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Yang-Mills theory as Schrödinger quantum mechanics on the space of gauge-group orbits

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Recent proposals to study Yang-Mills theory on the space of gauge-group orbits are reconsidered. In particular, it is shown that the right formal Hamiltonian is not given by $-\hbar^2/2$ times the Laplace-Beltrami operator plus the standard "magnetic field" potential, as was suggested, but has an additional potential term proportional to $\hbar^2$ and is expressible in terms of the geometry not only of the space of gauge-group orbits but also of the orbits themselves as embedded in the space of gauge fields. Formal discussion of the continuum fields is substantiated by a rigorous consideration of lattice gauge theory.

I. INTRODUCTION

In recent years it has been realized that gauge theories are the natural field for application of geometric ideas. Not only are gauge fields geometric objects (connections),[1] but geometric notions are indispensable in studying classical solutions of the field equations both in the relativistic case and in the Euclidean case.[2] These in turn might have a bearing on the quantum theory through a semiclassical approximation.[3] Recently, some problems directly related to the quantized theory where shown to have a natural formulation in geometric terms. Let us mention here the gauge-fixing ambiguities[4–6] and the problem of θ vacuums.[7–9] In particular, it has been argued that the configuration space for Yang-Mills theory should consist of the orbits under the group of local gauge transformations of gauge fields at fixed time rather than of the fields themselves. The space of orbits carries a natural Riemannian structure $\tilde{g}$ inherited from that of the space of fixed-time fields, the latter being given by the "electric field" part of the classical Lagrangian

$$\frac{1}{2}\int d\vec{x}\,\mathrm{tr}(F^{0k})^2 .$$

The "magnetic field" part of the Lagrangian

$$\frac{1}{4}\int d\vec{x}\,\mathrm{tr}(F_{kl})^2$$

defines a function $\tilde{V}$ on the space of orbits (the potential). Formal reasoning based on functional integration seems to suggest[10,11] that we should take $\tilde{H}=-(\hbar^2/2)\tilde{\Delta}_{\mathrm{LB}}+\tilde{V}$ as the formal quantum Hamiltonian in the space of gauge-invariant functionals with scalar product given by the (formal) Riemannian volume element on the space of orbits. $\tilde{\Delta}_{\mathrm{LB}}$ is the Laplace-Beltrami operator on the space of orbits. The reasoning uses the Faddeev-Popov standard trick[12,13] with gauge fixing depending only on the spatial components of the gauge field. One may come up with another proposition $\tilde{H}'$ for the quantum Hamiltonian which is worked out in the temporal gauge. $\tilde{H}'=\tilde{H}+\delta\tilde{V}$ and $\delta\tilde{V}$ may be expressed by the scalar curvature of the orbit space and the second fundamental form of the orbit. $\tilde{H}'$ seems to be the right Hamiltonian for Yang-Mills theory. Two different facts support this statement: First, $\tilde{H}'$ coincides with the Hamiltonian obtained by Schwinger[14] in the Coulomb gauge by postulating the commutation relations with the famous Schwinger terms for the energy density. Those commutation relations guarantee the Poincaré invariance of the quantum theory (on the formal level). Second, $\tilde{H}'$ is the (formal) limit of the logarithm of the lattice-gauge-theory transfer matrix. Notice that $\tilde{H}'$ is not given in terms of intrinsic geometry of the orbit space. This shows that doing the quantum Yang-Mills theory on the space of gauge-group orbits is not totally natural. The source of the discrepancy between $\tilde{H}$ and $\tilde{H}'$ is the arbitrariness in interpretation of the path-integral formulas on Riemannian manifolds connected to factor-ordering problems. We try to elucidate this point in the text.

The paper is organized as follows. Section II contains a heuristic discussion leading to two different expressions for the Hamiltonian, depending on whether we work in the temporal or in the spacelike gauge. The relation between them is discussed. Section III deals with lattice gauge theory where part of the reasoning of Section II may be made rigorous and the other part, if still formal, is more illuminating. Appendix A (formal) compares the Hamiltonian $\tilde{H}'$ with Schwinger's Hamiltonian showing that they coincide. Appendix B (rigorous) deals with the limit of lattice gauge theory when the time-direction lattice spacing tends to zero. Appendix C contains a formal computation of the $\delta\tilde{V}$ addition to the potential for the continuum case. It could be regularized by considering the lattice theory or by employing dimensional regularization.

It has to be pointed out that the problem of the relation between the temporal- and the spacelike-gauge canonical quantizations in connection to the ordering ambiguities has been addressed in numerous publications (see Refs. 15–19). In particular, the translation of the temporal-gauge quantum Hamiltonian into spacelike gauges, which is a by-product of our paper, has been advocated or carried out there. Nevertheless the arguments presented here based on a geometric approach are novel and offer in our opinion a "particularly simple and clear" picture on both the conceptual and the computational levels.

II. HEURISTIC DISCUSSION: THE CONTINUUM-GAUGE-THEORY CASE

Consider the pure gauge theory with the compact group $G=\mathrm{SU}(N)$. The basic fields $A\equiv(A_\mu(t,\vec{x}))$ take values in the Lie algebra $\mathscr{A}$ of G [$\mathscr{A}=\mathrm{su}(N)$]. The local gauge transformations $\gamma\equiv(\gamma(t,\vec{x}))$, $\gamma(t,\vec{x})\in G$ act by

$$^{\gamma}A_\mu=\gamma A_\mu\gamma^{-1}-i(\partial_\mu\gamma)\gamma^{-1}\ . \tag{2.1}$$

