a \ , \\ &\underline{F}_{\mu\nu}=\partial_\mu\underline{A}_\nu-\partial_\nu\underline{A}_\mu-ig[\underline{A}_\mu,\underline{A}_\nu] \ , \end{aligned} \qquad (2.3)$$

the quantum action is given by

$$\mathscr{S}=\int d^4x\left[-\tfrac{1}{4}\mathrm{Tr}\underline{F}_{\mu\nu}\underline{F}^{\mu\nu}-\frac{1}{2\alpha}\mathrm{Tr}(\partial^\mu\underline{A}_\mu)^2 -i\,\mathrm{Tr}(\partial_\mu\underline{\bar{C}})(\nabla^\mu\underline{C})\right] , \qquad (2.4)$$

where the covariant derivative is defined by

$$\nabla^\mu\underline{C}=\partial^\mu\underline{C}-ig[\underline{A}^\mu,\underline{C}] \ . \qquad (2.5)$$

The equations of motion read

$$0=-\frac{\delta\mathscr{S}}{\delta\underline{A}^\mu}=\nabla^\mu\underline{F}_{\mu\nu}-\frac{1}{\alpha}\partial_\mu\partial^\nu\underline{A}_\nu-g\{\partial_\mu\underline{\bar{C}},\underline{C}\} \ , \qquad (2.6)$$

$$0=-\frac{\delta\mathscr{S}}{\delta\underline{C}}=i\nabla^\mu\partial_\mu\underline{\bar{C}} \ , \qquad (2.7)$$

$$0=-\frac{\delta\mathscr{S}}{\delta\underline{\bar{C}}}=-i\partial^{\mu}\nabla_{\mu}\underline{C}\ , \tag{2.8}$$

and one has the general relation

$$0=\left\langle\frac{\delta\mathscr{F}}{\delta\phi}+i\mathscr{F}\frac{\delta\mathscr{S}}{\delta\phi}\right\rangle_{*}\ . \tag{2.9}$$

Here ϕ is A, C, or $\bar{C}$, $\mathscr{F}$ is any functional of the ϕ, and

$$\langle\mathscr{G}\rangle_{*}=\langle 0\,|\,T^{*}(\mathscr{G})\,|\,0\rangle \tag{2.10}$$

denotes the vacuum expectation value of the covariant T^* product.

The Lagrangian is invariant to the BRS transformation[9,11]

$$\begin{aligned}&\delta\underline{A}=\lambda\nabla_{\mu}\underline{C}\equiv\lambda(\partial_{\mu}\underline{C}-ig[\underline{A}_{\mu},\underline{C}])\ ,\\&\delta\underline{C}=\frac{i}{2}g\{\underline{C},\underline{C}\}\ ,\delta\underline{\bar{C}}=-i\lambda\frac{1}{\alpha}\partial^{\mu}\underline{A}_{\mu}\ ,\end{aligned} \tag{2.11}$$

where the Grassmann parameter λ anticommutes with itself,

$$\lambda\lambda'=-\lambda'\lambda\ , \tag{2.12}$$

and the invariance of the action,

$$\delta\mathscr{S}=0\ , \tag{2.13}$$

implies the Ward-Takahashi identities

$$\langle\delta\mathscr{F}\rangle_{*}=0\ . \tag{2.14}$$

It follows from (1.5) that if $\underline{U}(x)$ are $N\times N$ matrices and a gauge-transformed vector potential matrix is defined by

$$\begin{aligned}\underline{A}'_{\mu}(x)\equiv\ &\underline{U}(x)\underline{A}_{\mu}(x)\underline{U}^{-1}(x)\\&+\frac{i}{g}\underline{U}(x)\partial_{\mu}\underline{U}^{-1}(x)\ ,\end{aligned} \tag{2.15}$$

then the corresponding phase factor is

$$\begin{aligned}\underline{\Phi}'(\Gamma_{xy})&\equiv P\exp\left[g\int_{\Gamma_{xy}}dz^{\mu}\underline{A}'_{\mu}(z)\right]\\&=\underline{U}(x)\underline{\Phi}(\Gamma_{xy})\underline{U}^{-1}(y)\ .\end{aligned} \tag{2.16}$$

In particular, for a closed path (i.e., a loop),

$$\underline{\Phi}'(\Gamma_{xy})=\underline{U}(x)\underline{\Phi}(\Gamma_{xy})\underline{U}^{-1}(x)\ , \tag{2.17}$$

so that $\mathrm{tr}\underline{\Phi}(\Gamma_{xx})$ is gauge invariant and independent of the chosen starting point x on Γ_{xx}.

With $\underline{U}(x)=\exp[-i\underline{\lambda}(x)]$, the infinitesimal form

$$\begin{aligned}\underline{\Phi}'(\Gamma_{xy})=\ &\underline{\Phi}(\Gamma_{xy})-i\underline{\lambda}(x)\underline{\Phi}(\Gamma_{xy})\\&+i\underline{\Phi}(\Gamma_{xy})\underline{\lambda}(y)\end{aligned} \tag{2.18}$$

of (2.16) implies the transformation law

$$\delta\underline{\Phi}(\Gamma_{xx})=ig\lambda[\underline{C}(x),\underline{\Phi}(\Gamma_{xx})] \tag{2.19}$$

under the BRS transformation (2.11). It follows that

$$\delta\,\mathrm{tr}\underline{\Phi}(\Gamma_{xx})=0 \tag{2.20}$$

and

$$\delta\,\mathrm{tr}[\partial_{\nu}\underline{\bar{C}}(x)\Phi(\Gamma_{xx})]=-i\lambda\,\mathrm{tr}\left[\frac{1}{\alpha}\partial_{\nu}[\partial\cdot\underline{A}(x)]\underline{\Phi}(\Gamma_{xx})+g[\partial_{\nu}\underline{\bar{C}}(x)][\underline{C}(x),\underline{\Phi}(\Gamma_{xx})]\right]\ . \tag{2.21}$$

The latter transformation law gives us the very useful Ward-Takahashi identity[6,12]

$$\left\langle\mathrm{tr}\left[\left[\frac{1}{\alpha}\partial_{\nu}\partial\cdot\underline{A}(x)+g\{\partial_{\nu}\underline{\bar{C}}(x),\underline{C}(x)\}\right]\underline{\Phi}(\Gamma_{xx})\right]\right\rangle_{*}=0\ . \tag{2.22}$$

For open paths, the BRS transformation is

$$\delta\underline{\Phi}(\Gamma_{xy})=ig\lambda[\underline{C}(x)\underline{\Phi}(\Gamma_{xy})-\underline{\Phi}(\Gamma_{xy})\underline{C}(y)]\ , \tag{2.23}$$

so that

$$\delta[\underline{\bar{C}}(z)\underline{\Phi}(\Gamma_{xy})]=-i\lambda\left[\frac{1}{\alpha}[\partial\cdot\underline{A}(z)]\underline{\Phi}(\Gamma_{xy})+g\underline{\bar{C}}(z)[\underline{C}(x)\underline{\Phi}(\Gamma_{xy})-\underline{\Phi}(\Gamma_{xy})\underline{C}(y)]\right]\ . \tag{2.24}$$

We thus obtain the more general Ward-Takahashi identity

$$\frac{1}{\alpha}\langle[\partial\cdot\underline{A}(z)]\underline{\Phi}(\Gamma_{xy})\rangle_{*}=-g\langle\underline{\bar{C}}(z)[\underline{C}(x)\underline{\Phi}(\Gamma_{xy})-\underline{\Phi}(\Gamma_{xy})\underline{C}(y)]\rangle_{*}\ . \tag{2.25}$$

III. RENORMALIZATION

A. Rules and regulations

When the path-ordered phase factor

$$\underline{\Phi}(C)=P\exp\left[ig\oint_C dy^\mu \underline{A}_\mu(y)\right] \tag{3.1}$$

in the loop functional

$$W(C)=\frac{1}{N}\langle 0\,|\,T^*\mathrm{tr}\underline{\Phi}(C)\,|\,0\rangle \tag{3.2}$$

is expanded in a power series, one gets

$$W(C)=1+\sum_{n=2}^{\infty}\frac{1}{n}(ig)^n\oint_C\cdots\oint_C dx_1^{\alpha_1}\cdots dx_n^{\alpha_n}\theta_C(x_1,\ldots,x_n)\frac{1}{N}\,\mathrm{tr}\underline{G}_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n)\,, \tag{3.3}$$

where $\theta_C(x_1,\ldots,x_n)$ orders the points $x_1,\ldots,x_n$ along the contour C and

$$G_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n) =\langle 0\,|\,T^*\underline{A}_{\alpha_1}(x_1)\cdots\underline{A}_{\alpha_n}(x_n)\,|\,0\rangle \tag{3.4}$$

is the n-gluon Green's function. When (3.4) is expanded in perturbation theory, we obtain an expression for $W(C)$ as a power series in g given by Feynman rules based (in Euclidean four-space) on the usual bare propagators and vertices for Green's functions plus the new line vertex $ig\underline{\lambda}^a\int_0^1 d\tau\dot{y}^\mu(\tau)$. The x-space expressions for these quantities are given in Ref. 7.

The resultant integrations are usually formally divergent and so a regulation should be introduced which can be removed after renormalization. It is extremely convenient for our purposes to use the gauge-invariant dimensional regularization.[13] Then the (finite) expressions corresponding to the diagrams will be given by Feynman rules involving the previous vertices and the dimensionally regularized propagators, for example, the gluon propagator

$$D^{ab}_{\mu\nu}(x)=\delta^{ab}\left[\frac{1+\alpha}{8}\right]\frac{\Gamma((D-2)/2)}{\pi^{D/2}(x^2)^{(D-2)/2}}\delta_{\mu\nu} +\delta^{ab}\left[\frac{1-\alpha}{4}\right]\frac{\Gamma(D/2)}{\pi^{D/2}(x^2)^{D/2}}x_\mu x_\nu\,, \tag{3.5}$$

in which internal x-space integrations are taken over D dimensions, d^Dx, and contracted Lorentz indices run from 1 to D, so that $\delta_{\mu\mu}=D$. The resultant functions will be finite for a range of D and will have poles at $D=4$.

The poles at $D=4$ in the dimensionally regularized Green's functions are removed by the conventional renormalizations

$$\underline{A}\to\underline{A}_R=Z_3^{-1/2}\underline{A},\quad \underline{C}\to\underline{C}_R=\tilde{Z}_3^{-1/2}\underline{C}\,, \tag{3.6}$$

$$g\to g_R=Z_1^{-1}Z_3^{3/2}g\mu^{-\epsilon/2},\quad \alpha\to\alpha_R=Z_3^{-1}\alpha \qquad (\epsilon=4-D)$$

when use is made of the fact that dimensional regularization maintains the BRS symmetry and the consequent Ward-Takahashi identities and their implications such as

$$\frac{Z_1}{Z_3}=\frac{\tilde{Z}_1}{\tilde{Z}_3}\,. \tag{3.7}$$

Here Z_1 is the gluon-gluon-gluon vertex renormalization constant and $\tilde{Z}_1$ is the ghost-ghost-gluon vertex renormalization constant. In (3.6) the parameter μ is in general an arbitrary constant with dimensions of mass. For convenience we have chosen μ to be the momentum-space subtraction point for Green's functions. The power of μ in g_R has been chosen so that g_R is dimensionless for all ϵ.

The loop functions have additional poles at $D=4$ due to the coincident points in (3.3). It was shown by Dotsenko and Vergeles[6,14] that for smooth (i.e., differentiable) simple (i.e., non-self-intersecting) loops these additional poles cancel the ones mentioned above and it was shown in Ref. 7 that for loops with cusps and cross points these poles can be removed by further multiplicative renormalizations (with mixing). In the remainder of this section, these results will be discussed and the corresponding renormalization-group equations will be deduced.

B. Simple smooth loop

A given (unregularized) Feynman diagram corresponding to (3.3) will involve (internal) integrations d^4x over the internal x-space vertices and (line) integrations dx^μ over the line vertices. The divergences in these integrations arise from integration regions where two or more vertices are close together. If all the close vertices are internal, the divergences correspond to the divergences of the Green's functions (3.4) in (3.3). With dimensional regularization, these divergences are poles at $D=4$ (corresponding to the logarithmic divergences of cutoff regularization) and are removed by the conventional renormalizations (3.6). If some of the close vertices are line vertices, then (3.3) has additional divergences. These new primitive divergences are of two types: linear divergences arising from one-particle-irreducible[15] (1PI) subdiagrams with no external ghost or gluon lines, and logarithmic divergences arising from 1PI subdiagrams with one external gluon line and no external ghost lines. With dimensional regularization the linear divergences are absent as usual,[16] and the logarithmic divergences become additional poles at $D=4$. Dotsenko and Vergeles have shown that for smooth and simple loops C these poles plus the above Green's function poles plus the poles in g all cancel so that the dimensionally regularized expression (3.3) is finite at $D=4$ in perturbation theory when expressed in terms of the renormalized coupling constant $g_R=Z_1{}^{-1}Z_3{}^{3/2}g\mu^{-\epsilon/2}$.

The Dotsenko-Vergeles argument is based on the Ward-Takahashi identity (2.25) arising from the BRS symmetry.[12] When expressed in terms of the renormalized quantities (3.6), using (3.7), it reads

$$\frac{1}{\alpha_R}\langle \partial\cdot \underline{A}_R(z)\underline{\Phi}(\Gamma_{xy})\rangle_* = -\tilde{Z}_1 g_R \langle \underline{\bar{C}}_R(z)[\underline{C}_R(x)\underline{\Phi}(\Gamma_{xy})-\underline{\Phi}(\Gamma_{xy})\underline{C}_R(y)]\rangle_* . \tag{3.8}$$

The left-hand side of this equality will in general have poles at $D=4$ arising from those of $\underline{\Phi}$,[17] but a simple inductive argument in the Landau gauge shows that these poles are in fact absent in each order of perturbation theory in g_R. Thus $W=\langle \mathrm{tr}\underline{\Phi}(C)\rangle$, being gauge invariant, is finite in all orders of perturbation theory, in all gauges, when C is simple and smooth.

For a smooth simple loop C, the renormalized loop function is thus simply

$$W_R(C;g_R,\mu)=\lim_{\epsilon\to 0} W(C;g(g_R,\mu,\epsilon),\epsilon)\ , \tag{3.9}$$

where

$$\epsilon=4-D\ , \tag{3.10}$$

and $W(C;g,\epsilon)$ is the dimensionally regularized unrenormalized loop function. Note that, in order to maintain the dimensionlessness of $W(C;g,\epsilon)$, the bare coupling constant g acquires a mass dimension of $\epsilon/2$. It is a function $g(g_R,\mu,\epsilon)$ of g_R, μ, and ϵ given by $g=Z_1Z_3{}^{3/2}g_R\mu^{\epsilon/2}$. If we characterize a general loop C by its length $L=L(C)$ and various other dimensionless (angular) parameters $\chi=\chi(C)$, then we may write

$$W_R(C;g_R,\mu)=W_R(\chi,g_R,L\mu)\ . \tag{3.11}$$

With dimensional regularization, the renormalized and unrenormalized loop functions satisfy the same normalization conditions at $C=1\equiv$the trivial loop of length $L=0$:

$$\begin{aligned} W_R(1;g_R,\mu)&=W_R(\chi,g_R,0)\\ &=W(1;g,\epsilon)=1\ . \end{aligned} \tag{3.12}$$

This is because g and L always appear in the dimensionless combination g^2L^ϵ, which vanishes as $L\to 0$ with $\mathrm{Re}\epsilon>0$.

It follows from (3.9) that W_R satisfies the simple renormalization-group equation[18]

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}\right]W_R(C;g_R,\mu)=0\ , \tag{3.13}$$

where

$$\beta(g_R)=\lim_{\epsilon\to 0}\mu\frac{\partial g_R}{\partial\mu}\bigg|_g=-\tfrac{1}{2}bg_R{}^3+O(g_R{}^5)\ , \tag{3.14}$$

with

$$b=\frac{1}{8\pi^2}\frac{11}{3}C_1\ , \tag{3.15}$$

in terms of the adjoint-representation Casimir operator C_1 ($f^{acd}f^{bcd}=2C_1\delta^{ab}$).[19] The solution to

(3.13) is

$$W_R(C;g_R,\mu)\equiv W_R(\mathcal{X},g_R,L\mu)$$
$$=W_R(\mathcal{X},\bar{g}(g_R,t),1)\ ,$$
$$t\equiv -\ln L\mu\ , \quad (3.16)$$

where the effective coupling constant is defined as usual by

$$\frac{\partial \bar{g}(g_R,t)}{\partial t}=\beta(\bar{g}(g_R,t)),\ \ \bar{g}(g_R,0)=g_R\ . \quad (3.17)$$

Because of asymptotic freedom, the large-t behavior of $\bar{g}$ is known exactly,[10]

$$\bar{g}^2(g_R,t)\underset{t\to\infty}{\sim}\frac{1}{bt}\ , \quad (3.18)$$

which implies the small-L behavior

$$W_R(\mathcal{X},g_R,L\mu)\underset{L\to 0}{\longrightarrow}W_R(\mathcal{X},0,1)=1\ , \quad (3.19)$$

which was already known from (3.12). In the following subsections, a less trivial use will be made of (3.18).

C. Simple loop with cusp

If $C=C_\gamma$ is a loop which is smooth and simple except for a cusp of angle γ (see Fig. 1), then in perturbation theory the dimensionally regularized loop function will have poles at $D=4$ arising from integration regions in which subdiagrams with no external gluon or ghost lines have vertices near the cusp point. It is shown in Ref. 7 that these poles may be removed by a further multiplicative renormalization so that the renormalized loop function may be expressed as

$$W_R(C_\gamma;\gamma,g_R,\mu)$$
$$=\lim_{\epsilon\to 0} Z(\gamma,g,\mu,\epsilon)W(C_\gamma;g,\epsilon)\Big|_{g=g(g_R,\mu,\epsilon)}\ . \quad (3.20)$$

Here the renormalization constant may be written

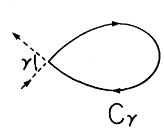


FIG. 1. A loop C_γ with a cusp of angle γ. The tangent vector to the loop jumps through angle γ at the cusp point.

$$Z(\gamma,g,\mu,\epsilon)=W^{-1}(\bar{C}_\gamma;g,\epsilon)\ , \quad (3.21)$$

where $\bar{C}_\gamma$ is an arbitrary but fixed loop with cusp γ. We characterize $\bar{C}_\gamma$ by the fixed length $1/\mu$ and dimensionless parameters $\bar{\mathcal{X}}$ (fixed) and γ. (Allowing the length of $\bar{C}$ to be given by a new parameter obviously leads to no increase in generality.) The choice (3.21) corresponds to the choice

$$W_R(\bar{C}_\gamma;\gamma,g_R,\mu)=1 \quad (3.22)$$

of normalization condition. In perturbation theory

$$Z=1+g_R{}^2\left[\eta(\gamma)\frac{1}{\epsilon}+\text{finite}\right]+O(g_R{}^4)\ , \quad (3.23)$$

where

$$\eta(\gamma)=\frac{N}{(2\pi)^2}(\gamma\cot\gamma-1)\le 0\ \ \text{for U}(N)\ . \quad (3.24)$$

The exact Z will be given below.

We characterize the general C_γ by a length $L=L(C_\gamma)$ and dimensionless parameters $\mathcal{X}=\mathcal{X}(C_\gamma)$ and γ so that dimensional analysis gives

$$W_R(C_\gamma;\gamma,g_R,\mu)=W_R(\mathcal{X},\gamma,g_R,L\mu)\ ,$$
$$W(C_\gamma;g,\epsilon)=W(\mathcal{X},\gamma,g_R,L\mu,\epsilon)\ , \quad (3.25)$$

and

$$Z(\gamma,g,\mu,\epsilon)=W^{-1}(\bar{\mathcal{X}},\gamma,g_R,1,\epsilon)$$
$$\equiv Z(\gamma,g_R,\epsilon)\ .$$

Then (3.20) reads

$$W_R(\mathcal{X},\gamma,g_R,L\mu)=\lim_{\epsilon\to 0} Z(\gamma,g_R,\epsilon)W(\mathcal{X},\gamma,g_R,L\mu,\epsilon)\ . \quad (3.26)$$

Note that, because Z is singular at $\epsilon=0$, even though

$$W(\mathcal{X},\gamma,g_R,0,\epsilon)=W(1;g,\epsilon)=1\ , \quad (3.27)$$

$W_R(\mathcal{X},\gamma,g_R,0)$ is singular and does not satisfy (3.12), but rather (3.22), or

$$W_R(\bar{\mathcal{X}},\gamma,g_R,1)=1\ . \quad (3.28)$$

Now W_R satisfies the nontrivial renormalization-group equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}+\Gamma(\gamma,g_R)\right]\times W_R(\chi,\gamma,g_R,L\mu)=0\ , \quad (3.29)$$

where the anomalous dimension function is given by

$$\Gamma(\gamma,g_R)=\lim_{\epsilon\to 0} Z\mu\frac{\partial}{\partial\mu}Z^{-1}\Big|_{\text{fixed}\,g,\epsilon}={g_R}^2\eta(\gamma)+O({g_R}^4)\ . \quad (3.30)$$

The solution

$$W_R(\chi,\gamma,g_R L\mu)=W_R(\chi,\gamma,\bar{g}(g_R,t),1)T\exp\left[\int_0^t dt'\Gamma(\gamma,\bar{g}(g_R,t'))\right],\ t\equiv-\ln L\mu\ , \quad (3.31)$$

combined with (3.18), enables us to calculate the exact[10] small-L behavior of W_R:

$$W_R(\chi,\gamma,g_R,L\mu)\underset{L\to 0}{\sim}\xi(g_R,\gamma)(-\ln L\mu)^{\eta(\gamma)/b}\approx 0\ , \quad (3.32)$$

where the χ and $L\mu$ independent dimensionless function $\xi(g_R,\gamma)$ is not computable. Note that, although (3.32) is dependent upon the cusp angle γ, it vanishes for all $0\le\gamma\le\pi$ except for $\gamma=0$ where the cusp disappears.

It follows from the definition (3.30) that Z^{-1} satisfies the renormalization-group equation

$$\left[\beta(g_R,\epsilon)\frac{\partial}{\partial g_R}-\Gamma(\gamma,g_R,\epsilon)\right]Z^{-1}(\gamma,g_R,\epsilon)=0\ , \quad (3.33)$$

where $\beta(g_R,\epsilon)$ and $\Gamma(\gamma,g_R,\epsilon)$ are given, respectively, by (3.14) and (3.30) without $\epsilon\to 0$. The solution is

$$Z^{-1}(\gamma,g_R,\epsilon)=\exp\left[\int_0^{g_R} dg'\Gamma(\gamma,g',\epsilon)/\beta(g',\epsilon)\right]\ , \quad (3.34)$$

from which, using

$$\begin{aligned}\beta(g_R,\epsilon)&=-\tfrac{1}{2}g_R\epsilon-\tfrac{1}{2}b{g_R}^3+O({g_R}^5)\ ,\\ \Gamma(g_R,\epsilon)&=\eta(\gamma){g_R}^2+O({g_R}^4)\ ,\end{aligned} \quad (3.35)$$

we obtain the exact[10] small-ϵ behavior

$$Z^{-1}\underset{\epsilon\to 0}{\sim}\epsilon^{\eta(\gamma)/b}\to\infty\ . \quad (3.36)$$

Thus the poles in Z at $\epsilon=0$ in finite orders of perturbation theory add up to a branch point.

It is of interest to compare Eqs. (3.26), (3.32), and (3.36). This comparison may be summarized by the diagram

$$Z(\gamma,g_R,\epsilon)W(\chi,\gamma,g_R,L\mu,\epsilon)\ \begin{array}{l}\overset{\nearrow}{\scriptstyle\epsilon\to 0}\ W_R(\chi,\gamma,g_R,L\mu)\underset{L\to 0}{\sim}(-\ln L\mu)^{\eta(\gamma)/b}\\ \underset{L\to 0}{\searrow}\ Z(\gamma,g_R,\epsilon)\underset{\epsilon\to 0}{\sim}\epsilon^{-\eta(\gamma)/b}\ .\end{array} \quad (3.37)$$

It is clearly the vanishing of $Z(\gamma,g_R,\epsilon)$ at $\epsilon=0$ which is responsible for the vanishing of $W_R(\chi,\gamma,g_R,L\mu)$ at $L=0$; and although the regularized unrenormalized W is finite (and unity) at $L=0$, it diverges at $\epsilon=0$ as

$$W(\chi,\gamma,g_R,L\mu,\epsilon)\underset{\epsilon\to 0}{\sim}Z^{-1}(\gamma,g_R,\epsilon)W_R(\chi,\gamma,g_R,L\mu)\cong\epsilon^{-\eta(\gamma)/b}W_R(\chi,\gamma,g_R,L\mu)\ . \quad (3.38)$$

The above comparison becomes more transparent if we use a conventional momentum space cutoff λ instead of dimensional regularization. For this purpose, we assume the existence of a gauge-invariant regularization in which the cutoff parameter λ has the dimension of mass. Then $g=g(g_R,\lambda/\mu)$ remains dimensionless and (3.26) is replaced by

$$W_R(\chi,\gamma,g_R,L\mu)=\lim_{\lambda\to\infty}Z(\gamma,g_R,\lambda/\mu)W(\chi,\gamma,g_R,L\lambda)\ . \quad (3.39)$$

Now, for large λ/μ, we have the renormalization-group equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-\Gamma(\gamma,g_R)\right]Z^{-1}(\gamma,g_R,\lambda/\mu)=0\ , \tag{3.40}$$

which implies

$$Z^{-1}\underset{\frac{\lambda}{\mu}\to\infty}{\sim}\left[\ln\frac{\lambda}{\mu}\right]^{-\eta(\gamma)/b}\ . \tag{3.41}$$

The large-λ and small-L limits are now directly comparable:

$$Z\left[\gamma,g_R,\frac{\lambda}{\mu}\right]W(\mathcal{X},\gamma,g_R,L\lambda)\ \begin{array}{l}\nearrow_{\lambda\to\infty}\ W_R(\mathcal{X},\gamma,g_R,L\mu)\underset{L\to 0}{\sim}(-\ln L\mu)^{\eta(\gamma)/b}\\ \searrow_{L\to 0}\ Z\left[\gamma,g_R,\frac{\lambda}{\mu}\right]\underset{\lambda\to\infty}{\sim}(\ln\lambda/\mu)^{\eta(\gamma)/b}\ .\end{array} \tag{3.42}$$

D. Loop with cross point

Suppose now that the loop $C=C_\gamma$ is smooth and simple except at a single point where it crosses itself one or more times. The parameter γ now stands for the set of independent crossing angles at the cross point. Examples are given in Figs. 2 and 3. Suppose that at the cross point there are N incoming and N outgoing lines. Then, as shown in Ref. 7, the renormalization of $W(C)$ must be considered in conjunction with the renormalization of the loop functions of all the sets of loops which are the same (in space and in direction) as C except at the cross point where (say) the N outgoing lines are permuted in any way. We denote the sets of so-defined loops by

$$\{C_1^i,\ldots,C_n^i\},\ \ i=1,\ldots,I_N\ , \tag{3.43}$$

where $I_N=(N+1)!$ is the number of possible distinct permutations and the integer n_i is the number of disconnected loops C_j resulting from the ith permutation; $1\le n_i\le N$. Examples are given in Figs. 2 and 3. We denote the corresponding unrenormalized dimensionally regularized loop functions by

$$W^i(\mathcal{X},\gamma,g_R,L\mu,\epsilon)\equiv W_{n_i}(C_1^i,\ldots,C_{n_i}^i;g,\epsilon)\ , \tag{3.44}$$

where

$$L=L(C)=\sum_{j=1}^{n_i}L(C_j^i) \tag{3.45}$$

is the length of C and $(\mathcal{X},\gamma)$ represent the dimensionless characteristics of C or the sets (3.43). We

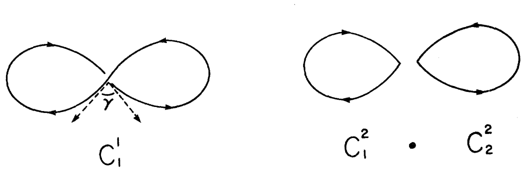


FIG. 2. A loop C_1^1 with a cross point of angle γ. γ is the angle between the two distinct tangent vectors to the loop at the cross point. C_1^2 and C_2^2 are the two subloops of C_1^1 which meet at the cross point. The loop functions $W_1(C_1^1)$ and $W_2(C_1^2,C_2^2)$ mix upon renormalization. The 2×2 renormalization matrix is given explicitly in Eqs. (3.64)–(3.66) to order $g_R{}^2$.

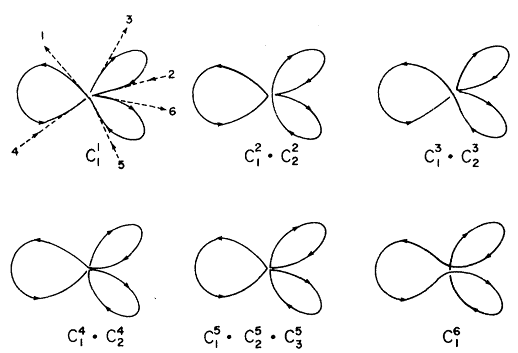


FIG. 3. A loop C_1^1 with a (double) cross point. The geometry of the loop at the cross point is characterized by the angles γ_{ij} between the ith and jth tangent vectors to the loop at the cross point. (For clarity some of the tangent vectors shown in the figure end at the cross point, but the angles γ_{ij} are defined to be the angles between the tangent vectors when drawn such that they *all* begin at the cross point.) Only five of the γ_{ij} are independent. The loops C_j^i $(i=1,\ldots,6;\ j=1,\ldots,n_i)$ are subloops of C_1^1 which meet at the cross point. The loop functions $W_{n_i}(C_1^i,\ldots,C_{n_i}^i)$ mix upon renormalization. The 6×6 renormalization matrix is given explicitly in Eqs. (3.74)–(3.76) to order $g_R{}^2$.

denote the corresponding renormalized loop functions by

$$W_R^i(\mathcal{X},\gamma,g_R,L\mu)\equiv W_{Rn_i}(C_1^i,\ldots,C_{n_i}^i;g_R,\mu)\ . \tag{3.46}$$

According to Ref. 7, there exists an $I_N\times I_N$ matrix $Z^{ij}(\gamma,g,\mu,\epsilon)=Z^{ij}(\gamma,g_R,\epsilon)$ such that[20]

$$W_R^i(\mathcal{X},\gamma,g_R,L\mu)=\lim_{\epsilon\to 0}Z^{ij}(\gamma,g_R,\epsilon)\times W^j(\mathcal{X},\gamma,g_R,L\mu,\epsilon)\ . \tag{3.47}$$

There is a considerable arbitrariness in the matrix elements Z^{ij}. We choose here to make the Z^{ij} unique by renormalizing the W^j according to the *minimal* subtraction scheme. This means that in each order of perturbation theory Z^{ij} is given by a sum of poles at $\epsilon=0$, with no finite part. More precisely, writing

$$Z^{ij}(\gamma,g_R,\epsilon)=\sum_{n=0}^{\infty}g_R{}^{2n}Z_n^{ij}(\gamma,\epsilon)\ , \tag{3.48}$$

we take

$$\begin{aligned}Z_n^{ij}(\gamma,\epsilon)&=-\mathrm{div}_i W_n^j(\mathcal{X},\gamma,L\mu,\epsilon)\\&=\sum_{r=1}^{n}\epsilon^{-r}\zeta_{nr}^{ij}(\gamma)\ ,\end{aligned} \tag{3.49}$$

exactly a sum of poles. Given g_R, the renormalization matrix has a unique expression of the form (3.49), independently of the choice of μ. The independence of Z_n^{ij} on the loop characteristics $\mathcal{X},L$ other than γ is proved in Ref. 7. Further discussion of the arbitrariness of Z^{ij} will be given in Sec. V.

The renormalized loop functions satisfy the renormalization-group equations

$$\left[\delta^{ij}\mu\frac{\partial}{\partial\mu}+\delta^{ij}\beta(g_R)\frac{\partial}{\partial g_R}+\Gamma^{ij}(\gamma,g_R)\right]W_R^j(\mathcal{X},\gamma,g_R,L\mu)=0\ , \tag{3.50}$$

where the anomalous dimension matrix is given by

$$\Gamma^{ij}(\gamma,g_R)=\lim_{\epsilon\to 0}\left[Z^{ik}(\gamma,g,\mu,\epsilon)\mu\frac{\partial}{\partial\mu}[Z^{-1}(\gamma,g,\mu,\epsilon)]^{kj}\right]_{\mathrm{fixed}\,g,\epsilon}\ . \tag{3.51}$$

From the perturbative expansion

$$Z^{ij}(\gamma,g_R,\epsilon)=\delta^{ij}+g_R{}^2\eta^{ij}(\gamma)\frac{1}{\epsilon}+O(g_R{}^4)\ , \tag{3.52}$$

we obtain

$$\Gamma^{ij}(\gamma,g_R)=g_R{}^2\eta^{ij}(\gamma)+O(g_R{}^4)\ . \tag{3.53}$$

The solution to (3.50) is ($t\equiv-\ln L\mu$ and $\overline{T}$ denotes anti-t ordering)

$$W_R^i(\mathcal{X},\gamma,g_R,L\mu)=\sum_j\left[\overline{T}\exp\left[\int_0^t dt'\Gamma(\gamma,\overline{g}(g_R,t'))\right]\right]^{ij}W_R^j(\mathcal{X},\gamma,\overline{g}(g_R,\epsilon),1) \tag{3.54}$$

in terms of the effective charge (3.17). The exact small-L behavior is thus[10]

$$W_R^i(\mathcal{X},\gamma,g_R,L\mu)\underset{L\to 0}{\sim}\sum_{j,k}\{\exp[\eta(\gamma)\ln t/b]\}^{ij}M^{jk}(\mathcal{X},\gamma,g_R)=\sum_{l=1}^{I_N}(-\ln L\mu)^{\eta_l(\gamma)/b}w_l^i(\mathcal{X},\gamma,g_R)\ , \tag{3.55}$$

where the eigenvalues $\eta_l(\gamma)$ of $\eta^{ij}(\gamma)$ are known from (3.53) but the coefficients w_l^i are not computable by these methods. In terms of the known projection operators $P_l^{ij}(\gamma)$ which decompose η^{ij},

$$\begin{aligned}&\eta^{ij}=\sum_l\eta_l P_l^{ij},\quad \sum_l P_l^{ij}=\delta^{ij}\ ,\\&P_l^{ij}P_m^{jk}=\delta_{lm}P_m^{ik}\ ,\end{aligned} \tag{3.56}$$

we have

$$w_l^i=\sum_{jk}P_l^{ij}M^{jk}\ , \tag{3.57}$$

but the mixing matrix M^{jk} is unknown. Some of the eigenvalues $\eta_l(\gamma)$ will in general be positive so that $W_R^i(\mathcal{X},\gamma,g_R,0)$ will be divergent unless $w_l^i(\mathcal{X},\gamma,g_R)$ accidentally vanishes.