We shall also consider the fixed-time picture: the space Q of fields $\vec{A}(x)\equiv(A_k(\vec{x}))$, the local gauge transformations $\gamma\equiv(\gamma(\vec{x}))$, and the gauge-invariant functionals ψ on Q having the interpretation of the Schrödinger-picture wave functions. According to the functional prescription the (Euclidean) matrix element

$$(\psi\mid e^{-(1/\hbar)T\,\mathrm{Hamiltonian}}\psi) \tag{2.2}$$

is proportional to the formal integral over paths $[0,T]\ni t\to A_\mu(t,\cdot)$,

$$\int\bar{\psi}(\vec{A}(0))\psi(\vec{A}(t))\times\exp\left[-\frac{1}{4\hbar}\int_0^T dt\int d\vec{x}\,\mathrm{tr}F_{\mu\nu}{}^2\right]\prod_t DA(t)\ , \tag{2.3}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-i[A_\mu,A_\nu] \tag{2.4}$$

and

$$DA(t)=\prod_{\mu,\vec{x}}dA_\mu(t,\vec{x})\ . \tag{2.5}$$

One of the many problems with (2.3) is the presence of redundant degrees of freedom due to the local gauge invariance of the integrand. This is dealt with by the standard Faddeev-Popov gauge-fixing procedure.[12,13] One multiplies (2.3) by the formally constant expression

$$\int\delta(F(^{\gamma}A))\Delta_F(^{\gamma}A)\prod_{t,\vec{x}}d\gamma(t,\vec{x})\ , \tag{2.6}$$

where F is a gauge-fixing functional of A, $\Delta_F(A)$ is the Jacobian of the transformation $\Lambda\to F(e^{i\Lambda}A)$ at $\Lambda=0$ with $\Lambda\equiv(\Lambda(t,\vec{x}))$, $\Lambda(t,\vec{x})\in\mathscr{A}$, and $d\gamma$ is the Haar measure on G. Changing the order of integration and using the gauge invariance of the underintegral expression in (2.3) and of the $\prod_{\mu,t,\vec{x}}dA_\mu(t,\vec{x})$ measure we arrive at an expression proportional to

$$\int\bar{\psi}(\vec{A}(0))\psi(\vec{A}(t))\exp\left[-\frac{1}{4\hbar}\int_0^T dt\int d\vec{x}\,\mathrm{tr}F_{\mu\nu}{}^2\right]\Delta_F(A)\delta(F(A))\prod_t DA(t)\ . \tag{2.7}$$

Consider two specific examples of this situation.

A. Temporal gauge $A_0=0$

Take $F(A)=A_0$. Then $\Delta_F(A)=\det(\partial_0-iA_0)=\mathrm{const}$ on the support of $\delta(A_0)$. Performing the A_0 integration we arrive at

$$(\psi\mid e^{-(1/\hbar)T\,\mathrm{Hamiltonian}}\psi)=\mathrm{const}\times\int\bar{\psi}(\vec{A}(0))\psi(\vec{A}(t))\exp\left\{-\frac{1}{\hbar}\int_0^T dt\int d\vec{x}\left[\tfrac{1}{2}\mathrm{tr}(\partial_0\vec{A})^2+\tfrac{1}{4}\mathrm{tr}F_{kl}{}^2\right]\right\}\times\prod_t D\vec{A}(t)\ , \tag{2.8}$$

where

$$D\vec{A}(t)=\prod_{k,\vec{x}} dA_k(t,\vec{x}) .$$

Remark: We could have obtained (2.8) directly from (2.3) by integrating first over $\vec{A}$ for A_0 fixed and gauge transforming A by an A_0-dependent gauge transformation such that $({}^{\gamma}A)_0=0$ which would make the $\vec{A}$ integral A_0 independent. This procedure of fixing the temporal gauge carries over to the lattice and will be used in the next section.

Comparison of (2.8) with the standard quantum-mechanical path-integral formula

$$\left[\psi \left| \exp\left[-\frac{1}{\hbar}T\left[-\frac{\hbar^2}{2}\Delta+V\right]\right]\right| \psi\right] = \text{const}\times\int \bar{\psi}(x(0))\psi(x(t))\exp\left\{-\frac{1}{\hbar}\int_0^T dt\left[\frac{1}{2}\left[\frac{dx^i}{dt}\right]^2+V(x)\right]\right\}\prod_t dx(t) \quad (2.9)$$

motivates the choice of

$$H=-\frac{\hbar^2}{2}\Delta_{\text{LB}}+V \quad (2.10)$$

as the formal gauge-field Hamiltonian,

$$\Delta_{\text{LB}}=\int d\vec{x}\,\text{tr}\frac{\delta^2}{\delta A_k(\vec{x})\delta A_k(\vec{x})} \quad (2.11)$$

and

$$V=\tfrac{1}{4}\int d\vec{x}\,\text{tr}F_{kl}{}^2 \quad (2.12)$$

are the operators acting on the (gauge-invariant) functionals of $\vec{A}(\vec{x})$ whose scalar product is given by flat volume element $C\times D$ on the space Q of gauge fields $\vec{A}(\vec{x})$, C is a constant. For the matrix element of H we have

$$(\psi\,|\,H\psi)=C\int\left[\frac{\hbar^2}{2}g(d\bar{\psi},d\psi)+\bar{\psi}V\psi\right]D\vec{A}\,, \quad (2.13)$$

where

$$g(d\bar{\psi},d\psi)(\vec{A})=\int d\vec{x}\,\text{tr}\frac{\delta\bar{\psi}}{\delta A_k(\vec{x})}\frac{\delta\psi}{\delta A_k(\vec{x})} \quad (2.14)$$

gives the (flat) Riemannian metric on Q.

Fixing the temporal gauge left us with the residual gauge freedom to perform time-independent gauge transformations. This is the source of presence in (2.13) of redundant gauge degrees of freedom. We may get rid of them again by the Faddeev-Popov procedure writing

$$(\psi\,|\,H\psi)=\int\left[\frac{\hbar^2}{2}g(d\bar{\psi},d\psi)+\bar{\psi}V\psi\right]\times\Delta_f(\vec{A})\delta(f(\vec{A}))D\vec{A}\,, \quad (2.15)$$

where $\Delta_f(\vec{A})$ is the spacelike version of $\Delta_F(A)$ and

$$||\psi||^2=\int\bar{\psi}\psi\Delta_f(\vec{A})\delta(f(\vec{A}))D\vec{A}\,. \quad (2.16)$$

We have chosen the constant C so that it cancels the constant factor produced by the gauge-fixing procedure. The choice of C does not have a physical meaning since it does not appear in the Green's functions (vacuum matrix elements). We claim that H is the right (formal) Hamiltonian of the pure Yang-Mills theory since it coincides with the Schwinger Hamiltonian[14] and may be obtained by taking formal continuum limit of the lattice transfer matrix.