As in Sec. III C, Eq. (3.51) (without $\epsilon\to 0$) can be recast as the renormalization-group equation

$$\left[\delta^{ij}\beta(g_R,\epsilon)\frac{\partial}{\partial g_R}-\Gamma^{ji}(\gamma,g_R,\epsilon)\right]\times[Z^{-1}(\gamma,g_R,\epsilon)]^{kj}=0\ , \quad (3.58)$$

whose solution is the anti-g-ordered exponential

$$[Z^{-1}(\gamma,g_R,\epsilon)]^{ij}=\left[\bar{G}\exp\left[\int_0^{g_R}dg'\frac{\Gamma(\gamma,g',\epsilon)}{\beta(g',\epsilon)}\right]\right]^{ij} \quad (3.59)$$

which gives the exact small-ϵ behavior

$$[Z^{-1}(\gamma,g_R,\epsilon)]^{ij}\underset{\epsilon\to 0}{\sim}\sum_l \epsilon^{\eta_l(\gamma)/b}z_l^{ij}(\gamma,g_R) \quad (3.60)$$

in terms of the unknown projected coefficients z_l^{ij}. Comparison with (3.55) is again facilitated by use of a momentum cutoff λ instead of dimensional regularization. Then (3.47) is replaced by

$$W_R^i(X,\gamma,g_R,L\mu)=\lim_{\lambda\to\infty}Z^{ij}(\gamma,g_R,\lambda/\mu)W^j(X,\gamma,g_R,\lambda L,\lambda/\mu)\ , \quad (3.61)$$

and (3.58) is replaced by (for large λ)

$$\left[\delta^{ij}\mu\frac{\partial}{\partial\mu}+\delta^{ij}\beta(g_R)-\Gamma^{ji}(\gamma,g_R)\right]\times[Z^{-1}(\gamma,g_R,\lambda/\mu)]^{kj}=0 \quad (3.62)$$

whose solution has the large-λ behavior

$$[Z^{-1}(\gamma,g_R,\lambda/\mu)]^{ij}\underset{\lambda/\mu\to\infty}{\sim}(\ln\lambda/\mu)^{\eta_l(\gamma)/b}z_l^{ij}(\gamma,g_R)\ . \quad (3.63)$$

We illustrate the above results first for the mixing group of Fig. 2, in which $C_1{}^1$ is a loop which is smooth everywhere and simple everywhere except at the single cross point, and $C_1{}^2$ and $C_2{}^2$ are the two subloops of $C_1{}^1$ which meet at the cross point. There is then only one independent angle γ necessary to characterize the crossing. We find that the matrix elements of the renormalization-constant matrix are

$$Z^{ij}(\gamma,g_R,\epsilon)=\delta^{ij}-\frac{Ng_R{}^2}{(2\pi)^2}\lambda^{ij}(\gamma)\frac{1}{\epsilon}+O(g_R{}^4)\ , \quad (3.64)$$

where

$$(\lambda^{ij})=2\begin{bmatrix}0 & a\\ \frac{1}{N^2}(a-b) & b\end{bmatrix} \quad (3.65)$$

with

$$a=\pi\cot\gamma,\quad b=\gamma\cot\gamma-1\ . \quad (3.66)$$

Thus the anomalous dimension matrix is

$$\underline{\Gamma}(\gamma,g_R)=\underline{Z}\mu\frac{\partial}{\partial\mu}\underline{Z}^{-1}\Big|_{g,\epsilon\text{ fixed}}=\frac{Ng_R{}^2}{(2\pi)^2}\underline{\lambda}+O(g_R{}^4)\ , \quad (3.67)$$

which has eigenvalues $Ng_R{}^2\lambda_\pm/(2\pi)^2$ with

$$\lambda_\pm=b\pm[b^2+4a(a-b)/N^2]^{1/2}\ , \quad (3.68)$$

the eigenvalues of $\underline{\lambda}$. Now a is positive for $0\le\gamma<\pi/2$ and negative for $\pi/2<\gamma\le\pi$, but for all $0\le\gamma\le\pi$ we have the inequalities

$$b\le 0,\quad a-b\ge 0\ ,$$

$$\tfrac{1}{4}b^2+\frac{1}{N^2}a(a-b)=\left[\frac{b}{2}-\frac{a}{N^2}\right]^2+\left[1-\frac{1}{N^2}\right]\frac{a^2}{N^2}\ge 0\ . \quad (3.68')$$

Thus for $0\le\gamma<\pi/2$ we have $\lambda_-\le 0$ and $\lambda_+\ge 0$, and for $\pi/2<\gamma\le\pi$ we have $\lambda_-\le\lambda_+\le 0$. It follows that λ_+ gives the small-L behavior of the loop functions,

$$W_R^i(X,\gamma,g_R,L\mu)\underset{L\to 0}{\sim}(-\ln L\mu)^{N\lambda_+(\gamma)/(2\pi)^2b}\times w_+^i(X,\gamma,g_R)\ , \quad (3.69)$$

and λ_- gives the small-ϵ behavior of the renormalization constants,

$$(Z^{-1})^{ij}(\gamma,g_R,\epsilon)\underset{\epsilon\to 0}{\sim}\epsilon^{N\lambda_-(\gamma)/(2\pi)^2b}\times z_-^{ij}(\gamma,g_R)\ . \quad (3.70)$$

The eigenvalues $\lambda_{\pm}(\gamma)$ are plotted in Fig. 4. Note that for $N\rightarrow\infty$ with $Ng_R{}^2\equiv g_\infty$ fixed, $\lambda_-\rightarrow b$, $\lambda_+\rightarrow 0$, and $\underline{\lambda}$ ceases to be diagonalizable. We still have the expansion

$$\underline{\lambda}=2\begin{bmatrix}0 & a\\ 0 & b\end{bmatrix}=2b\underline{P}_+ + 0\underline{P}_- \qquad (3.71)$$

in term of the projection matrices

$$\underline{P}_+=\begin{bmatrix}0 & a/b\\ 0 & 1\end{bmatrix},\quad \underline{P}_-=\begin{bmatrix}1 & -a/b\\ 0 & 0\end{bmatrix}, \qquad (3.72)$$

and may conclude that the small-L behaviors of the renormalized infinite N-loop functions W^i_∞ are

$$W^i_\infty(X,\gamma,g_\infty,L\mu)\underset{L\to 0}{\sim}(-\ln L\mu)^{b(\gamma)/(2\pi)^2 b}\times w^i_\infty(X,\gamma,g_\infty)\,. \qquad (3.73)$$

As a second example, we consider the six loop functions W^i $(i=1-6)$ associated with the loops of Fig. 3 ($C_1{}^1$ and the sets of loops with which it mixes). There are six tangent vectors at the cross point and we call γ_{ij} the angle between the ith and jth (outgoing) tangent vector. We find

$$Z^{ij}(\gamma,g_R,\epsilon)=\delta^{ij}+\frac{g_R{}^2N}{(2\pi)^2}\Lambda^{ij}(\underline{\gamma})\frac{1}{\epsilon}+O(g_R{}^4)\,, \qquad (3.74)$$

where

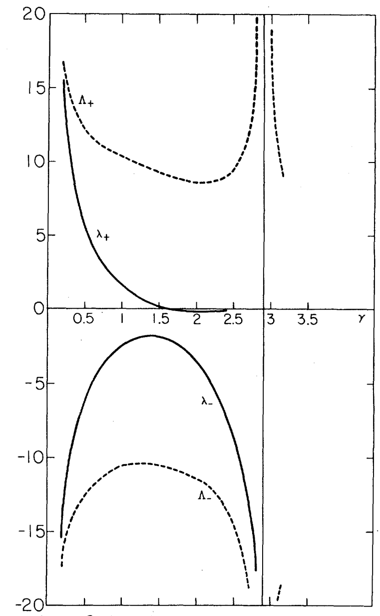


FIG. 4. Plot of the eigenvalues $\lambda_{\pm}(\gamma)$, Eq. (3.68), of the 2×2 matrix $\underline{\lambda}$ given in Eq. (3.65), and of the maximum and minimum eigenvalues $\Lambda_{\pm}(\gamma)$ of the 6×6 matrix $\underline{\Lambda}$ given in Eq. (3.75). The lowest-order anomalous dimension matrix for the mixing set of Fig. 2 is $\underline{\Gamma}(\gamma,g_R{}^2)=Ng_R{}^2\underline{\lambda}/2\pi^2$ and for the mixing set of Fig. 3 is $\underline{\Gamma}(\underline{\gamma},g_R{}^2)=Ng_R{}^2\underline{\Lambda}(\gamma)/2\pi^2$ when $\gamma\equiv\gamma_{13}$ varies between 0 and 3.5 with γ_{1i} fixed for $i\neq 3$. [The angle γ_{15} is held fixed at 2.9 and so $\Lambda_{\pm}(\gamma)$ diverge at $\gamma=2.9$.] The greater eigenvalue $\lambda_+(\gamma)$ is positive for $0\le\gamma<\pi/2$ [with $\lambda_+(0)=+\infty$] and negative (and numerically very close to zero) for $\pi/2<\gamma<\pi$ and zero for $\gamma=\pi/2$ and π. The lesser eigenvalue $\lambda_-(\gamma)$ is negative for all $0\le\gamma\le\pi$ and minus infinity for $\gamma=0$ and π. The maximum eigenvalue $\Lambda_+(\gamma)$ is positive for all γ, and the minimum eigenvalue $\Lambda_-(\gamma)$ is negative for all γ, with $\Lambda_{\pm}(\gamma)=\pm\infty$ for $\gamma=0$ and 2.9. The maximum eigenvalues determine the exact small-length behavior of the renormalized loop functions according to Eqs. (3.69) and (3.77), and the minimum eigenvalues determine the exact small-ϵ behavior of the renormalization constants according to Eqs. (3.70) and (3.78).

$$\underline{\Lambda}=\begin{bmatrix}
N^2a_{152634} & N^2b_{1345}c_{1435} & N^2b_{2436}c_{2346} & N^2b_{1625}c_{1256} & 0 & 0\\
b_{1345}c_{1534} & N^2a_{142635} & 0 & 0 & N^2b_{2536}c_{2356} & b_{1624}c_{4612}\\
b_{2436}c_{2634} & 0 & N^2a_{152346} & 0 & N^2b_{1645}c_{1456} & b_{1325}c_{1235}\\
b_{1625}c_{1526} & 0 & 0 & N^2a_{123456} & N^2b_{1324}c_{1423} & b_{3645}c_{3546}\\
0 & b_{2536}c_{2635} & b_{1645}c_{1546} & b_{1324}c_{1234} & N^2a_{142356} & 0\\
0 & N^2b_{16}c_{1426} & Nb_{1325}c_{1523} & N^2b_{3645}c_{3456} & 0 & N^2a_{123546}b_{24}
\end{bmatrix} \qquad (3.75)$$

with

$$a_{ij}=\gamma_{ij}\cot\gamma_{ij}-1\,,\quad b_{ij}=(\pi-\gamma_{ij})\cot\gamma_{ij}\,,$$
$$c_{ij}=\gamma_{ij}\cot\gamma_{ij}\,, \qquad (3.76)$$

and $a_{ijkl}=a_{ij}+a_{kl}$, etc. We have computed the eigenvalues of $\underline{\Lambda}(\underline{\gamma})$ numerically and found both positive and negative eigenvalues for all γ and also complex eigenvalues for some $\underline{\gamma}$. In Fig. 4 we have plotted the maximum eigenvalue $\Lambda_+(\gamma)$ and the minimum eigenvalue $\Lambda_-(\underline{\gamma})$ as a function of $\gamma\equiv\gamma_{13}$ at constant γ_{1i} for $i\neq 3$. The maximum eigenvalue $\Lambda_+(\underline{\gamma})$ determines the leading small-L behavior

$$W_R^i(X,\underline{\gamma},g_R,L\mu) \underset{L\to 0}{\sim} (-\ln L\mu)^{N\Lambda_+(\underline{\gamma})/4\pi^2 b} \times w_+^i(X,\underline{\gamma},g_R) , \quad (3.77)$$

of W_R^i and the minimum eigenvalue $\Lambda_-(\underline{\gamma})$ determines the leading small-ϵ behavior

$$(Z^{-1})^{ij}(\underline{\gamma},g_R,\epsilon) \underset{\epsilon\to 0}{\sim} \epsilon^{N\Lambda_-(\underline{\gamma})/4\pi^2 b} z_-^{ij}(\underline{\gamma},g_R) \quad (3.78)$$

of $\underline{Z}^{-1}$.

IV. THE MAKEENKO-MIGDAL EQUATIONS

A. Formal derivation

We take the gauge group G to be U(N) or SU(N). The first step in the formal derivation of the MM equation is to use (2.9) to obtain an expression for the insertion of the local field operator $\nabla^\mu \underline{F}_{\mu\nu}(x)$ into a loop function. We take $\mathscr{F}=\underline{\Phi}(\Gamma_{xx})$ and $\phi=\underline{A}_\mu(x)$ and easily obtain

$$\left\langle \mathrm{tr}\left[\left[\nabla^\mu \underline{F}_{\mu\nu}(x)-\frac{1}{\alpha}\partial_\nu\partial\cdot\underline{A}(x)-g\{\partial_\nu\underline{\bar{C}}(x),\underline{C}(x)\}\right]\underline{\Phi}(\Gamma_{xx})\right]\right\rangle_* = -ig\oint_{\Gamma_{xx}} dy_\nu \delta^4(x-y)\langle[\mathrm{tr}\underline{\Phi}(\Gamma_{xy})][\mathrm{tr}\underline{\Phi}(\Gamma_{yx})]\rangle_* + \left[\frac{ig}{N}\oint_{\Gamma_{xx}} dy^\nu \delta^4(x-y)\langle \mathrm{tr}\underline{\Phi}(\Gamma_{xx})\rangle_*\right]_{\mathrm{SU}(N)} . \quad (4.1)$$

The left-hand side of (4.1) is $i\langle \mathrm{tr}\underline{\Phi}\,\underline{\lambda}^a\delta\phi/\delta A_\nu^a\rangle_*$ and the right-hand side, in which the relation

$$\lambda_{ij}^a\lambda_{kl}^a=\delta_{il}\delta_{jk} \text{ for U}(N) , \quad \lambda_{ij}^a\lambda_{kl}^a=\delta_{il}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{kl} \text{ for SU}(N) , \quad (4.2)$$

has been used, is $\langle \mathrm{tr}\underline{\lambda}^a\delta\underline{\Phi}/\delta A_\nu^a\rangle_*$. There is a contribution to the right-hand side for each decomposition of Γ_{xx} into the product $\Gamma_{xy}\Gamma_{yx}$ of [closed because of $\delta^4(x-y)$] paths Γ_{xy} and Γ_{yx}. From now on we will for simplicity take $G=\mathrm{U}(N)$ so that the second term on the right-hand side of (4.1) is absent. The gauge-fixing and ghost field contributions to the left-hand side can be eliminated by use of the Ward-Takahashi identity (2.22). The result is

$$\langle \mathrm{tr}\{[\nabla^\mu \underline{F}_{\mu\nu}(x)]\underline{\Phi}(\Gamma_{xx})\}\rangle_* = -ig\oint_{\Gamma_{xx}} dg_\nu \delta^4(x-y)\langle[\mathrm{tr}\underline{\Phi}(\Gamma_{xy})][\mathrm{tr}\underline{\Phi}(\Gamma_{yx})]\rangle_* . \quad (4.3)$$

The right-hand side of (4.3) involves only $W_2(\Gamma_{xy},\Gamma_{yx})$ and the object now is to also express the left-hand side in terms of functional derivatives of loop functions. To accomplish this, MM used the old Mandelstam[1] area derivative

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\Psi(C_x)\equiv \lim_{|\delta\sigma_{\mu\nu}|\to 0} |\delta\sigma_{\mu\nu}|^{-1} \times[\Psi(C_x\delta C_{\mu\nu})-\Psi(C_x)] \quad (4.4)$$

on functionals Ψ of loops $C_x=\Gamma_{xx}$ (or paths). Here $\delta C_{\mu\nu}$ is a little loop around x in the μ-ν plane of area $|\delta\sigma_{\mu\nu}|$. Mandelstam states that[21]

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\mathrm{tr}\underline{\Phi}(C)=ig\,\mathrm{tr}[\underline{F}_{\mu\nu}(x)\underline{\Phi}(C_x)] . \quad (4.5)$$

To make this more precise, we first take the added loop $\delta C_{\mu\nu}$ to be a square in the μ-ν plane beginning at x with sides of length ϵ and with the same orientation as C. Thus $\delta C_{\mu\nu}$ starts at x, goes a distance ϵ in the μ direction, then goes a distance ϵ in the ν direction, then goes a distance ϵ in the $-\mu$ direction, and finally goes a distance ϵ in the $-\nu$ direction. This is illustrated in Fig. 5. Then $|\delta\sigma_{\mu\nu}|=\epsilon^2\to 0$ as $\epsilon\to 0$. Using (1.5) and *assuming* that the local field products like $\underline{A}_\mu(x)\underline{A}_\nu(x)$, which occur in (3.4), are not singular, we arrive at (4.5). This smoothness assumption is crucial in the derivation of (4.5). Without it, one would obtain additional contributions to the right-hand side such as the symmetric term

$$\underline{S}_{\mu\nu}(x)\equiv \lim_{|\delta\sigma|\to 0}\frac{1}{|\delta\sigma_{\mu\nu}|}\left[\oint_{\delta C_{\mu\nu}}\underline{A}(y)\cdot dy\right]^2 . \quad (4.6)$$

If $\underline{A}(x)\underline{A}(x)$ is finite, this limit vanishes since the line integrals are of order $\delta\sigma$, but in unregularized perturbation theory $\underline{A}(x)\underline{A}(x)$ is singular and (4.6)

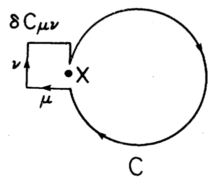


FIG. 5. The loop $C\cdot\delta C_{\mu\nu}$ used to define the area derivative $\delta/\delta\sigma_{\mu\nu}(x)$.

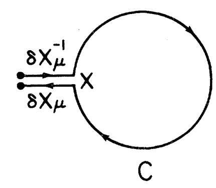


FIG. 7. The loop $\delta x_\mu{}^{-1}\cdot C\cdot\delta x_\mu$ used to define the derivative $\partial/\partial x_\mu$.

does not vanish. [See Eq. (4.70) for the two-dimensional analog of this.] We must therefore always assume that a suitable regularization is present which renders the local field products finite. Furthermore, to be as safe and general as possible, we will *define* $\delta/\delta\sigma_{\mu\nu}$ to be antisymmetric in μ and ν. That is, we take

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\Psi(C_x)\equiv\lim_{|\delta\sigma_{\mu\nu}|\to 0}|\delta\sigma_{\mu\nu}|^{-1}\times\tfrac{1}{2}[\Psi(C_x\delta C_{\mu\nu})-\Psi(C_x\delta\widetilde{C}_{\mu\nu})]\ , \tag{4.7}$$

where $\delta C_{\mu\nu}$ is as above and $\delta\widetilde{C}_{\mu\nu}$ covers the same points as $\delta C_{\mu\nu}$ but begins in the ν direction and ends in the $-\mu$ direction. Thus (4.7) is manifestly antisymmetric in μ and ν. This is illustrated in Fig. 6.

Given (4.5), MM next take the ordinary derivative $\partial/\partial x_\mu$ defined on functionals of loops by

$$\partial_\mu\Psi(C_x)=\lim_{|\delta x_\mu|\to 0}|\delta x_\mu|^{-1}\times[\Psi(\delta x_\mu{}^{-1}C\delta x_\mu)_{x+\delta x}-\Psi(C_x)]\ , \tag{4.8}$$

where δx_μ is a line in the μ-direction length $|\delta x_\mu|$. This is illustrated in Fig. 7. If $\Psi(C_x)$ is just an ordinary function $f(x)$ of x, this reduces to the ordinary derivative $\partial_\mu f(x)$. From the definition (1.5), one finds

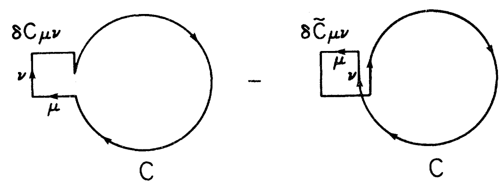


FIG. 6. The loops $C\cdot\delta C_{\mu\nu}$ and $C\cdot\delta\widetilde{C}_{\mu\nu}$ used to define the manifestly antisymmetric area derivative $\delta/\delta\sigma_{\mu\nu}(x)$.

$$\partial_\mu\underline{\Phi}(C_x)=ig[\underline{A}_\mu(x),\underline{\Phi}(C_x)]\ . \tag{4.9}$$

It then follows from (4.5) that

$$\partial_\mu\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\text{tr}\underline{\Phi}(C)=ig\,\text{tr}[\nabla^\mu\underline{F}_{\mu\nu}(x)\underline{\Phi}(C_x)]\ . \tag{4.10}$$

Combining (4.3) and (4.10), we arrive at the first MM equation

$$\partial_\mu\frac{\delta}{\delta\sigma_{\mu\nu}(x)}W_1(C)=g^2N\oint_C dy^\nu\delta^4(x-y)\times W_2(C_{xy},C_{yx})\ . \tag{4.11}$$

All reference to local gauge fields has been eliminated as desired. This is the first of the infinite set of equations which relate the loop functions $W_m(C_1,\ldots,C_m)$. These equations, together with the Bianchi identities

$$\epsilon^{\mu\nu\kappa\lambda}\partial_\nu\frac{\delta}{\delta\sigma_{\kappa\lambda}}W_m=0\ , \tag{4.12}$$

and suitable constraints, hopefully provide a complete system of equations for the loop functions.

So far the formal derivation of (4.11) only suggest its validity for points x on or near the loop C. Although this is all that is needed, it is interesting to ask if (4.11) can be extended off of C. There is no difficulty with the right-hand side, which is well defined as a distribution in x with singular support on C [assuming that $W_2(C_{xy},C_{yx})$ is not singular]. We may try to define $[\delta/\delta\sigma_{\mu\nu}(x)]\Psi(C)$ for x not on C by using (4.4) with C replacing C_x and with $\delta C_{\mu\nu}$ defined as a loop starting at a point z on C, proceeding to x along a line γ_{zx}, proceeding back to x about the old square $(\mu,\nu,-\mu,-\nu)$ contour, and finally returning back to z along the line $\gamma_{zx}{}^{-1}\equiv\gamma_{xz}$. This is illustrated in Fig. 8. The problem of course is that the resultant area derivative might depend on the choice of z on C and on the choice of the line γ_{zx} from z to x. Since we have no control over this at present, we will only use (4.11) for x on C in this paper.

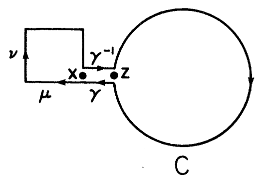


FIG. 8. The loop $\gamma^{-1}\cdot C\cdot\gamma\cdot\delta C_{\mu\nu}$ used to define the area derivatives $\delta/\delta\sigma_{\mu\nu}(x)$ for x not on C.

B. Formal properties

In this subsection we will discuss some of the properties of the MM equation (4.11). Our discussion will be formal in that we will ignore the regularization which is necessary to make the operations and functions in (4.11) well defined, and we will take x to lie on the loop C.

The general oriented loop C on the left-hand side of (4.11) may be parametrized as

$$C=\{\tau\to y(\tau)\ ,\ 0\le\tau\le 1\ ,\ y(0)=y(1)\}\ . \tag{4.13}$$

The orientation of C is defined by the path ordering in (1.5). The loops C_{xy} and C_{yx} on the right-hand side are parts of C running, respectively, from x to y and from y back to x ($x=y$ because of the δ function). These loops have the same path ordering (orientation) as C. Without loss of generality we may take $x=y(0)=y(1)$ so that

$$C=C_{xy}\cdot C_{yx}\ . \tag{4.14}$$

For each τ such that $x=y(\tau)$ there is a contribution to the right-hand side of (4.11). This includes the trivial case $\tau=0$ or 1 as well as other τ's if x is a point of self-intersection of C.

There are two cases when (4.11) simplifies. These are $N=1$ and $N=\infty$. $N=1$ is the (free if there are no charged fields) Abelian gauge theory. In this case the $N\times N$ matrices in (1.5) are just numbers and $W_2(C_{xy},C_{zx})=W_1(C)$. Then (4.11) reduces to

$$\mathscr{D}_x^\mu W_1(C)=g^2 I_x^\mu(C)W_1(C)\quad(N=1)\ , \tag{4.15}$$

where we have introduced the abbreviations

$$\mathscr{D}_x^\mu\equiv\partial^\nu\frac{\delta}{\delta\sigma_{\mu\nu}(x)} \tag{4.16}$$

and

$$I_x^\mu(C)\equiv\oint_C dy^\mu\delta^4(x-y)\ . \tag{4.17}$$

The solution to (4.15) which satisfies the normalization condition

$$W(1_x)=1 \tag{4.18}$$

[$C=1_x$ is the trivial loop $\{\tau\to y(\tau)\equiv x\}$—the unit element of $\mathscr{G}_x$], and the Abelian constraint[22]

$$W_1(C_1\cdot C_2)=W_2(C_1,C_2)\ , \tag{4.19}$$

is

$$W_1(C)=\exp\left[ig^2\oint_C\oint_C dy_1^\alpha dy_{2\alpha}\Delta(y_1-y_2)\right]\quad(N=1), \tag{4.20}$$

where

$$\Delta(x)\equiv\frac{-i}{(2\pi)^2}\frac{1}{x^2} \tag{4.21}$$

is the massless free-field Green's function, which satisfies

$$\Box\Delta(x)=-\delta^4(x)\ . \tag{4.22}$$

The MM equation also simplifies in the large-N (fixed $g_\infty\equiv Ng^2$) limit.[23] There[1-3]

$$W_2(C_{xy},C_{yx})=W_1(C_{xy})W_1(C_{yx})\quad(N=\infty)\ , \tag{4.23}$$

so that (4.11) becomes

$$\mathscr{D}_x^\mu W_1(C)=g_\infty\oint_C dy^\mu\delta^4(x-y)W_1(C_{xy})W_1(C_{yx})\quad(N=\infty)\ , \tag{4.24}$$

a closed equation for W_1. In two dimensions, (4.24) (with $\delta^4\to\delta^2$) can be solved exactly and reproduces the known solution (see Sec. IV F). In four dimensions, (4.24) has not been solved, but the simplicity of (4.24) has rekindled the hope[23] that the $N=\infty$ gauge theory may be (at least, approximately) soluble. However, for $N\to\infty$ the constraint equations become very complicated and therein might lie the undoing of this hope (see Sec. V).

For $N\neq1$ or ∞, we must deal with (4.11) and the infinite chain of equations to which it is coupled. However, if C is a loop without self-intersection, then the factorization (4.14) is possible only if one of the factors is C and the other is trivial:

$$C=C\cdot 1_x\ . \tag{4.25}$$

Then, with (4.18), (4.11) reduces to

$$\mathscr{D}_x^\mu W_1(C)=Ng^2 I_x^\mu(C)W_1(C) \quad (C \text{ simple}) . \tag{4.26}$$

This is the same as the Abelian limit equation (4.15) and that is why it is insufficient to consider only simple loops. Note, however, that (4.20) is not the correct solution to (4.26) for $N\neq1$. There are (at least) two good reasons for this: (i) (4.20) satisfies the wrong constraint (4.19) (Refs. 22 and 24); (ii) (4.20) is not a finite function of the renormalized charge. (see Secs. III and IV E).

C. Area derivative of multiple line integrals

In this subsection we will derive an expression for the area derivative of multiple line integrals of the form

$$\begin{aligned}\mathscr{F}(C_{xx})&\equiv \oint_{C_{xx}} dx_1^{\alpha_1}\cdots dx_n^{\alpha_n}\theta_{C_{xx}}(x_1,\ldots,x_n)f_{\alpha_1\cdots\alpha_n}(x_1,\ldots,x_n)\\&\equiv \int_0^1 d\tau_1\cdots d\tau_n\theta(\tau_1-\tau_2)\cdots\theta(\tau_{n-1}-\tau_n)\dot{x}^{\alpha_1}(\tau_1)\cdots\dot{x}^{\alpha_n}(\tau_n)f_{\alpha_1\cdots\alpha_n}(x(\tau_1),\ldots,x(\tau_n)) ,\end{aligned} \tag{4.27}$$

where C_{xx} is the loop $[\tau\to x^\alpha(\tau),\ 0\le\tau\le1,\ x(0)=x(1)=x]$. We must assume that the function f is a nonsingular function of its arguments. The expression for $W(C)$ is of this form in each order of perturbation theory with dimensional regularization. We will now see that the area derivative (4.7) is well defined for such functionals of loops.

It immediately follows from the definition (4.7) that

$$\begin{aligned}\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\mathscr{F}(C_{xx})=&\lim_{|\delta\sigma_{\mu\nu}|\to0}|\delta\sigma_{\mu\nu}|^{-1}\\&\times\tfrac12\sum_{i=1}^n\int_1^2 d\tau_1\cdots d\tau_i\theta(\tau_1,\ldots,\tau_i)\\&\times\int_0^1 d\tau_{i+1}\cdots d\tau_n\theta(\tau_{i+1},\ldots,\tau_n)\\&\times[\dot{x}_\delta^{\alpha_1}(\tau_1)\cdots\dot{x}_\delta^{\alpha_i}(\tau_i)f_{\alpha_1\cdots\alpha_n}(x_\delta(\tau_1),\ldots,x_\delta(\tau_i),x(\tau_{i+1}),\ldots,x(\tau_n))\\&-\dot{x}_{\bar\delta}^{\alpha_1}(\tau_1)\cdots\dot{x}_{\bar\delta}^{\alpha_i}(\tau_i)f_{\alpha_1\cdots\alpha_n}(x_{\bar\delta}(\tau_1),\ldots,x_{\bar\delta}(\tau_i),x(\tau_{i+1}),\ldots,x(\tau_n))]\\&\times\dot{x}^{\alpha_{i+1}}(\tau_{i+1})\cdots\dot{x}^{\alpha_n}(\tau_n) ,\end{aligned} \tag{4.28}$$

where $x_\delta(\tau)$ represents $\delta C_{\mu\nu}[\tau\to x_\delta^\alpha(\tau),\ 1\le\tau\le2,\ x_\delta(1)=x_\delta(2)=x]$ and $x_{\bar\delta}(\tau)$ represents $\delta\widetilde{C}_{\mu\nu}[\tau\to x_{\bar\delta}^\alpha(\tau),\ 1\le\tau\le2,\ x_{\bar\delta}(1)=x_{\bar\delta}(2)=x,\ x_{\bar\delta}(\tau)=x_\delta(3-\tau)]$. Taking $\delta C_{\mu\nu}$ a rectangular in the μ-ν plane with the sides of order ϵ, the individual terms in this sum can be easily evaluated up to the desired orders ϵ^2. The result is

$$\begin{aligned}\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\mathscr{F}(C_{xx})=&\int_0^1 d\tau_1\cdots d\tau_{n-1}\theta(\tau_1,\ldots,\tau_n)\dot{x}^{\alpha_1}(\tau_1)\cdots x^{\alpha_{n-1}}(\tau_{n-1})\\&\times[\partial_\mu f_{\nu\alpha_1\cdots\alpha_{n-1}}(x,x(\tau_1),\ldots,x(\tau_{n-1}))-\partial_\nu f_{\mu\alpha_1\cdots\alpha_{n-1}}(x,x(\tau_1),\ldots,x(\tau_{n-1}))]\\&-\int_0^1 d\tau_1\cdots d\tau_{n-2}\theta(\tau_1,\ldots,\tau_{n-2})\dot{x}^{\alpha_1}(\tau_1)\cdots\dot{x}^{\alpha_{n-2}}(\tau_{n-2})\\&\times[f_{\mu\nu\alpha_1\cdots\alpha_{n-2}}(x,x,x(\tau_1),\ldots,x(\tau_{n-2}))\\&-f_{\nu\mu\alpha_1\cdots\alpha_{n-2}}(x,x,x(\tau_1),\ldots,x(\tau_{n-2}))] .\end{aligned} \tag{4.29}$$

The first term in (4.29) comes from the $i=1$ term in (4.28) and follows from an application of Stokes's theorem. The second term in (4.29) comes from the $i=2$ term in (4.28). The $i>2$ terms in (4.28) are at least of order ϵ^3 since we have assumed that the function f is sufficiently smooth.