B. Spacelike gauges

In (2.7) we choose $F(\vec{A})=(f(\vec{A}(t)))$. [The Coulomb gauge with $f(\vec{A})=-\partial_k A_k$ is an example of such a gauge.] Then

$$(2.7)=\int\bar{\psi}(\vec{A}(0))\psi(\vec{A}(T))\exp\left\{-\frac{1}{\hbar}\int_0^T dt\int d\vec{x}\,[\tfrac{1}{2}\text{tr}(\nabla_k A_0-\partial_0 A_k)^2+\tfrac{1}{4}\text{tr}F_{kl}{}^2]\right\}\times\prod_t(\Delta_f(\vec{A}(t))\delta(f(\vec{A}(t)))DA(t))\,, \quad (2.17)$$

where

$$\nabla_k = \partial_k - i[A_k, \cdot\,] \tag{2.18}$$

is the covariant derivative acting on $\mathscr{A}$ valued fields. But the A_0 integral is now a Gaussian one and we may evaluate it obtaining

$$(\psi \mid e^{-(1/\hbar)T\,\text{Hamiltonian}}\psi)$$
$$= \text{const} \times \int \bar{\psi}(\vec{A}(0))\psi(\vec{A}(T)) \exp\left[-\frac{1}{\hbar}\int_0^T dt \int d\vec{x}\{\tfrac{1}{2}\,\text{tr}[\partial_0 A_k(\delta_{kl} - \nabla_k \Delta^{-1}\nabla_l)\partial_0 A_l] + \tfrac{1}{4}\,\text{tr}F_{kl}{}^2\}\right]$$
$$\times \prod_t (\det(-\Delta)^{-1/2}\Delta_f \delta(f) D\vec{A}), \tag{2.19}$$

where

$$\Delta \equiv \Delta(\vec{A}) = \nabla_k \nabla_k \tag{2.20}$$

is the covariant Laplacian.

Equation (2.19) was given the following geometric interpretation in Ref. 11 (see also Refs. 9 and 20). The group $\mathscr{G}$ of local gauge transformations $\gamma = (\gamma(\vec{x}))$ acting on Q [the space of fields $\vec{A} \equiv (A_k(\vec{x}))$] preserves the Riemannian metric g [see Eq. (2.14)]. Moreover, if we exclude $\vec{A}$'s having nontrivial local gauge symmetries (the so-called reducible ones), which form some sort of boundary set, restricting ourselves to the set Q_0 of irreducible $\vec{A}$'s, $\mathscr{G}$ acts without fixed points and $Q_0/\mathscr{G} \equiv \tilde{Q}_0$ is a decent space (may be given a manifold structure[21]). The metric g projects down giving a Riemannian structure $\tilde{g}$ on $\tilde{Q}_0$. By definition the Riemannian square of a vector $(d/dt)[\vec{A}]$ tangent to $\tilde{Q}_0$ is obtained by taking a lift of this vector to Q_0, e.g., $\partial_0 \vec{A}$, and computing the scalar square according to g of its projection onto the subspace normal to the orbit. Hence

$$\tilde{g}\left[\frac{d}{dt}[\vec{A}], \frac{d}{dt}[\vec{A}]\right] = \int d\vec{x}\,\text{tr}[\partial_0 \vec{A}_k(\delta_{kl} - \nabla_k \Delta^{-1}\Delta_l)\partial_0 \vec{A}_l]\,. \tag{2.21}$$

Let $\tilde{v}([\vec{A}])$ denote the volume of the $\mathscr{G}$ orbit $[\vec{A}]$:

$$\tilde{v}([\vec{A}]) = \text{const} \times \int \det(-\Delta)^{1/2}({}^{\gamma}A) \prod_x d\gamma(\vec{x})$$
$$= \text{const} \times \det(-\Delta)^{1/2} \tag{2.22}$$

since $-\Delta$ gives the pullback of g by the exponential map

$$\Lambda \to \frac{d}{dt}\bigg|_{t=0} {}^{e^{it\Lambda}}\vec{A}\,.$$

We have used the gauge invariance of $\det(-\Delta)$. For a gauge-invariant function ψ on Q_0 let $\tilde{\psi}$ denote the corresponding function on $\tilde{Q}_0$. Let $d\tilde{\mu}([A])$ be the measure on $\tilde{Q}_0$ defined by the Riemannian volume element. We have

$$\int \psi D\vec{A} = \int \tilde{\psi}\tilde{v}\, d\tilde{\mu}$$
$$= \text{const} \times \int \tilde{\psi} \det(-\Delta)^{1/2} d\tilde{\mu}\,. \tag{2.23}$$

On the other hand, by the Faddeev-Popov trick

$$\int \psi D\vec{A} = \text{const} \times \int \psi \Delta_f \delta(f) D\vec{A}\,. \tag{2.24}$$

Hence

$$\int \tilde{\psi}\, d\tilde{\mu} = \text{const} \times \int \psi \det(-\Delta)^{-1/2}\Delta_f \delta(f) D\vec{A}\,. \tag{2.25}$$

Using (2.21) and (2.25) we may rewrite (2.19) as

$$(\psi \mid e^{-(1/\hbar)T\,\text{Hamiltonian}}\psi) = \text{const} \times \int \bar{\tilde{\psi}}([\vec{A}](0))\tilde{\psi}([\vec{A}](T)) \exp\left\{-\frac{1}{\hbar}\int_0^T dt \left[\tfrac{1}{2}\tilde{g}\left[\frac{d}{dt}[A], \frac{d}{dt}[A]\right] + \tilde{V}([A])\right]\right\}$$
$$\times \prod_t d\tilde{\mu}([A](t))\,, \tag{2.26}$$

where V is given by (2.12).

This way (2.26) becomes an analog of the quantum-mechanical path-integral expression on a Riemannian manifold

$$\text{const}\times\int\bar{\psi}(x(0))\psi(x(t))\exp\left\{-\frac{1}{\hbar}\int_0^T dt\left[\tfrac{1}{2}g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}+V(x)\right]\right\}\prod_t[g(x(t))]^{1/2}dx(t)\ . \tag{2.27}$$

However, the intepretation of (2.27) is not as straightforward as that of the right-hand side of (2.9) since there is a factor-ordering problem involved. One might understand by (2.27) the matrix element

$$\left[\psi\left|\exp\left[-\frac{1}{\hbar}T\left[\frac{-\hbar^2}{2}\Delta_{\rm LB}+V\right]\right]\right|\psi\right]$$

where $\Delta_{\rm LB}$ is the Laplace-Beltrami operator on the Riemannian manifold and the scalar product is defined with the use of the Riemannian volume element. Based on this it was recently argued[10] that (2.27) gives rise to the quantum Hamiltonian

$$\tilde{H}=-\frac{\hbar^2}{2}\tilde{\Delta}_{\rm LB}+\tilde{V}\ , \tag{2.28}$$

where $\tilde{\Delta}_{\rm LB}$ is the Laplace-Beltrami operator on $\tilde{Q}_0$. $\tilde{H}$ acts in the space of functionals on $\tilde{Q}_0$ with the scalar product defined by $d\tilde{\mu}$. This definition of $\tilde{H}$ was at the origin of an attempt[10] to construct rigorously cut-off versions of $\tilde{H}$. The choice of (2.28) is based, however, on an arbitrary convention for factor ordering in (2.26). The right convention should be compatible with symmetry principles. It is known, for example, that the postulate of the conformal symmetry forces one to interpret (2.28) as

$$\left[\psi\left|\exp\left\{-\frac{1}{\hbar}T\left[-\frac{\hbar^2}{2}(\Delta_{\rm LB}+\alpha r)\right]\right\}\right|\psi\right]$$

for $V=0$ where α is the dimension-dependent constant and r is the scalar curvature. In our case the postulate of the Poincaré invariance is the symmetry principle at hand. Let us compare $\tilde{H}$ to the Hamiltonian H chosen in Sec. II which, according to the analysis of Ref. 14 and of Appendix A is compatible with the Poincaré symmetry principle. H and $\tilde{H}$ act on spaces with different scalar product which may be related by a unitary transform U,

$$U\tilde{\psi}=e^{-h/2}\psi\ , \tag{2.29}$$

where

$$e^h=\text{const}\times\det(-\Delta)^{1/2} \tag{2.30}$$

and the constant is chosen so that

$$||U\tilde{\psi}||^2=\int\bar{\tilde{\psi}}\tilde{\psi}\,d\tilde{\mu} \tag{2.31}$$

see (2.16) and (2.25)]. Denote by $\tilde{H}'$ the operator $U^{-1}HU$ acting as $\tilde{H}$ in the space of functionals on $\tilde{Q}_0$ with the scalar product defined by $d\tilde{\mu}$:

$$(\tilde{\psi}\,|\,\tilde{H}'\tilde{\psi})=(U\tilde{\psi}\,|\,HU\tilde{\psi})=\int\left[\frac{\hbar^2}{2}g(d(e^{-h/2}\bar{\tilde{\psi}}),d(e^{-h/2}\psi))+\bar{\psi}V\psi e^{-h}\right]\Delta_f\delta(f)D\vec{A} \tag{2.32}$$

$$=\int\left[\frac{\hbar^2}{2}g(d\bar{\psi},d\psi)-\frac{\hbar^2}{4}g(d(\bar{\psi}\psi),dh)+\frac{\hbar^2}{8}\bar{\psi}g(dh,dh)\psi+\bar{\psi}V\psi\right]e^{-h}\Delta_f\delta(f)D\vec{A}$$

$$=\int\left[\frac{\hbar^2}{2}\tilde{g}(d\bar{\tilde{\psi}},d\tilde{\psi})-\frac{\hbar^2}{4}\tilde{g}(d(\bar{\tilde{\psi}},\tilde{\psi}),d\tilde{h})+\frac{\hbar^2}{8}\bar{\tilde{\psi}}\tilde{g}(d\tilde{h},d\tilde{h})\tilde{\psi}+\bar{\tilde{\psi}}\tilde{V}\tilde{\psi}\right]d\tilde{\mu}$$

$$=\int\bar{\tilde{\psi}}\left[-\frac{\hbar^2}{2}\tilde{\Delta}_{\rm LB}+\tilde{V}+\delta\tilde{V}\right]\tilde{\psi}\,d\tilde{\mu}\ , \tag{2.33}$$

where

$$\delta\tilde{V}=\frac{\hbar^2}{4}\tilde{\Delta}_{\rm LB}\tilde{h}+\frac{\hbar^2}{8}\tilde{g}(d\tilde{h},d\tilde{h})\ . \tag{2.34}$$

Hence

$$\tilde{H}'=\tilde{H}+\delta\tilde{V}\ .$$

In Appendix B we give explicit but formal computation of the $\delta\tilde{V}$ addition to the potential.

For the matrix element of $e^{-(T/\hbar)H}$ one obtains

$$(\psi\,|\,e^{-(T/\hbar)H}\psi)=(e^{\tilde{h}/2}\tilde{\psi}\,|\,e^{-(T/\hbar)\tilde{H}'}e^{\tilde{h}/2}\tilde{\psi})$$
$$=(e^{\tilde{h}/2}\tilde{\psi}\,|\,e^{-(T/\hbar)(\tilde{H}+\delta\tilde{V})}e^{\tilde{h}/2}\tilde{\psi})\ , \tag{2.35}$$

which we claim is the right interpretation of (2.26) as contrasted with $(\widetilde{\psi} \mid e^{-(1/\hbar)T\widetilde{H}}\widetilde{\psi})$. The presence of the $e^{\hbar/2}$ factors is not physically relevant [they do not appear in the (Euclidean) Green's functions]. We shall moreover be able to elucidate their origin to a certain extent going to the lattice. The additional "quantum" potential $\delta\widetilde{V}$ seems, however, relevant.