It must be stressed that one cannot neglect the $i>2$ terms in (4.28) unless a proper regularization is introduced in perturbation theory. In the dimensional regularization scheme which we have adopted, we are effectively integrating over D dimensions and the meaning of the area derivative in an arbitrary direction is unclear since the general function $f_{\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n)$ is not defined in the entire D-dimensional space where the indices α range from 1 to D. Fortunately, for the perturbative expansions of loop functions, the functions f which are encountered are expressed in terms of propagators and derivatives and these can be unambiguously extended to D dimensions.

Our next task is to calculate the ordinary derivative $\partial/\partial x_\mu$ defined by

$$\partial_\mu \Psi(C_{xx},x) \equiv \lim_{|\delta x_\mu| \to 0} |\delta x_\mu|^{-1}[\Psi((\delta x_\mu{}^{-1} C_{xx} \delta x_\mu)_{x+\delta x, x+\delta x}, x+\delta x) - \Psi(C_{xx},x)] \ . \tag{4.30}$$

Here δx_μ is a line which starts from x, lies in μ direction and has a length $|\delta x_\mu|$ and $\delta x_\mu{}^{-1}$ is the same line with the opposite orientation. Then $(\delta x_\mu{}^{-1} C_{xx} \delta x_\mu)_{x+\delta x, x+\delta x}$ is a loop which runs from $x+\delta x$ to x along δx_μ, around C_{xx} and goes from x back to $x+\delta x$ along $\delta x_\mu{}^{-1}$. This is illustrated in Fig. 7. If Ψ were just an ordinary function of x, this reduces to the ordinary derivative $\partial/\partial x_\mu$.

Using the definition (4.30), the derivative for a multiple line integral,

$$\mathscr{F}(C_{xx},x) \equiv \int_{C_{xx}} dx_1^{\alpha_1} \cdots dx_n^{\alpha_n} \theta_{C_{xx}}(x_1, \ldots, x_n) f_{\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n; x) \ ,$$

is given by

$$\begin{aligned}\partial_\mu \mathscr{F}(C_{xx},x) \equiv \lim_{|\delta x_\mu| \to 0} |\delta x_\mu|^{-1} & \\ \times \Bigg[\sum_{i=0}^{n} \sum_{j=0}^{n-i} \int_{\delta x} & dx_1^{\alpha_1} \cdots dx_i^{\alpha_i} \theta_{C_{xx}}(x_1, \ldots, x_i)_{C_{xx}} dx_{i+1}^{\alpha_{i+1}} \cdots dx_{n-j}^{\alpha_{n-j}} \theta_{C_{xx}}(x_{i+1}, \ldots, x_{n-j}) \\ & \times \int_{\delta x^{-1}} dx_{n-j+1}^{\alpha_{n-j+1}} \cdots dx_n^{\alpha_n} \theta_{\delta x^{-1}}(x_{n-j+1}, \ldots, x_n) \\ & \times f_{\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n; x+\delta x) - \mathscr{F}(C_{xx},x) \Bigg] \ ,\end{aligned} \tag{4.31}$$

where each line integral is defined with an appropriate parametrization as in (4.28). The terms in (4.31) are easily evaluated up to the order $|\delta x_\mu|$ and the result is

$$\begin{aligned}\partial_\mu \mathscr{F}(C_{xx},x) = & \int_{C_{xx}} dx_1^{\alpha_1} \cdots dx_n^{\alpha_n} \theta_{C_{xx}}(x_1, \ldots, x_n) \partial_\mu f(x_1, \ldots, x_n; x) \\ & + \int_{C_{xx}} dx_1^{\alpha_1} \cdots dx_{n-1}^{\alpha_{n-1}} \theta_{C_{xx}}(x_1, \ldots, x_{n-1}) [f_{\mu\alpha_1 \cdots \alpha_{n-1}}(x, x_1, \ldots, x_{n-1}; x) \\ & - f_{\alpha_1 \cdots \alpha_{n-1}\mu}(x_1, \ldots, x_{n-1}, x; x)] \ .\end{aligned} \tag{4.32}$$

We can use the results (4.29) and (4.32) to give a second formal derivation of the MM equations. Equation (4.5) follows from the application of (4.29) to (1.5) and Eq. (4.10) then follows from the application of (4.32). In the next subsection, we will use (4.29) and (4.32) to prove the MM equations in all orders of perturbation theory.

D. Proof of the Makeenko-Migdal equations in perturbation theory

We consider the unrenormalized dimensionally regularized perturbative expansion (3.3) of the loop functional $W(C)$ and the corresponding expansion of the loop functional

$$V_\nu(C_x;x) \equiv \langle \mathrm{tr}[\nabla^\mu \underline{F}_{\mu\nu}(x) \Phi(C_x)] \rangle_* \tag{4.33}$$

with the $\nabla \cdot \underline{F}$ insertion. The expansion of $\nabla \cdot \underline{F}$ in powers of $\underline{A}$ includes linear, quadratic, and cubic terms. These terms may be represented graphically by the new vertices shown to the left of the equalities in Fig. 9. The linear term

$$L_\nu^a \equiv \partial^\mu \partial_\mu A_\nu^a - \partial_\nu \partial^\mu A_\mu^a \tag{4.34}$$

gives rise to the propagator

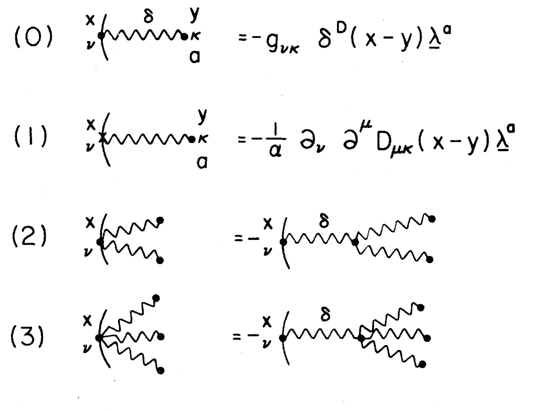


FIG. 9. Diagrams (on the left-hand sides of the equalities) representing the vertices which arise when $\nabla^\mu \underline{F}_{\mu\nu}(x)$ is expressed in terms of A. The (0) and (1) vertices arise from the terms linear in $\underline{A}$, the (2) vertex arises from the term quadratic in $\underline{A}$, and the (3) vertex arises from the term cubic in $\underline{A}$. A wavy line between two dots represents the usual gluon propagator (3.5). Such a line with a δ on top represents the expression $-\delta^{ab}g_{\nu\kappa}\delta^D$ of Eq. (4.35). A wavy line between an x vertex and a dot represents the expression $-\alpha^{-1}\delta^{ab}\partial_\nu\partial^\mu D_{\mu\kappa}$ of Eq. (4.35). The resultant contributions to diagrams are given on the right side of the equalities in diagrams (0) and (1). The equalities illustrated in diagrams (2) and (3), in which the three- and four-gluon vertices on the right are the usual ones occurring in diagrammatic expansions of Green's functions, follow from straightforward but lengthy combinatorics.

$$\langle L_\nu^a(x) A_\kappa^b(y)\rangle_* |_{g=0} = -\delta^{ab} g_{\nu\kappa}\delta^D(x-y) - \frac{1}{\alpha}\delta^{ab}\partial_\nu\partial^\mu D_{\mu\kappa}(x-y) . \tag{4.35}$$

We have used the equation

$$\Box D_{\mu\nu} - \partial^\lambda\partial_\mu D_{\lambda\nu} + \frac{1}{\alpha}\partial^\lambda\partial_\mu D_{\lambda\nu} = -g_{\mu\nu}\delta^D(x) \tag{4.36}$$

satisfied by the gluon propagator (3.5) to obtain this result.

The two terms in (4.35) correspond to the diagrams (0) and (1) of Fig. 9. An elementary but tedious calculation shows that the new vertex corresponding to the quadratic term in the expansion of $\nabla\cdot\underline{F}$, illustrated on the left of the equality in diagram (2) of Fig. 9, satisfies the equality given in Fig. 9, where to the right of the equality the δ propagator is given in diagram (0) and the three-gluon vertex is the usual one corresponding to the cubic coupling $gf^{abc}A_\mu^a A_\nu^b \partial^\mu A_\nu^c$ in the action (2.4). Another elementary but tedious calculation shows that the new vertex corresponding to the cubic term in the expansion of $\nabla\cdot\underline{F}$, illustrated to the left of the equality in diagram (3) of Fig. 9, satisfies the equality given in Fig. 9, where to the right of the equality the four-gluon vertex is the usual one corresponding to the quartic coupling $-\frac{1}{4}g^2 f_{abe} f_{cde} A_\mu^a A_\nu^b A_\mu^c A_\nu^d$ in the action (2.4)

We can also use the vertices (0) and (1) of Fig. 9 to obtain a graphical statement of the Ward-Takahashi identity (2.22). This is illustrated in Fig. 10. The first figure employs the one-gluon vertex (1) and the second figure, which employs the δ propagator (0), amounts to a new two-ghost vertex

$$G^{ab}(x;y,z) \equiv -igf^{abc} \times [\partial_\nu \Delta(x-y)]\Delta(x-z)\underline{\lambda}^a . \tag{4.37}$$

The decomposition of (4.33) corresponding to the expansion of $\nabla\cdot\underline{F}$ in powers of $\underline{A}$ reads (the ν index is suppressed)

$$V = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} , \tag{4.38}$$

where $V^{(r)}$ corresponds to the vertex (r) of Fig. 9 with $r = 0,1,2,3$. This is illustrated in Fig. 11. We can further expand the general diagram corresponding to $V^{(0)}$, as illustrated in Fig. 12, into diagrams in which the δ propagator ends at a gluon-ghost-ghost vertex [Fig. 12(a)], a three-gluon vertex [Fig. 12(b)], a four-gluon vertex [Fig. 12(c)], or a gluon-line vertex [Fig. 12(d)]:

$$V^{(0)} = V_a^{(0)} + V_b^{(0)} + V_c^{(0)} + V_d^{(0)} . \tag{4.39}$$

Now, the Ward-Takahashi identity (Fig. 10) tells us that

$$V^{(1)} = -V_a^{(0)} , \tag{4.40a}$$

the equality of Fig. 9(2) tells us that

$$V^{(2)} = -V_b^{(0)} , \tag{4.40b}$$

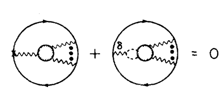


FIG. 10. Diagrammatic representation of the Ward-Takahashi identity of Eq. (2.22). The vertices on the left sides of the loops are given in Fig. 9(0) and (1).

and the equality of Fig. 9(3) tells us that

$$V^{(3)} = -V_c^{(0)} \ . \tag{4.40c}$$

Equation (4.38) thus becomes simply

$$V = V_d^{(0)} \ , \tag{4.41}$$

and it is clear from Fig. 12(d) that (4.33) is therefore given by

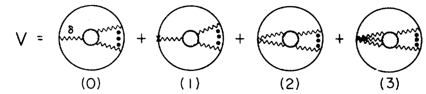


FIG. 11. Diagrammatic representation of the expansion of the functional $V \equiv \langle \mathrm{tr} \nabla \cdot \underline{F} \Phi \rangle_*$ corresponding to the expansion of $\nabla \cdot \underline{F}$ into powers of $\underline{A}$. The vertices on the left sides of the loops are given in Fig. 9.

$$\begin{aligned} V_\nu(C_x;x) &= -g_{\nu\mu} \oint_C dy^\mu \delta^D(x-y) \langle \mathrm{tr}[\underline{\lambda}^a \underline{\Phi}(C_{xy}) ig \underline{\lambda}^a \underline{\Phi}(C_{yx})] \rangle_* \\ &= -ig \oint_C dy^\nu \delta^D(x-y) \langle [\mathrm{tr}\underline{\Phi}(C_{xy})][\mathrm{tr}\underline{\Phi}(C_{yx})\rangle_* \ . \end{aligned} \tag{4.42}$$

This is precisely the dimensionally regularized form of the result (4.3) previously derived formally. If the gauge group were SU(N) instead of U(N), we would have also obtained the second term on the right-hand side of Eq. (4.1).

To complete the proof of the MM equations, it remains to show that

$$\partial^\mu \frac{\delta}{\delta \sigma^{\mu\nu}(x)} W(C) = V_\nu(C_x;x) \ . \tag{4.43}$$

Given our expressions (4.29) and (4.32) for derivatives of multiple line integrals and the equalities of Fig. 9, it is just a matter of combinatorics to verify (4.43). We will omit the simple but tedious details and simply outline the calculations.

Let G represent the contribution of the diagrams (written as multiple line integrals over the loop C and multiple D-dimensional integrals of a product of dimensionally regularized gluon and ghost propagators) for the nth term in the expansion of $\underline{\Phi}(C)$ to $W(C) = \langle (1/N) \mathrm{tr} \underline{\Phi}(C) \rangle_*$. We have

$$\frac{\delta}{\delta \sigma^{\mu\nu}(x)} G = G_{\mu\nu}^1(x) + G_{\mu\nu}^2(x) \ , \tag{4.44}$$

where $G_{\mu\nu}^r(x)$ is the contribution of the rth (antisymmetric) term in (4.29) for $r=1$ and 2. Next

$$\partial^\mu G_{\mu\nu}^r(x) = G_\nu^{r1}(x) + G_\nu^{r2}(x) \ , \tag{4.45}$$

where $G_\nu^{rs}(x)$ is the contribution of the sth term in (4.32) for $s=1$ and 2 and μ runs from 1 to D. $G_\nu^{11}(x)$ is given by putting one of the line vertices in G at the point x and taking the second derivative of the propagator attached to x with respect to x. $G_\nu^{12}(x)$ and $G_\nu^{21}(x)$ are given by putting the neighboring two-line vertices in G at x and taking the first derivative. $G_\nu^{22}(x)$ is given by putting the neighboring three-line vertices in G at x. We find that $G_\nu^{11}(x)$ is exactly the term in $V_\nu(C_x;x)$ linear in the A dependence of $\nabla \cdot F$, $G_\nu^{12}(x) + G_\nu^{21}(x)$ is exactly the term in $V_\nu(C_x;x)$ quadratic in the A dependence of $\nabla \cdot F$, and $G_\nu^{22}(x)$ is exactly the term in $V_\nu(C_x;x)$ cubic in the A dependence of $\nabla \cdot F$. That is how (4.43) comes about in perturbation theory. Notice that (4.43) is valid in any regularization scheme; we did not use specific properties of dimensional regularization to derive it.

The first MM equation (1.1) is thus verified in each order of dimensionally regularized perturbation theory. It has the surprisingly simple form

$$\partial_\nu \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W_1(C;g,\epsilon) = Ng^2 \oint_C dy^\mu \delta^{4-\epsilon}(x-y) \times W_2(C_{xy}, C_{yx}; g, \epsilon) \ . \tag{4.46}$$

The higher equation may be similarly verified. Let us emphasize the great advantage of dimensional regularization in our analysis. If we had used a different regularization, such as the Pauli-Villars regularization accompanied by higher derivative terms in the action, then the analog of (4.36) does not hold and/or there appear additional vertices which come from the modification of the action. This would spoil the above derivation and give rise to additional terms on the right-hand side of (4.46) which could not be represented in terms of loop functions.

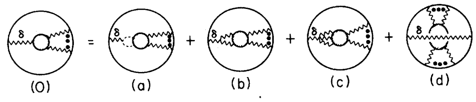


FIG. 12. Diagrammatic representation of the expansion of diagram $V^{(0)}$ of Fig. (11) corresponding to the various ways in which the δ propagator can couple in the blob.

E. Renormalization

In this subsection we will point out some problems one encounters in attempting to formulate a renormalized MM equation. We start with a smooth and simple loop C. In Sec. III B we saw that the corresponding dimensionally regularized loop function $W(C;g_R,\mu,\epsilon)$ is finite for $\epsilon\to 0$,

$$W(C;g_R,\mu,\epsilon) \underset{\epsilon\to 0}{\to} W(C;g_R,\mu,0) \equiv W_R(C;g_R,\mu)\ , \qquad (4.47)$$

and in Sec. IV D we saw that is satisfies the regularized MM equation

$$\mathscr{D}_x^\nu(\epsilon)W(C;g_R,\mu,\epsilon)=\alpha(g_R,\mu,\epsilon)I_x^\nu(C;\epsilon) \times W(C;g_R,\mu,\epsilon)\ , \qquad (4.48)$$

where

$$\mathscr{D}_x^\nu(\epsilon)=\sum_{\mu=1}^{4-\epsilon}\partial^\mu\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\ ,$$

$$\alpha(g_R,\mu,\epsilon)=N[g(g_R,\mu,\epsilon)]^2 \qquad (4.49)$$

$$=N[g_R Z_1(g_R,\epsilon)Z_3^{-3/2}(g_R,\epsilon)\mu^{\epsilon/2}]^2\ ,$$

$$I_x^\nu(C;\epsilon)=\oint_C dy^\nu\delta^{4-\epsilon}(x-y)\ .$$

Thus, if the $\epsilon\to 0$ limit commutes with the differential operator $\mathscr{D}_x^\nu$, the renormalized loop functional will satisfy the same MM equation (4.48) as the unrenormalized one satisfies:

$$\mathscr{D}_x^\nu W_R(C;g_R,\mu)=\alpha(g_R,\mu,0)I_x^\nu(C)W_R(C;g_R,\mu) \quad (C\text{ simple})\ . \qquad (4.50)$$

This is a remarkable result. In perturbation theory, where the unrenormalized coupling constant g is logarithmically divergent, it says that the operation of $\mathscr{D}_x^\nu$ on the finite function W_R leads to a divergent result. This should be compared with equations of the form

$$\Box_x G_R(x)=\delta^4(x)+g_R F_R(x) \qquad (4.51)$$

which relate conventional renormalized Green's functions G_R and F_R. Since it is the renormalized coupling constant which appears, (4.51) says that $\Box_x$ is a finite operation, in contrast to the behavior of $\mathscr{D}_x$ in (4.50). In the exact theory where $\alpha(g_R,\mu,0)=0$, Eq. (4.50) is even more surprising.