III. RIGOROUS DISCUSSION: THE LATTICE-GAUGE-THEORY CASE

Most of the previous formal arguments (except the one based on the Poincaré invariance) can be made rigorous by considering the lattice gauge theory in the manner of Wilson[22]) in a finite volume. $A_\mu(t,\vec{x})$ gets replaced by the Lie-group-valued fields $g_b \in G$, $g_{b^{-1}}=g_b{}^{-1}$, with b running through the bonds of the lattice $\Lambda=\Lambda_t\times\Lambda_s$ where $\Lambda_t=a_tZ\cap[0,T]$, and Λ_s is, e.g., a cube in the spacelike lattice a_sZ^d. The local gauge transformations $\gamma\equiv(\gamma_x)$ are specified by giving a group element γ_x for each $x\in\Lambda$ and act by

$$^{\gamma}g_b=\gamma_y g_b \gamma_x{}^{-1} \quad \text{for } b=\langle xy\rangle\ . \tag{3.1}$$

In the fixed-time picture we consider the space Q_l of $g\equiv(g_b)_{b\subset\Lambda_s}$, $g_{b^{-1}}=g_b{}^{-1}$, the local gauge transformations $\gamma\equiv(\gamma_{\vec{x}})_{\vec{x}\in\Lambda_s}$ forming the group $\mathscr{G}_l$, and gauge-invariant wave functions ψ on Q_l. The regularized version of (2.3) reads now (T/a_t is assumed to be an integer)

$$\int \bar{\psi}(g(0))\psi(g(T))\exp\left[\frac{1}{\hbar}\mathrm{Re\,tr}\left[\sum_{p\subset\Lambda}{}' a_t{}^{-1}a_s{}^{d-2}(g_p-1)+\sum_{p\subset\Lambda}{}'' a_ta_s{}^{d-4}(g_p-1)\right]\right]\prod_{b\subset\Lambda_T} dg_b\ , \tag{3.2}$$

where $\sum'_{p\subset\Lambda}$ runs over the plaquettes of Λ containing timelike bonds, $\sum''_{p\subset\Lambda}$ over the spacelike plaquettes, $g_p=\prod_{b\subset\partial p}g_b$, and dg_b is the Haar measure on G. Equation (3.2) is perfectly well defined for ψ in $\mathscr{H}\equiv L^2(Q_l,dg)$ where $dg=\prod_{b\subset\Lambda_s}dg_b$. We shall show that, after proper normalization, its $a_t\to0$ limit exists and defines the standard Hamiltonian of the lattice gauge theory.[23] To this end transform (3.2) by fixing the temporal gauge. This is done by performing the local gauge transformation γ with

$$\gamma(t,\vec{x})=\prod_{b\subset[0,t]\times\{\vec{x}\}} g_b,\quad \gamma(0,\vec{x})=1\ .$$

[See Remark preceding Eq. (2.9)].] The result is

$$\int \bar{\psi}(g(0))\psi(g(T))\exp\left[\frac{1}{\hbar}\mathrm{Re\,tr}\left[\sum_{\substack{t\in\Lambda_t\\ t\neq T}}\sum_{b\subset\Lambda_s} a_t{}^{-1}a_s{}^{d-2}(g_b(t+a_s)g_b(t)^{-1}-1) + \sum_{t\in\Lambda_t}\sum_{p\subset\Lambda_s} a_ta_s{}^{d-4}(g_p-1)\right]\right]\prod_{\substack{t\in\Lambda_t\\ b\subset\Lambda_s}} dg_b(t)\ . \tag{3.3}$$

Denote by $\mathscr{T}_{a_t}$ the operator in $\mathscr{H}$ with the kernel

$$\mathscr{T}_{a_t}(g',g)=\exp\left[\frac{1}{\hbar}\mathrm{Re\,tr}\left[\sum_{b\subset\Lambda_s} a_t{}^{-1}a_s{}^{d-2}(g_b'g_b{}^{-1}-1)\right]\right]W_{a_t}(g)\Big/ \int \exp\left[\frac{1}{\hbar}\mathrm{Re\,tr}\left[\sum_{b\subset\Lambda_s} a_t{}^{-1}a_s{}^{d-2}(g_b{}^{-1}-1)\right]\right]dg\ , \tag{3.4}$$

where

$$W_{a_t}(g)=\exp\left[\frac{1}{\hbar}\mathrm{Re\,tr}\left[\sum_{p\subset\Lambda_s} a_t a^{d-4}(g_p-1)\right]\right]. \tag{3.5}$$

$\mathscr{T}_{a_t}$ is the transfer matrix of the lattice theory. Equation (3.3) is proportional to

$$(W_{a_t}\psi\,|\,\mathscr{T}_{a_t}^{T/a_t}\psi)\,. \tag{3.6}$$

In Appendix C we prove that $\mathscr{T}_{a_t}^{T/a_t}$ converges strongly as $a_t\to 0$ to

$$\exp\left[-\frac{1}{\hbar}T\left[-\frac{\hbar^2}{2}\Delta+V\right]\right]$$

where Δ is the Laplace operator on the group $\times_{b\subset\Lambda_s}G$ $(=Q_l)$ corresponding to the invariant Riemannian structure g defined by the scalar product

$$\langle A,A\rangle=\sum_{b\subset\Lambda_s} a^{d-2}\mathrm{tr}A_b{}^2\,, \tag{3.7}$$

$A\in\times_{b\subset\Lambda_s}\mathscr{A}$, and

$$V=-\mathrm{Re\,tr}\left[\sum_{b\subset\Lambda_s} a^{d-4}(g_p-1)\right].$$

This gives immediately convergence of (3.6) to

$$\left[\psi\left|\exp\left[-\frac{1}{\hbar}T\left[-\frac{\hbar^2}{2}\Delta+V\right]\right]\right|\psi\right]$$

and produces the lattice version of the definition (2.10) of the quantum Hamiltonian

$$H_l\equiv-\frac{\hbar^2}{2}\Delta+V \tag{3.8}$$

in $\mathscr{H}_{\mathrm{inv}}$, the gauge-invariant subspace of $\mathscr{H}$. This is the Kogut-Susskind Hamiltonian.[23] For gauge-invariant smooth ψ's (such ψ's constitute an essential domain for H_l in $\mathscr{H}_{\mathrm{inv}}$)

$$(\psi\,|\,H_l\psi)=\int\left[\frac{\hbar^2}{2}g(d\bar\psi,d\psi)+\bar\psi V\psi\right]dg\,. \tag{3.9}$$

The group of local gauge transformations $\mathscr{G}_l$ acts freely on Q_l except for a closed subset of measure zero composed of reducible g's. The quotient space $\tilde{Q}_{l_0}\equiv Q_{l_0}/\mathscr{G}_l$ of classes of irreducible g's is a manifold.

The Riemannian metric g on Q_{l_0} projects down to $\tilde{Q}_{l_0}$ giving rise to the metric $\tilde{g}$. The two natural measures on $\tilde{Q}_{l_0}$, the projection $d[g]$ of the measure dg and the Riemannian measure $d\tilde{\mu}([g])$, are related by

$$d[g]=e^{\tilde{h}([g])}d\tilde{\mu}([g])\,, \tag{3.10}$$

where $e^{\tilde{h}([g])}$ is proportional to the Riemannian volume of the $\mathscr{G}_l$ orbit $[g]$ in Q_l. The unitary mapping $U{:}\tilde{\mathscr{H}}=L^2(Q_{l_0},d\tilde{\mu})\to\mathscr{H}_{\mathrm{inv}}$,

$$U\tilde{\psi}=e^{-h/2}\psi\,, \tag{3.11}$$

allows one to define the Hamiltonian $\tilde{H}'_l=U^{-1}H_lU$ in $\tilde{\mathscr{H}}$ such that

$$\begin{aligned}(\psi\,|\,e^{-(1/\hbar)TH_l}\psi)&=(U^{-1}\psi\,|\,e^{-(1/\hbar)T\tilde{H}'_l}U^{-1}\psi)\\&=(e^{\tilde{h}/2}\tilde{\psi}\,|\,e^{-(1/\hbar)T\tilde{H}'_l}e^{\tilde{h}/2}\tilde{\psi})\,.\end{aligned} \tag{3.12}$$

Moreover, for smooth invariant functions ψ vanishing in a neighborhood of reducible g's

$$\begin{aligned}(\tilde{\psi}\,|\,\tilde{H}'_l\tilde{\psi})&=(e^{-h/2}\psi\,|\,H_le^{-h/2}\psi)\\&=\left[\tilde{\psi}\left|\left[-\frac{\hbar^2}{2}\tilde{\Delta}_{\mathrm{LB}}+\tilde{V}+\delta\tilde{V}\right]\right|\tilde{\psi}\right],\end{aligned} \tag{3.13}$$

where

$$\delta\tilde{V}=\frac{\hbar^2}{4}\tilde{\Delta}_{\mathrm{LB}}\tilde{h}+\frac{\hbar^2}{8}\tilde{g}(d\tilde{h},d\tilde{h}) \tag{3.14}$$

by virtue of (3.10) and of a (rigorous) computation analogous to (2.33). Let us denote by $\nabla\equiv\nabla(g)$ the covariant derivative ∇: $\times_{x\subset\Lambda_s}\mathscr{A}\to\times_{b\subset\Lambda_s}\mathscr{A}$,

$$\begin{aligned}(\nabla\Lambda)_b&=\left.\frac{d}{dt}\right|_{t=0}({}^{e^{it\Lambda}}g)_bg_b{}^{-1}\\&=\Lambda_y-g_b\Lambda_xg_b{}^{-1}\quad\text{for } b=\langle xy\rangle\,.\end{aligned} \tag{3.15}$$

Consider in $\times_{b\subset\Lambda_s}\mathscr{A}$ the scalar product (3.7) and in $\times_{x\subset\Lambda_s}\mathscr{A}$ the one given by

$$\langle\Lambda,\Lambda\rangle=\sum_{x\in\Lambda_s} a_s^d\mathrm{tr}\Lambda_x{}^2\,. \tag{3.16}$$

Let $\nabla^\dagger$ be the adjoint of ∇ and let

$$-\nabla^\dagger\nabla=\Delta \tag{3.17}$$

be the covariant Laplacian. Then

$$e^h=\mathrm{const}\times\det(-\Delta)^{1/2} \tag{3.18}$$

or

$$h=\tfrac{1}{2}\,\mathrm{Tr}\ln(-\Delta)+\mathrm{const}\,. \tag{3.19}$$

Notice that e^h vanishes at reducible g's possessing continuous subgroups of symmetries and does not vanish on irreducible g's. The additional "quantum" potential $\delta\widetilde{V}$ of (3.14) could be rigorously computed based on formulas (C15) and (C16). The result is an involved expression and we do not give it here.

The second approach we have followed in our heuristic discussion of the continuum gauge theory which consists of fixing a spacelike gauge and integrating out A_0 has its lattice counterpart also only on a formal level. However, the lattice case is more illuminating than the continuum one. We shall first take the formal $a_t \to 0$ limit of the properly normalized (3.2) by setting $g_{\langle (t,\vec{x})(t+a_t,\vec{x})\rangle} = e^{ia_t A_0(t,\vec{x})}$. This gives

$$\int \overline{\psi}(g(0))\psi(g(T))\exp\left[\frac{1}{\hbar}\int_0^T dt\left[-\tfrac{1}{2}\|\dot{g}-\nabla A_0\|^2+\mathrm{Re\,tr}\sum_{p\subset\Lambda_s} a_s{}^{d-4}(g_p-1)\right]\right]\prod_t dg(t)dA_0(t)\,, \tag{3.20}$$

where

$$\dot{g}_b=\frac{1}{i}\frac{dg_b}{dt}g_b{}^{-1} \tag{3.21}$$

and $\|\ \|^2$ is given by (3.7). Formal integration over A_0 yields now [compare (2.19)–(2.26)]