To see what is going on more clearly, let us attempt to explicitly calculate the area derivative $\delta/\delta\sigma$ of W_R. (We suppress the parameters g_R and μ and the indices μ and ν.) Referring to the loop $C\cdot\delta C$ of Fig. 5, we have

$$W_R(C\cdot\delta C_\gamma)=\lim_{\epsilon\to 0}\zeta(\gamma,\epsilon)W(C\cdot\delta C_\gamma;\epsilon)\ , \qquad (4.52)$$

where γ stands for the set of cusp angles $\{\gamma_i\}$ in $C\cdot\delta C_\gamma$ so that, according to (3.23) and the locality[7] of cusp renormalization,

$$\zeta(\gamma,\epsilon)=\prod_i Z(\gamma_i,\epsilon)\ . \qquad (4.53)$$

Thus

$$\frac{\delta}{\delta\sigma}W_R(C)=\lim_{\sigma\to 0}\frac{1}{\sigma}[W_R(C\cdot\delta C_\gamma)-W_R(C)]$$

$$=\lim_{\sigma\to 0}\lim_{\epsilon\to 0}\left\{\zeta(\gamma,\epsilon)\frac{1}{\sigma}[W(C\cdot\delta C_\gamma;\epsilon)-W(C;\epsilon)]+\frac{1}{\sigma}[\zeta(\gamma,\epsilon)-1]W(C;\epsilon)\right\}\ . \qquad (4.54)$$

If the $\sigma\to 0$ and $\epsilon\to 0$ limits can be interchanged, this becomes

$$\frac{\delta}{\delta\sigma}W_R(C)=\lim_{\epsilon\to 0}\left\{\lim_{\sigma\to 0}\zeta(\gamma,\epsilon)\cdot\frac{\delta}{\delta\sigma}W(C;\epsilon)+\lim_{\sigma\to 0}\left[\frac{\zeta(\gamma,\epsilon)-1}{\sigma}\right]\cdot W(C;\epsilon)\right\}\ . \qquad (4.55)$$

Here we have encountered a serious problem since we have no control over the limit of $\zeta(\gamma,\epsilon)$ or $[\zeta(\gamma,\epsilon)-1]\sigma^{-1}$ for $\sigma\to 0$. These expressions are, however, independent of x and so if the derivative $\partial/\partial x$ can be taken inside of the $\epsilon\to 0$ and $\sigma\to 0$ limits, we arrive at

$$\mathscr{D}_x W_R(C)=\left[\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\zeta(\gamma,\epsilon)\alpha(\epsilon)\right]I_x(C)W_R(C)\ . \qquad (4.56)$$

This result and its derivation are sufficiently ambiguous to cast doubt on the existence or usefulness of a renormalized MM equation. One can, of course, insist that the loop δC used in the definition of the area derivative is such that $C\cdot\delta C$ has no cusps. Then $\zeta(\gamma,\epsilon)\equiv 1$ and, assuming the limit interchanges are valid, we arrive back at (4.50). This procedure seems to us to be very artificial and so we have concluded that it is best to first solve (or approximately solve) the regularized unrenor-

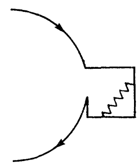


FIG. 13. Example of a diagram which does not contribute to the area derivative $\delta/\delta\sigma_{\mu\nu}$ defined by the diagrams of Fig. 6. It also does not contribute to the area derivative by the diagram of Fig. 5 if the gluon propagator is nonsingular at short distances.

malized equation (4.48) and then take the $\epsilon \to 0$ limit (4.47).

If the loop C has a cusp, there are additional problems. Consider such a loop C_γ with a cusp of angle γ at a point z but with no cross points. Then the renormalized loop function is

$$W_R(C_\gamma;\gamma,g_R,\mu)=\lim_{\epsilon\to 0} Z(\gamma,g_R,\epsilon) \times W(C_\gamma;g_R,\mu,\epsilon)\ , \qquad (4.57)$$

where the renormalization constant Z is given in (3.23). If $x\neq z$, then the loops $C_\gamma\cdot\delta C_{\mu\nu}$ used in the definition (4.4) of $[\delta/\delta\sigma_{\mu\nu}(x)]W_R$ have the same cusp angle γ as does C_γ, and so $W(C_\gamma\cdot\delta C_{\mu\nu})$ is renormalized by the same factor $Z(\gamma)$. (To circumvent the previous difficulties, we are assuming that δC does not introduce new cusps.) We therefore again obtain (4.50),

$$\mathscr{D}_x^\nu W_R(C_\gamma,g_R,\mu)=\alpha(g_R,\mu,0)I_x^\nu(C_\gamma)W_R(C_\gamma;\gamma,g_R,\mu)\quad (x\neq z)\ , \qquad (4.58)$$

and the same discussion applies. For $y=z$, $C_\gamma\cdot\delta C_{\mu\nu}$ can have a cusp angle $\gamma+\gamma'$ different from γ. Then

$$W_R(C_\gamma\cdot\delta C_{\mu\nu};\gamma+\gamma',g_R,\mu)=\lim_{\epsilon\to 0} Z(\gamma+\gamma',g_R,\epsilon)W(C_\gamma\cdot\delta C_{\mu\nu};g_R,\mu,\epsilon)\ , \qquad (4.59)$$

and so (suppressing irrelevent variables)

$$\frac{\delta}{\delta\sigma_{\mu\nu}}W_R(\gamma)=Z(\gamma)\frac{\delta}{\delta\sigma_{\mu\nu}}W(\gamma)+\lim_{|\delta\sigma_{\mu\nu}|\to 0}\left[\frac{Z(\gamma+\gamma')-Z(\gamma)}{|\delta\sigma_{\mu\nu}|}\right]\cdot W(\gamma)\ . \qquad (4.60)$$

Here again we have a big problem since the indicated limit is uncontrollable. $|\delta\sigma_{\mu\nu}|$ can approach zero with γ' remaining finite so that (4.60) does not exist. We could attempt to restrict $\delta C_{\mu\nu}$ such that the limit does exist, but this is unnatural. It is perhaps more reasonable to assume that the divergence $\partial/\partial x^\mu$ of the limit in (4.60) is zero. This is formally true, but is difficult to formulate rigorously. The safest approach is to use (4.58) only for $x\neq z$ and treat $x=z$ as a limit. Of course one must still contend with the previously encountered difficulties associated with (4.54).

We consider finally the case where the loop $C=C_a\cdot C_b$ has a single cross point z, where C_a and C_b are the two subloops which meet at z. Now $W^1(C)$ mixed with $W^2(C_a,C_b)$ by both renormalization,

$$W_R^i=Z^{ij}W^j\ ,$$

and by the MM equations,

$$\begin{aligned}\mathscr{D}_x W^1(C)&=\alpha I_x(C_a)W^1(C)\\&\quad+\alpha J_x(C_a;z)W^2(C_a,C_b)\ ,\\ \mathscr{D}_x W^2(C_a,C_b)&=\alpha I_x(C_a)W^2(C_a,C_b)\\&\quad+\frac{1}{N}\alpha J_x(C_a;z)W^1(C)\ ,\end{aligned} \qquad (4.61)$$

where formally

$$J_x^\mu(C;z)\equiv\lim_{\epsilon\to 0}\int_{\tau_1-\epsilon}^{\tau_1+\epsilon}d\tau\dot{y}^\mu(\tau)\delta^4(x-y(\tau))\ ,$$
$$z=y(\tau_1)\ , \qquad (4.62)$$

and we have taken $x\in C_a$ for definiteness. For $x\neq z$, if the previously discussed difficulties are circumvented, we have simply

$$\mathscr{D}_x W_R^i=\alpha I_x(C)W_R^i\quad (x\neq z) \qquad (4.63)$$

just as in (4.50). For $x=z$, the same problem as in (4.60) precludes us from reaching definitive conclusions, but at least formally, (4.63) then generalizes to

$$\mathscr{D}_x W_R^i=\alpha I_x(C)W_R^i+\alpha J_x(C;z)X^{ij}W_R^j\ , \qquad (4.64)$$

where

$$X^{ij}\equiv Z^{i1}(Z^{-1})^{2j}+\frac{1}{N}Z^{i2}(Z^{-1})^{1j} \qquad (4.65)$$

is the cusp angle dependent—divergent in perturbation theory—combination of renormalization constants.

The above considerations lead us to conclude that the renormalized loop functions do not satisfy useful functional differential equations. The need for assumptions about limit interchanges and for restrictions on contour deformations severely limits our confidence in such equations. It is clearly more sensible to first determine the regularized unrenormalized loop functions as solutions to the MM equations and then perform the renormalization.

F. Two dimensions

The pure Yang-Mills theory in two dimensions is trivial and the loop functions $W(C)$ are functions $F(A_1, \ldots, A_n)$ of only the areas A_i subtended by the subloops of C. For a simple loop C of area A, one has simply[25]

$$W(C)=F(A)=\exp(-\tfrac{1}{2}g^2A) \quad (C \text{ simple}) \tag{4.66}$$

but for more complicated loops, the A_i dependence is more complicated and difficult to determine by conventional methods,[26] even for $N\to\infty$. It is far simpler to determine the loop functions using the MM equation, as has been shown by Kasakov and Kostov (KK).[27] We will briefly review this work here, partly for illustration and comparison with the four-dimensional case, and partly because we can simplify the KK treatment.

Using the definition

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}W(C)\equiv \lim_{\delta\sigma_{\mu\nu}\to 0}\frac{1}{\delta\sigma_{\mu\nu}}[W(C\cdot\delta C_{\mu\nu})-W(C)] \tag{4.67}$$

of area derivative, where $\delta\sigma_{\mu\nu}$ is the oriented area of $\delta C_{\mu\nu}$ (so that, e.g., $\delta\sigma_{12}>0$ if $\delta C_{\mu\nu}$ is clockwise oriented), KK show that

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}W(C)=\left\langle \operatorname{tr}P\left[\underline{F}_{\mu\nu}(x)\exp\left[ig\oint_{C_{xx}}\underline{A}\cdot dy\right]\right]\right\rangle-\tfrac{1}{2}g^2n_{\mu\nu}(x)W(C)\,, \tag{4.68}$$

where $n_{\mu\nu}(x)$ is the orientation tensor of $\delta C_{\mu\nu}(x)$:

$$|\,\delta\sigma(x)\,|=\tfrac{1}{2}n_{\mu\nu}(x)\delta\sigma_{\mu\nu}(x)\,. \tag{4.69}$$

Note that the second term in (4.68) is not present in the regularized (or cutoff) four-dimensional theory.[28] At this point, KK wanted to avoid using C's which added new self-intersections and so demanded that $\delta C(x)$ changed orientation as x crossed through C. Then the term $-\frac{1}{2}g^2\partial^\mu n_{\mu\nu}(x)W(C)$ in the divergence of (4.68) exactly cancels the term in (4.3) (with $\delta^4\to\delta^2$) arising from the trivial coincidence of x and y. We, on the contrary, find it simpler and more natural to demand that $\delta C(x)$ does not change orientation as x crosses through C, so that $\partial^\mu n_{\mu\nu}(x)=0$. We thus obtain in the $N\to\infty$ limit[25]

$$\partial^\mu\frac{\delta}{\delta\sigma_{\mu\nu}(x)}W(C)=g^2\oint dy_\nu\delta^2(x-y)W(C_{xy})\times W(C_{yx})\,, \tag{4.70}$$

wherein the line integral receives contributions both from trivial and nontrivial coincidences $x=y$. This actually makes it easier to solve (4.70) since it immediately yields (4.66) for a simple loop, whereas the KK equation reads $0=0$ for a simple loop and then (4.66) can only be obtained by considering a simple loop as a limit of nonsimple loops.

We will illustrate the solution of (4.70) for the three loops of Fig. 14. We assume that the corresponding loop functions are functions of the areas:

$$W(C_1)=F(A)\,,\quad W(C_2)=F(A+A')\,,\quad W(C_3)=F(A,A')\,, \tag{4.71}$$

with the boundary conditions

$$F(0)=1\,,\quad F(A,0)=F(A)\,, \tag{4.72}$$

and show that this gives a solution with determined F's. For the simple loop C_1, we use (4.70) to relate the $\delta\sigma\to 0$ limits of the two loops shown in Fig. 15, and obtain

$$-2F'(A)=g^2F(A)\,, \tag{4.73}$$

whose solution [subject to (4.72)] is (4.66). For the loop C_3, we use (4.70) to relate the $\delta\sigma\to 0$ limits of the two loops shown in Fig. 16, and obtain

$$\frac{\partial}{\partial A'}F(A,A')-\frac{\partial}{\partial A}F(A,A')=g^2F(A)F(A')\,. \tag{4.74}$$

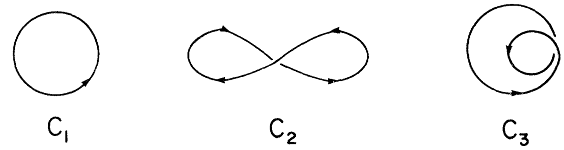
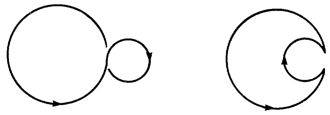


FIG. 14. Three loops. The loop functions are $W(C_1)=F(A)$, $W(C_2)=F(A+A')$, $W(C_3)=F(A,A')$, in terms of the areas of the subloops.



(a) (b)

FIG. 15. (a) The loop $C_1 \cdot \delta C_{xx}$ for x just outside of C_1. The loop function is $F(A+\delta\sigma)$. (b) The loop $C_1 \cdot \delta C_{xx}$ for x just inside of C_1. The loop function is $F(A-\delta\sigma)$.

[There is no $F(A,A')$ contribution on the right-hand side since we chose $\nu=2$ in (4.70) in the coordinate system shown in Fig. 16. The $F(A,A')$ contribution is purely $\nu=1$, whereas the $F(A)F(A')$ contribution is both $\nu=1$ and $\nu=2$.] The solution to (4.74) subject to (4.72) is

$$F(A,A')=(1-g^2A')\exp[-\tfrac{1}{2}g^2(A+A')] \; . \tag{4.75}$$

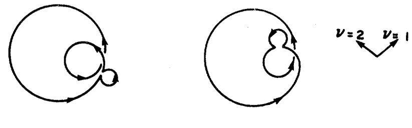
As in KK, we can similarly determine the solution for an arbitrary loop C.

V. THE MANDELSTAM CONSTRAINTS

The Mandelstam constraints are the loop-space analogs of the choice of the gauge group G. They guarantee classically that the loop functions involve traces of $N \times N$ matrices. Their general form, given by Giles,[5] follows from the identity

$$0=\sum_{\pi \in S_{N+1}} (-1)^{p_\pi} \delta_{n_1 m_{\pi_1}} \cdots \delta_{n_{N+1} m_{\pi(N+1)}} \; , \tag{5.1}$$

where the indices n_i, m_i range from 1 to N and the



(a) (b)

FIG. 16. (a) The loop $C_3 \cdot \delta C_{xx}$ for x just outside of C_3 and just below the cross point. The loop function is $F(A+\delta\sigma, A')$. (b) The loop $C_3 \cdot \delta C_{xx}$ for x just inside of C_3 above the cross point. The loop function is $F(A, A'+\delta\sigma)$. In the indicated coordinate system, the $\nu=2$ direction is one of the tangents to the loop at the cross point, and the $\nu=1$ direction is perpendicular to $\nu=2$.

sum is taken over all permutations π in the symmetric group S_{N+1} of order $N+1$ and $(-1)^{p_\pi}$ is the parity of π. The constraints read[29]

$$0=\sum_{\pi \in S_{N+1}} (-1)^{p_\pi} N^{c_\pi} \times W_{c_\pi}(C_{n_{11}} \cdots C_{n_1 s_1}, \ldots, C_{n_{c_\pi 1}} \cdots C_{n_{c_\pi s_{c_\pi}}}) \; , \tag{5.2}$$

where

$$\pi=(n_{11}, \ldots, n_{1s_1}) \cdots (n_{c_\pi 1}, \ldots, n_{c_\pi s_{c_\pi}})$$

is the cyclic decomposition of π, with c_π the number of cycles. Giles has proven that, in the classical case where $W_n(C_1, \ldots, C_n) = W_1(C_1) \cdots W_1(C_n)$, the constraints (1.12) and (5.2) are sufficient to allow reconstruction of the $N \times N$ matrices $\underline{\Phi}$ and hence $\underline{A}$ (up to gauge transformations).