$$\mathrm{const}\times\int \overline{\psi}(g(0))\psi(g(T))\exp\left[\frac{1}{\hbar}\int_0^T dt\left[-\tfrac{1}{2}\langle \dot{g},[1-\nabla(-\Delta)^{-1}\nabla^\dagger]\dot{g}\rangle+\mathrm{Re\,tr}\sum_{p\subset\Lambda_s} a_s{}^{d-4}(g_p-1)\right]\right]$$

$$\times\prod_t \det(-\Delta)^{1/2}dg(t)$$

$$=\mathrm{const}\times\int \widetilde{\psi}([g](0))\widetilde{\psi}([g](T))\exp\left\{-\frac{1}{\hbar}\int_0^T dt\left[\tfrac{1}{2}\widetilde{g}\left[\frac{d}{dt}[g],\frac{d}{dt}[g]\right]+\widetilde{V}\right]\right\}\prod_t d\widetilde{\mu}([g])\,, \tag{3.22}$$

which we might be tempted to interpret as

$$\left[\psi\left|\exp\left[-\frac{1}{\hbar}T\left[-\frac{\hbar^2}{2}\widetilde{\Delta}_{\mathrm{LB}}+\widetilde{V}\right]\right]\right|\psi\right]_{\mathscr{H}}, \tag{3.23}$$

whereas it should be interpreted as (3.12). We may illuminate the source of this discrepancy setting a finite difference approximation scheme for the last steps leading to (3.22) in which we keep the lattice spacing small but nonzero:

$$\int \overline{\psi}(g(0))\psi(g(T))\exp\left[\frac{1}{\hbar}\sum_{\substack{t\in\Lambda_t\\ t\neq T}} a_t\left[-\tfrac{1}{2}\|\dot{g}(t+a_t/2)-\nabla(g(t+a_t/2))A_0(t+a_t/2)\|^2\right.\right.$$

$$\left.\left.+\mathrm{Re\,tr}a_s{}^{d-4}\sum_{p\subset\Lambda_s}[g_p(t+a_t/2)-1]\right]\right]\prod_{t\in\Lambda_t} dg(t)\prod_{\substack{t\in\Lambda_t\\ t\neq T}} dA_0(t+a_t/2)\,, \tag{3.24}$$

where

$$g_b(t+a_t/2)=\exp[\tfrac{1}{2}ia_t\dot{g}(t+a_t/2)]g_b(t)=\exp[-\tfrac{1}{2}ia_t\dot{g}(t+a_t/2)]g_b(t+a_t)\,,$$

seems to be a good approximation to (properly normalized) (3.2) when $a_t\to 0$. The version of (3.24) with A_0 integrated out, projected to $\widetilde{Q}_0$, reads

$$\text{const}\times\int\tilde{\tilde{\bar{\psi}}}([g](0))\psi([g](T))\exp\left[\frac{1}{\hbar}\sum_{\substack{t\in\Lambda_t\\ t\neq T}}(-1)a_t[\tfrac{1}{2}\tilde{g}([\dot g],[\dot g])(t+a_t/2)+\tilde{V}([g](t+a_t/2))]\right]$$

$$\times\prod_{\substack{t\in\Lambda_t\\ t\neq T}}\det(-\Delta(g(t+a_t/2)))^{-1/2}\prod_{t\in\Lambda_t}\det(\Delta(g(t)))^{1/2}\prod_{t\in\Lambda_t}d\tilde{\mu}([g](t))\,. \tag{3.25}$$

Equation (3.25) contains certain ordering prescription for (a finite difference approximation of)

$$\int_0^T dt\,\tilde{g}\left[\frac{d}{dt}[g],\frac{d}{dt}[g]\right].$$

Moreover, the cancellation of the determinants does not occur completely which allows one to understand the appearance of the factors $e^{h/2}=\text{const}\times\det(-\Delta)^{1/2}$ in (3.12) and is another reason why the $a_t\to 0$ limit of (3.2) should be (3.12) rather then (3.23).

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APPENDIX A

We compare the Hamiltonian (2.15) to the formal expression obtained by Schwinger in Ref. 14. In the Coulomb gauge (2.15), (2.16) reads

$$(\psi\,|\,H\psi)=\int\left[\frac{\hbar^2}{2}g(d\bar{\psi},d\psi)+\bar{\psi}V\psi\right]\det(-\partial_k\nabla_k)\delta(\partial_k A_k)D\vec{A}\,, \tag{A1}$$

$$||\psi||^2=\int\bar{\psi}\psi\det(-\partial_k\nabla_k)\delta(\partial_k A_k)D\vec{A}\,. \tag{A2}$$

Consider another scalar product

$$||\psi||'^2=\int\bar{\psi}\psi\delta(\partial_k A_k)D\vec{A}\,, \tag{A3}$$

which may be related to (A2) by a unitary transform U,

$$U\psi=\det(-\partial_k\nabla_k)^{-1/2}\psi\equiv e^{u/2}\psi\,. \tag{A4}$$

The space with scalar product (A3) is the one Schwinger implicitly used. The canonical transverse operators $A_k^T=P_{kl}A_l$ and $\pi_k=P_{kl}(\hbar/i)\delta/\delta A_l$, where $P_{kl}=\delta_{kl}-\partial_k\Delta_0{}^{-1}\partial_l$, $\Delta_0=\partial_n\partial_n$, are symmetric with respect to (A3). Upon the similarity transform H goes to $H'=U^{-1}HU$. By virtue of (A1), we have

$$(\psi\,|\,H'\psi)'=(U\psi\,|\,HU\psi)=\int\left[\frac{\hbar^2}{2}g(e^{-u/2}d(e^{u/2}\bar{\psi}),e^{-u/2}d(e^{u/2}\psi))+\bar{\psi}V\psi\right]\delta(\partial_k\nabla_k)D\vec{A}$$

$$=\int\left\{\frac{\hbar^2}{2}\int d\vec{x}\,\mathrm{tr}\left[\left[\frac{\delta\bar{\psi}}{\delta A_k}+\tfrac{1}{2}\bar{\psi}\frac{\delta u}{\delta A_k}\right]\left[\frac{\delta\psi}{\delta A_k}+\tfrac{1}{2}\psi\frac{\delta u}{\delta A_k}\right]\right]+\bar{\psi}V\psi\right\}\delta(\partial_l A_l)D\vec{A}\,. \tag{A5}$$