We can formally extend Gile's proof to quantum field theory by use of the representation

$$W_n(C_1, \ldots, C_n)=\int_{\hat{\mathscr{G}}} dW\, e^{-S(W)} \times W(C_1) \cdots W(C_n) \; , \tag{5.3}$$

where $W(C) \equiv tr\underline{\Phi}(C)$ are the characters of the representations $C \to \underline{\Phi}(C)$ of the loop group $\mathscr{G}$ (see Sec. I), $S(W)$ is the classical action expressed as a function of W, and the integration is over the set $\hat{\mathscr{G}}$ of characters of representations of $\mathscr{G}$. This representation is formal in that the existence of a measure on $\hat{\mathscr{G}}$ has not been established.[30] We thus have a suggestion, but not a proof, that the Mandelstam constraints are sufficient to determine $\underline{A}$ [i.e., the Green's functions (3.4)] up to gauge transformations in quantum field theory.

It is remarkable that the sets of loops which are related in (5.2) are precisely the sets of loops which mix upon renormalization. [They are also the sets of loops which are related by the MM equations, but in this section neither this fact, nor the equations themselves, will be used.] This will enable us to give a simple renormalized version of the constraints. We may use the summation convention and suppress all irrelevant variables and write the constraint (5.2) as

$$a^i W^i = 0 \; , \tag{5.4}$$

and the renormalization relation (3.47) as

$$W_R^i = Z^{ij} W^j \; , \tag{5.5}$$

with i and j ranging from 1 to $I_N \equiv (N+1)!$, the number of elements in S_{N+1}.

It is first of all clear that the precise form of the renormalized constraints must depend on certain details of the renormalization conventions. Thus, if a given renormalized set $\{W_R^i\}$ of functions satisfies a constraint

$$a^i W_R^i = 0 , \tag{5.6}$$

then the equivalent functions

$$\overline{W}_R^i \equiv (K)^{ij} W_R^j = (K)^{ij} (Z)^{jk} W^k , \tag{5.7}$$

where $\{K^{ij}\}$ is an arbitrary finite invertible matrix, will satisfy the different constraint

$$\overline{a}^i \overline{W}_R^i \equiv a^j (K^{-1})^{ji} \overline{W}_R^i = 0 . \tag{5.8}$$

The most we can expect is therefore that the renormalization matrix Z^{ij} can be chosen such that (5.6) is satisfied.

To avoid unnecessary complications, we will always choose

$$W_R^i = W^i = 1, \text{ i.e., } Z^{ij} = \delta^{ij} \quad (g=0) \tag{5.9}$$

in the lowest order of perturbation theory. This is not necessary. We could always take $Z^{ij} = K^{ij}$, an arbitrary invertible finite matrix or even $K^{ij} = c^i a^j + \cdots$ with c^i divergent. However, (5.9) is the simplest and most natural choice. In some sense it associates W_R^i with the ith loop set in all orders. Relatedly, it also gives us $a^i W_R^i = 0$ in lowest order.

Let us now ask if the Z^{ij} can be chosen such that $a^i W_R^i = 0$ in all orders of perturbation theory; e.g., $a^i Z^{ij} \propto a^j$. It is easy to see that this condition need not be satisfied in general. With (we suppress irrelevant variables)

$$\begin{aligned} Z^{ij}(\epsilon) = \delta^{ij} &+ g_R{}^2(\epsilon^{-1}\zeta_{11}^{ij} + \zeta_{10}^{ij}) \\ &+ g_R{}^4(\epsilon^{-2}\zeta_{22}^{ij} + \epsilon^{-1}\zeta_{21}^{ij} + \zeta_{20}^{ij}) \\ &+ O(g_R{}^6) \end{aligned} \tag{5.10}$$

and[31]

$$\begin{aligned} W^j(C;\epsilon) = 1^j &+ g_R{}^2\epsilon^{-1}\rho_{11}^j + R_1^j(C;\epsilon) \\ &+ g_R{}^4\epsilon^{-2}\rho_{22}^j + \epsilon^{-1}\rho_{21}^j + R_2^j(C;\epsilon) \\ &+ O(g_R{}^4) , \end{aligned} \tag{5.11}$$

where $R_n^j(C;\epsilon)$ are regular at $\epsilon = 0$, we obtain the finiteness relations

$$\begin{aligned} &\underline{\zeta}_{11}\cdot 1 + \rho_{11} = 0 , \\ &\underline{\zeta}_{22}\cdot 1 + \underline{\zeta}_{11}\cdot\rho_{11} + \rho_{22} = \underline{\zeta}_{21}\cdot 1 + \underline{\zeta}_{11}\cdot R_1(C;0) \\ &\qquad\qquad + \underline{\zeta}_{10}\cdot\rho_{11} + \rho_{21} = 0 , \end{aligned} \tag{5.12}$$

and the expression

$$\begin{aligned} W_R^i(C) = 1^i &+ g_R{}^2[\zeta_{10}^{ij} 1^j + R_1^i(C;0)] \\ &+ g_R{}^4[\zeta_{11}^{ij} R_{11}^j(C;0) + \zeta_{10}^{ij} R_1^j(C;0) \\ &\quad + \zeta_{20}^{ij} 1^j + R_2^i(C;0) + O(g_R{}^6)] , \end{aligned} \tag{5.13}$$

where

$$R_{11}^j(C;0) = \frac{\partial}{\partial\epsilon} R_1^j(C;\epsilon) \bigg|_{\epsilon=0} . \tag{5.14}$$

The dimensionally regularized unrenormalized function (5.11) will satisfy (5.4), so that

$$\begin{aligned} 0 &= a^j 1^j = a^j \rho_{11}^j = a^j R_1^j(C;\epsilon) = a^j \rho_{22}^j = a^j \rho_{21}^j \\ &= a^j \rho_{20}^j = a^j R_2^j(C;\epsilon) , \end{aligned} \tag{5.15}$$

but this does not imply that $a^i W_R^i = 0$ except in lowest order:

$$\begin{aligned} a^i W_R^i(C) = g_R{}^2 a^i \zeta_{10}^{ij} 1^j & \\ + g_R{}^4[a^i\zeta_{11}^{ij} R_{11}^j(C;0) &+ a^i \zeta_{10}^{ij} R_1^j(C;0) \\ &+ a^i \zeta_{20}^{ij} 1^j] + O(g_R{}^6) . \end{aligned} \tag{5.16}$$

Of course, we can try to choose the ζ_{rs}^{ij} such that (5.16) vanishes, e.g., $a^i \zeta_{rs}^{ij} \propto a^j$, but it is not clear that this is consistent with the finiteness of the W_R^i.

Another, related, question of interest is whether the $I_N \times I_N$ components of Z^{ij} can be uniquely specified by imposing normalization conditions, e.g., of the form

$$W_R^i(\overline{C}^k) \equiv W_{Rn_i}(\overline{C}_1^{ik}, \ldots, \overline{C}_{n_i}^{ik}) = f^{ik} , \tag{5.17}$$

on the renormalized loop functions at I_N specified sets $\{\overline{C}_1^{ik}, \ldots, \overline{C}_{n_i}^{ik}\}$ of subtraction loops. Given (5.5), the general answer is no since the $I_N \times I_N$ matrix $W^i(\overline{C}^k)$ is not invertible because of its zero eigenvalue implied by (5.4). In other words, if Z^{ij} is such that (5.5) is finite and (5.17) is satisfied, then so is

$$Z'^{ij} = Z^{ij} + c^i a^j . \tag{5.18}$$

But we can still ask if some of the components of Z^{ij} can be fixed by normalization conditions.

In order to answer the above questions, it is convenient to perform the linear coordinate transformation

$$\begin{aligned} &\widetilde{W}^i = T^{ij} W^j, \quad \widetilde{W}_R^i = T^{ij} W_R^j , \\ &\widetilde{Z}^{ij} = T^{ik} Z^{kl} (T^{-1})^{lj} , \end{aligned} \tag{5.19}$$

where T^{ij} is any (and there are clearly many) *invertible* matrix such that

$$T^{1j}=a^j \quad (j=1,\ldots,I_N)\ . \qquad (5.20)$$

Then the renormalization relation (5.5) becomes

$$\widetilde{W}_R^i=\widetilde{Z}^{ij}\widetilde{W}^j \qquad (5.21)$$

and the Mandelstam constraint (5.4) becomes simply

$$\widetilde{W}^1=0\ . \qquad (5.22)$$

In the original basis the loops were diagonal in the sense that each W^i corresponds to a specific loop set, whereas in the new basis the constraint is diagonal. The zero coupling conditions (5.9) become

$$\widetilde{W}_R^i=\widetilde{W}^i=T^{ij}1^j,\ \ \widetilde{Z}^{ij}=\delta^{ij}\ \ (g=0)\ . \qquad (5.23)$$

It follows from (5.22) that the I_N matrix elements $\widetilde{Z}^{i1}$ are completely arbitrary, and the I_N-1 matrix elements $\widetilde{Z}^{1j}$ $(j\neq 1)$ may be chosen to vanish:

$$\widetilde{Z}^{i1}\ \text{arbitrary},\ \ \widetilde{Z}^{1j}=0\ ;$$
$$i=1,\ldots,I_N,\ \ j=2,\ldots,I_N\ . \qquad (5.24)$$

The arbitrariness of $\widetilde{Z}^{il}$ in the new basis corresponds to the arbitrariness (5.18) in the old basis. We may use this arbitrariness to choose

$$\widetilde{Z}^{11}=1\ , \qquad (5.25)$$

consistently with (5.23). The choice $\widetilde{Z}^{1j}=0$ $(j\neq 1)$ implies that the renormalized loop functions $\widetilde{W}_R^i$ satisfy

$$\widetilde{W}_R^1=0\ , \qquad (5.26)$$

just like the unrenormalized constraint (5.22). In the original basis, (5.26) reads

$$a^iW_R^i=0\ ; \qquad (5.27)$$

and (5.24), with (5.25), implies that

$$a^iZ^{ik}=a^k\ . \qquad (5.28)$$

Of course, we could impose (5.28) in the original basis and deduce (5.27); but it would then not be obvious that (5.28) is consistent with the finiteness of the $\widetilde{W}_R^i$. The equivalent choice (5.24) is, on the other hand, clearly consistent with the finiteness of $\widetilde{W}_R^i$. Indeed, if $\widetilde{Z}_1^{ij}$ is such that $\widetilde{Z}_1^{ij}\widetilde{W}^j$ is finite, then the definition

$$\widetilde{Z}^{ij}=\begin{cases}0, & i=1,\ j\neq 1\ ,\\ \widetilde{Z}_1^{ij}, & \text{otherwise}\end{cases} \qquad (5.29)$$

is obviously such that $\widetilde{Z}^{ij}\widetilde{W}^j$ is finite: $\widetilde{Z}^{ij}\widetilde{W}^j=\widetilde{Z}_1^{ij}\widetilde{W}^j$ for $i\neq 1$ and $\widetilde{Z}^{1j}\widetilde{W}^j=0$. Note that it is crucial here that $\widetilde{W}^1=0$. Otherwise we would have to take also $\widetilde{Z}^{11}=0$ to get $\widetilde{W}_R^1=0$, and that would render $\widetilde{Z}^{ij}$ to be not invertible and therefore unacceptable.

We could now attempt to specify the remaining $(I_N-1)\cdot(I_N-1)$ matrix elements $\widetilde{Z}^{ij}$ $(i,j\neq 1)$ by normalization conditions such as

$$\widetilde{W}_R^i(\bar{C}^k)=\tilde{f}^{ik}\quad (i,k=2,\ldots,I_N)\ . \qquad (5.30)$$

The problem with this is that it will still not uniquely specify $\widetilde{Z}^{ik}$ because the zero-order contribution $T^{ij}1^j$ to $\widetilde{W}^i$ is not invertible. We could eliminate this contribution, e.g., by using normalization conditions of the form

$$\left[L\frac{\partial}{\partial L}\right]^k\widetilde{W}_R(\bar{C}_L^i)=\tilde{f}^{ik}\quad (i,k=2,\ldots,I_N)\ , \qquad (5.31)$$

where $\bar{C}_L^i$ is a fixed loop set of type i and length L. This would lead to a unique expression

$$\widetilde{Z}^{ik}(\epsilon)=\tilde{f}^{ij}[\widetilde{W}^{-1}(\epsilon)]^{jk}\quad (i,j,k=2,\ldots,I_N) \qquad (5.32)$$

for Z if we assume that the matrix

$$\widetilde{W}^{ik}(\epsilon)\equiv\left[L\frac{\partial}{\partial L}\right]^k\widetilde{W}(C_L^i;\epsilon)\quad (i,k=2,\ldots,I_N) \qquad (5.33)$$

is invertible, but (5.32) is unsatisfactory since the $(n+1)$th order $\widetilde{W}^{ik}$ is necessary to determine the nth order $\widetilde{Z}^{ik}$. Furthermore, the expression (5.32) is of the complicated form (5.10) plus an infinite power series in ϵ. We conclude, therefore, that it is best to specify the remaining matrix elements of $\widetilde{Z}$ by direct conditions, such as the minimal-subtraction statement (3.49), rather than by conditions on $\widetilde{W}_R^i$.

To summarize, we have shown that the $I_N\cdot I_N$ components of Z^{ij} can be chosen such that the I_N conditions

$$a^iZ^{ij}=a^j \qquad (5.34)$$

are satisfied, and I_N-1 conditions are arbitrary so that Z^{ij} and $Z^{ij}+c^ia^j$ are equivalent. [Only I_N-1 components of c^i are arbitrary since (5.34) requires that $a^ic^i=0$.] The remaining $(I_N-1)\cdot(I_N-1)$ components are best determined by a specified renormalization prescription such as the minimal one.

There is another, instructive, way to interpret the above analysis. According to Ref. 7, the I_N unrenormalized function W^i can be independently renormalized, in the matrix multiplicative form (5.5), without consideration of the constraint (5.4). Then the I_N renormalized functions W_R^i need not satisfy (5.4). Alternatively, we need only renormalize I_N-1 of the W^i, say $W^2, \ldots, W^{I_N}$:

$$W_R^i = \tilde{Z}^{ij} W^j, \quad i,j=2,\ldots,I_N , \tag{5.35}$$

and define

$$W_R^1 \equiv -\frac{1}{a_1}\sum_{i=2}^{I} a^i W_R^i , \tag{5.36}$$

analogously to

$$W^1 = -\frac{1}{a_1}\sum_{i=2}^{I} a^i W' . \tag{5.37}$$

The W^j for $j=2,\ldots,I_N$ are independent [assuming that (5.4) is the only linear constraint], so that $\tilde{Z}^{ij}$ in (5.35) has no arbitrary components, and by construction $a^i W_R^i = 0$. If we now define

$$\tilde{Z}^{j1} = 0$$

and (5.38)

$$\tilde{Z}^{1j} = -\frac{1}{a_1}\sum_{i=2}^{I_N} a^i \tilde{Z}^{ij} \quad (j=1,\ldots,I_N) ,$$

then

$$W_R^i = \tilde{Z}^{ij} W^j \quad (i,j=1,\ldots,I_N) \tag{5.39}$$

and

$$a^i \tilde{Z}^{ij} = 0 . \tag{5.40}$$

We see that the $I_N \cdot I_N$ $\tilde{Z}$ is not invertible, although the $(I_N-1)\cdot(I_N-1)$ $\tilde{Z}$ is invertible. But the $I_N \cdot I_N$ $\tilde{Z}$ has the arbitrariness (5.19) and this may be used to make it invertible and, in fact, equal to the preceding Z:

$$Z^{ij} = \tilde{Z}^{ij} + \delta^{i1} a^j / a^1 . \tag{5.41}$$

As a final remark, let us comment on the large-N limit of the Mandelstam constraints. As N increases, the number of terms in the constraint equations increases rapidly, and the degree of line crossing at the cross point also increases. It is not clear if the constraints have a nontrivial large-N limit. The simplest possibility is of course that the infinite-N theory is unconstrained. This is plausible since it should be sufficient to consider only loops with a finite degree of crossing, but we have no formal proof of this.

VI. OBSERVABLES

In this section we will illustrate how observable matrix elements of physical currents may be expressed as functional integrals over loop functions. For simplicity we consider first the case in which the color singlet physical currents J_μ^r, which carry flavor (index r), arise from covariantly coupled scalar fields ϕ_α^i which carry both color (index α) and flavor (index i):

$$J_\mu^r = i \phi_\alpha^{\dagger i} \overleftrightarrow{D}_\mu^{\alpha\beta} t_{ij}^r \phi_\beta^j . \tag{6.1}$$

Here t_{ij}^r are the flavor matrices and

$$D_\mu^{\alpha\beta} = \delta^{\alpha\beta}\partial_\mu - igA_\mu^{\alpha\beta} \tag{6.2}$$

is the appropriate color covariant derivative matrix. The multicurrent Green's functions can be expressed as

$$\langle 0 | T^*[J_{\mu_1}^{r_1}(x_1)\cdots J_{\mu_n}^{r_n}(x_n)] | 0 \rangle = g^{-n} e^{-iU(a)} \left[i\frac{\delta}{\delta a_{\mu_1}^{r_1}(x_1)} \right] \cdots \left[i\frac{\delta}{\delta a_{\mu_n}^{r_n}(x_n)} \right] e^{iU(a)} \Bigg|_{a=0} , \tag{6.3}$$

in terms of the generating functional

$$e^{iU(a)} \equiv \int dA\, dC\, d\bar{C}\, d\phi\, d\phi^\dagger \exp\left[i \left[\mathscr{S}(A,C,\bar{C}) + \int d^4x [\, |(D_\mu - iga_\mu^r t^r)\phi|^2 - im^2 |\phi|^2] \right] \right] , \tag{6.4}$$

where $\mathscr{S}$ is the pure Yang-Mills action (2.4) and $a_\mu^r(x)$ are external flavor vector fields.

The Gaussian scalar field integrations may be performed to give

$$e^{iU(a)} = \int dA\, dC\, d\bar{C}\, e^{i\mathscr{S}} \det{}^{-1}[(D_\mu - iga_\mu^a t^a)^2 + m^2] , \tag{6.5}$$

in which $D_\mu - iga_\mu^a t^a$ is a matrix with respect to both color and flavor. Following standard methods, we obtain the representation

$$\det{}^{-1}[\ ]=\exp\left[\int_0^\infty \frac{d\tau}{\tau}e^{-m^2\tau/2}\int d\Gamma_\tau f(\Gamma_\tau)\mathrm{Tr}\Phi(A;\Gamma_\tau)\mathrm{tr}\Phi(a\cdot t;\Gamma_\tau)\right], \tag{6.6}$$

where the functional integral is taken over all loops Γ_τ of parameter length τ,

$$\Gamma_\tau:\ \tau'\to z(\tau'),\ \ z(0)=z(\tau), \tag{6.7}$$

and

$$f(\Gamma_\tau)\equiv\exp\left[-\frac{i}{2}\int_0^\tau d\tau'\dot{z}^2(\tau')\right], \tag{6.8}$$

and

$$\Phi(A;\Gamma)\equiv P\exp\left[ig\int_\Gamma dz^\mu A_\mu(z)\right] \tag{6.9}$$

are the classical path-order phase-factor matrices. The trace Tr is over color indices and tr is over flavor indices. The proof of (6.6) is essentially the same as the corresponding one[32] in the Abelian case.

We thus obtain

$$e^{iU(a)}=\int dA\,dC\,d\bar{C}\exp\left[i\left[\mathscr{S}(A,C,\bar{C})+\int\frac{d\tau}{\tau}e^{-im^2\tau/2}\int d\Gamma_\tau f(\Gamma_\tau)\mathrm{Tr}\Phi(A;\Gamma_\tau)\mathrm{tr}\Phi(a\cdot t;\Gamma_\tau)\right]\right]. \tag{6.10}$$

Expansion of $\exp[i(\cdots)]$ now results in an expression in terms of the loop functions (1.4):

$$e^{iU(a)}=\sum_{n=0}^\infty\frac{1}{n!}\prod_{i=1}^n\left[\int\frac{d\tau_i}{\tau_i}e^{-im^2\tau_i/2}\int d\Gamma_{\tau_i}f(\Gamma_{\tau_i})N\,\mathrm{tr}\Phi(a\cdot t;\Gamma_{\tau_i})\right]W_n(\Gamma_{\tau_1},\ldots,\Gamma_{\tau_n}). \tag{6.11}$$

In the large-N limit the loop functions W_n factorize and we can resume the series to obtain the remarkably simple result

$$iU(a)\underset{N\to\infty}{\sim}N\int\frac{d\tau}{\tau}e^{-im^2\tau/2}\int d\Gamma_\tau f(\Gamma_\tau)[\mathrm{tr}\Phi(a\cdot t;\Gamma_\tau)]W(\Gamma_\tau). \tag{6.12}$$

We consider next the realistic case in which, instead of (6.1), the color singlet flavor currents are bilinear in spin-$\frac{1}{2}$ (quark) colored and flavored fields:

$$J_\mu^r=\bar{\psi}\lambda^r\gamma_\mu\psi. \tag{6.13}$$

The generating functional satisfying (6.3) which replaces (6.4) is then given by

$$e^{iU(a)}\equiv\int dA\,dC\,d\bar{C}\,d\psi\,d\bar{\psi}\exp\left[i\left[\mathscr{S}(A,C,\bar{C})+i\int d^4x[\bar{\psi}(\not{D}-ig\not{a}^r\lambda^r-m)\psi]\right]\right], \tag{6.14}$$

where

$$D_\mu=\partial_\mu-igA_\mu^aT^a,\ \ \not{D}\equiv\gamma^\mu D_\mu, \tag{6.15}$$

is the appropriate color covariant derivative matrix.

We find that the representations (6.10)–(6.12) remain formally valid in this case, with

$$f(\Gamma_\tau)\equiv-\int d\Pi_\tau P\exp\left[i\int_0^\tau d\tau'[\not{\pi}(\tau')-\pi(\tau')\cdot\dot{z}(\tau')]\right], \tag{6.16}$$

instead of (6.8). Here the functional integration $d\Pi_\tau$ is taken over all (momentum space paths)

$$\Pi_\tau:\ \tau'\to\pi_\mu(\tau'),\ \ 0\le\tau'\le\tau, \tag{6.17}$$

and the path ordering P is with respect to the γ matrices in $\not{\pi}\equiv\gamma^\mu\pi_\mu$.

The proof of these results is essentially the same as the proof of the Abelian analog given in Ref. 33. The logarithm of the fermionic determinant which results when the integrations over the quark fields are performed in (6.14) can be written

$$\mathrm{Tr}\ln(\not{D}-iga^r\lambda^r)=-\int d^4x\int_0^\infty \frac{d\tau}{\tau}\langle x\mid \mathrm{tr}\exp[-\tau(\not{D}-iga^r\lambda^r)]\mid x\rangle\ , \tag{6.18}$$

where Tr is the complete trace and tr is the color-flavor-spin trace. Now, according to the Trotter formula,

$$\begin{aligned}\langle x\mid \exp[-\tau(\not{D}-iga^r\lambda^r)]\mid y\rangle\\ =\lim_{n\to\infty}\int d^4x_1\cdots\int d^4x_{n-1}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_n}{(2\pi)^4}\exp\left[i\Delta g(\not{A}\cdot T+\not{a}\cdot\lambda)\left[\frac{x+x_{n-1}}{2}\right]\right]\\ \times\exp[-ip_n\cdot(x-x_{n-1})]\exp[i\Delta\not{p}_n]\\ \times\cdots\times\exp\left[i\Delta g(\not{A}\cdot T+\not{a}\cdot\lambda)\left[\frac{x_1+y}{2}\right]\right]\\ \times\exp[-ip_1\cdot(x_1-y)]\exp[i\Delta\not{p}_1]\ ,\end{aligned} \tag{6.19}$$

where $\Delta\equiv\tau/n$. We next formally change integration variables from p_i to $\pi_i\equiv p_i+(A\cdot T+a\cdot\lambda)(\bar{x}_i)$ where $\bar{x}_i=(x_i+x_{i-1})/2$. The integrand becomes

$$\begin{aligned}\exp(i\Delta\not{\pi})\exp[-i\pi_n\cdot(x-x_{n-1})]\exp\left[ig(x-x_{n-1})\cdot(A\cdot T+a\cdot\lambda)\left[\frac{x+x_{n-1}}{2}\right]\right]\\ \times\cdots\times\exp(i\Delta\not{\pi}_1)\exp[-i\pi_1\cdot(x_1-y)]\exp\left[ig(x_1-y)\cdot(A\cdot T+a\cdot\lambda)\left[\frac{x_1+y}{2}\right]\right]\ .\end{aligned} \tag{6.20}$$

Although in general the meaning of this transformation is unclear since p_i^μ is a number while $(A^\mu\cdot T+a^\mu\cdot\lambda)$ is a matrix, we can prove that the integral of (6.20) actually coincides with (6.19) to order $\Delta=\tau/n$. This can be seen by explicitly performing the p integration in each expression. For (6.19) we get

$$\begin{aligned}\int\frac{d^4p_i}{(2\pi)^4}e^{-ip_i\cdot(x_i-x_{i-1})}e^{i\Delta(\not{A}\cdot T+\not{a}\lambda)(\bar{x}_i)}e^{i\Delta\not{p}_i}\\ =\int\frac{d^4p_i}{(2\pi)^4}e^{-ip_i\cdot(x_i-x_{i-1})}\{1+i\Delta[(A^\mu\cdot T+a^\mu\cdot\lambda(\bar{x}_i)+p_i^\mu]\gamma_\mu+O(\Delta^2)\}\\ =\{1+i\Delta\gamma_\mu[(A^\mu\cdot T+a^\mu\cdot\lambda)(\bar{x}_i)-\partial^\mu_{x_i}]\}\delta^4(x_i-x_{i-1})+O(\Delta^2)\ ,\end{aligned} \tag{6.21}$$

and for (6.20) we get

$$\begin{aligned}\int\frac{d^4p_i}{(2\pi)^4}e^{-ip_i\cdot(x_i-x_{i-1})}e^{i\Delta\not{p}_i}e^{i(x_i-x_{i-1})(A\cdot T+a\cdot\lambda)(\bar{x}_i)}\\ =\int\frac{d^4p_i}{(2\pi)^4}e^{-ip_i\cdot(x_i-x_{i-1})}[1+i\Delta p_i^\mu\gamma_\mu+O(\Delta^2)]e^{i(x_i-x_{i-1})(A\cdot T+a\cdot\lambda)(\bar{x}_i)}\\ =\{\delta^4(x_i-x_{i-1})-\Delta\gamma_\mu[\partial^\mu_{x_i}\delta^4(x_i-x_{i-1})]+O(\Delta^2)\}e^{i(x_i-x_{i-1})(A\cdot T+a\cdot\lambda)(\bar{x}_i)}\ .\end{aligned} \tag{6.22}$$

If we now use

$$[\partial^\mu\delta^4(x)]f(x)=\partial^\mu\delta^4(x)f(0)-\delta^4(x)[\partial^\mu f(x)]_{x=0}\ , \tag{6.23}$$

we confirm the equality of (6.21) and (6.22). We are thus permitted to use (6.20) in (6.19). This is important since the matrices $\not{p}$, $\not{A}\cdot T$, and $\not{a}\cdot\lambda$ in (6.19) do not commute with each other, whereas the matrices $\not{p}$, $A\cdot T$, and $a\cdot\lambda$ in (6.20) all commute.

We thus arrive at the representation

$$\langle x\,|\exp[-\tau(\nabla\!\!\!\!/ - ig a\cdot\lambda)]\,|\,y\rangle=\int d\Gamma_\tau(x,y)F(\Gamma_\tau(x,y))P_c\exp\left[ig\int T\cdot A\cdot dz\right]P_f\exp\left[ig\int \lambda\cdot a\cdot dz\right], \quad (6.24)$$

where the functional integration is taken over all paths,

$$\Gamma_\tau(x,y):\ \tau'\to z(\tau');\ \ 0\le\tau'\le\tau;\ \ z(0)=x,\ \ z(\tau)=y \quad (6.25)$$

and

$$F(\Gamma_\tau)\equiv\int d\Pi_\tau P_s\exp\{i\int_0^\tau d\tau'[\pi\!\!\!/(\tau')-\pi(\tau')\cdot\dot z(\tau')]\}\ . \quad (6.26)$$

In these expressions, P_c denotes color path ordering, P_f denotes flavor path ordering, and P_s denotes spin path ordering. Substitution of (6.24) into (6.18) now gives (6.10)–(6.12) with (6.16) and (6.17). We have confirmed that these results are correct in perturbation theory.

VII. CONCLUSIONS

We have seen that the dimensionally regularized unrenormalized loop functions $W(C;g,\epsilon)$ are more tractable than the renormalized loop functions $W_R(C;\gamma,g_R,\mu)$, for a general loop C with cusps and cross points characterized by angles γ. The W's are unique, whereas the W_R's depend on various renormalization conventions. The W's approach unity when the length $L(C)$ approaches zero, whereas the W_R's approach the complicated but known expression $(-\ln L\mu)^{\eta(\gamma)/b}$. The W's satisfy the MM equations, whereas it is doubtful if the W_R's satisfy similar useful equations. The W's automatically satisfy the Mandelstam constraints, whereas such constraints must be enforced upon the W_R's. We are therefore led to conclude that one should formulate and solve non-Abelian gauge theories in terms of the W's, and afterwards calculate the W_R's and observables.

In terms of the dimensionally regularized unrenormalized functions $W(C;g,\epsilon)$, our main results are the exact[10] determination of the $\epsilon\to 0$ behavior (3.38), etc., and the proof of the dimensional regularized MM equation (4.45). Explicit examples of the $\epsilon\to 0$ behaviors follow from Eqs. (3.70) and (3.78). We do not know how to solve the MM equation (4.45), even for large N [Eq. (4.24)], except in perturbation theory. Reversing our arguments in Sec. IV D, we can show that the MM equations have a power-series (in g) solution which coincides with the conventionally calculated expressions for the W's in perturbation theory. This indicates that the MM equations are at least consistent.

The program should now be to find nonperturbative solutions to the dimensionally regularized MM equations, subject to the Mandelstam constraints (1.12) and (5.2), and then renormalize according to (3.47). This exact renormalization is easy since we know the exact small-ϵ behavior (3.60) of the renormalization matrices and since the MM, the Mandelstam, and the renormalization mixing sets are all the same. The final step would be to calculate observables such as the flavor current generating functionals (6.11). In the large-N limit, the MM equation and current generating functional enormously simplify to (4.24) and (6.12), respectively. If, also, the Mandelstam constraints disappear in this limit, loop space may be the best hope for solving QCD in the large-N limit.

ACKNOWLEDGMENTS

One of us (R.A.B.) thanks the particle theory group of Brookhaven National Laboratory for its hospitality. This work was supported in part by National Science Foundation Grant No. PHY78-21503.

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[16]Even if this were not the case, the linear divergences could be removed by a further multiplicative renormalization. See Ref. 6.

[17]There are additional singularities, coming from the end points x and y, which go away when $x=y$ so that the loop is closed.

[18]The equivalent to this equation seems to have been first stated by W. Fischler, Nucl. Phys. B129, 157 (1977), before the finiteness of W_R was established. See also M. E. Peskin, Phys. Lett. 94B, 161 (1980) and A. J. McKane and M. Stone, *ibid.* 103B, 365 (1981).

[19]If fermions were present, (3.15) would be replaced by $(1/8\pi^2)(\frac{11}{3}C_1-\frac{4}{3}C_2)$, and our conclusions would be unchanged in the asymptotically free cases $C_2/C_1<\frac{11}{4}$. Here C_2 is given by $\mathrm{tr}\,t^a t^b=2C_2\delta^{ab}$ in terms of the fermion representation matrices t^a.

[20]We use the summation convention for the loop set indices i,j,k so that repeated indices sum from 1 to I_N.

[21]Mandelstam's (Ref. 1) notation is different, but his result is equivalent to (4.5).

[22]More precisely, (4.20) is the solution to the infinite coupled chain of MM equations which satisfies (4.19).

[23]G. 't Hooft, Nucl. Phys. B72, 461 (1974); S. Coleman, in *Pointlike Structure Inside and Outside Hadrons,* proceedings of the Seventeenth International School of Subnuclear Physics, Erice, 1979, edited by A. Zichichi (Plenum, New York, 1982), and references therein.

[24]This assumes that the constant (4.22) implies that the gauge group is Abelian. This is true classically (Ref. 5), but has not been proved in quantum field theory.

[25]In this section we replace Ng^2 by g^2.

[26]N. E. Bralic, Phys. Rev. D 22, 3090 (1980).

[27]V. A. Kasakov and I. K. Kostov, Nucl. Phys. B176, 199 (1980).

[28]This term arises from diagrams in which the same gluon propagator starts and ends in $\delta C_{\mu\nu}$, as illustrated in Fig. 13. If the propagator is nonsingular at short distances, this diagram is of order $|\delta\sigma|^2$, by Stokes's theorem, and so does not contribute to the area derivative. Furthermore, since it is manifestly symmetric in μ and ν, it does not contribute to the manifestly antisymmetric area derivative (4.7) even if the propagator has a short-distance singularity.

[29]Our notation and normalizations are different from those of Ref. 5.

[30]B. Durhuus and J. M. Leinaas, Phys. Ser. 25, 504 (1982).

[31]$1^j\equiv 1$ for all j.

[32]R. P. Feynman, Phys. Rev. 80, 440 (1950).

[33]R. A. Brandt, F. Neri, and D. Zwanziger, Phys. Rev. D 19, 1153 (1979).