Now

$$\hbar\frac{\delta\psi}{\delta A_k}=i\pi_k\psi+\partial_k\Lambda \tag{A6}$$

and $\nabla_k \, \delta\psi/\delta A_k = 0$ since ψ is gauge invariant. This gives

$$\Lambda = -D\nabla_k i\pi_k \psi = iD[A_k, i\pi_k\psi] , \tag{A7}$$

where $D = (\nabla_k \partial_k)^{-1} = (\partial_k \nabla_k)^{-1}$ and

$$\hbar \frac{\partial\psi}{\delta A_k} = (\delta_{kl} + i\partial_k D[A_l, \cdot])i\pi_l\psi . \tag{A8}$$

Introducing a set of generators t^a in the Lie algebra $\mathscr{A}$, $\mathrm{tr} t^a t^b = \delta^{ab}$, $[t^a, t^b] = ic^{dab} t^d$ with structure constants c^{dab} completely antisymmetric and writing $A_k = A_k^a t^a$ we may rewrite (A8) in components as

$$\begin{aligned}\hbar \frac{\delta\psi}{\delta A_k^a(\vec{x})} &= \int d\vec{y}[\delta_{kl}\delta^{ad}\delta(\vec{x}-\vec{y}) - \partial_k D^{ab}(\vec{x},\vec{y})c^{bcd}A_l^c(\vec{y})]i\pi_l^d(\vec{y})\psi \\ &= \int d\vec{y}[\delta_{kl}\delta^{ad}\delta(\vec{x}-\vec{y}) - \partial_k D^{ab}(\vec{x},\vec{y})c^{bcd}A_l^c(\vec{y})].i\pi_l^d(\vec{y})\psi \\ &\quad + \tfrac{1}{2}\int d\vec{y}[i\pi_l^d(\vec{y}), \partial_k D^{ab}(\vec{x},\vec{y})c^{bcd}A^c(\vec{y})]\psi ,\end{aligned} \tag{A9}$$

where we have used Schwinger's notation $A.B = \frac{1}{2}AB + \frac{1}{2}BA$ and $\partial D(\vec{x},\vec{y})$ $(D(\vec{x},\vec{y})\overleftarrow{\partial})$ denotes the derivation with respect to the first (second) set of variables.

Now

$$[i\pi_l^d(\vec{y}), D^{ab}(\vec{x},\vec{y})] = -\int d\vec{u}\, d\vec{v}\, D^{ae}(\vec{x},\vec{u})[i\pi_l^d(\vec{y}), (\nabla_n\partial_n)^{ef}(\vec{u},\vec{v})]D^{fb}(\vec{v},\vec{y}) \tag{A10}$$

and

$$(\nabla_n\partial_n)^{ef}(\vec{u},\vec{v}) = \delta^{ef}\Delta_0\delta(\vec{u}-\vec{v}) + c^{egf}A_n^g(\vec{u})\partial_n\delta(\vec{u}-\vec{v}) . \tag{A11}$$

Hence

$$\hbar^{-1}[i\pi_l^d(\vec{y}), (\nabla_n\partial_n)^{ef}(\vec{u},\vec{v})] = c^{edf}[\delta(\vec{y}-\vec{u})\partial_l\delta(\vec{u}-\vec{v}) - \partial_l\partial_n\Delta_0{}^{-1}(\vec{y},\vec{u})\partial_n\delta(\vec{u}-\vec{v})] , \tag{A12}$$

$$\hbar^{-1}[i\pi_l^d(\vec{y}), D^{ab}(\vec{x},\vec{y})] = -c^{edf}D^{ae}(\vec{x},\vec{y})\partial_l D^{fb}(\vec{y},\vec{y}) + \int d\vec{u}\, c^{edf}D^{ae}(\vec{x},\vec{u})\partial_l\partial_n\Delta_0{}^{-1}(\vec{y},\vec{u})\partial_n D^{fb}(\vec{u},\vec{y}) , \tag{A13}$$

and

$$\begin{aligned}\tfrac{1}{2}\hbar^{-1}\int d\vec{y}[i\pi_l^d(\vec{y}), \partial_k D^{ab}(\vec{x},\vec{y})c^{bcd}A_l^c(\vec{y})] &= -\tfrac{1}{2}\int d\vec{y}\, c^{bcd}c^{edf}\partial_k D^{ae}(\vec{x},\vec{y})\partial_l D^{fb}(\vec{y},\vec{y})A_l^c(\vec{y}) \\ &\quad + \tfrac{1}{2}\int d\vec{y}\, d\vec{u}\, c^{bcd}c^{edf}\partial_k D^{ae}(\vec{x},\vec{u})\partial_l\partial_n\Delta_0{}^{-1}(\vec{y},\vec{u})\partial_n D^{fb}(\vec{u},\vec{y})A_l^c(\vec{y}) .\end{aligned} \tag{A14}$$

Next we shall compute

$$\frac{\delta u}{\delta A_k^a(\vec{x})} = -\frac{\delta}{\delta A_a^k(\vec{x})} \mathrm{tr}\ln(-\partial_n\nabla_n) .$$

We have

$$\begin{aligned}[i\pi_l^d(\vec{y}), \mathrm{tr}\ln(-\partial_n\nabla_n)] &= \mathrm{tr}(D[i\pi_l^d(\vec{y}), \partial_n\nabla_n]) \\ &= \hbar\int d\vec{u}\, d\vec{v}\, c^{edf}D^{fe}(\vec{v},\vec{u})[\delta_{ln}\delta(\vec{y}-\vec{u}) - \partial_l\partial_n\Delta_0{}^{-1}(\vec{y},\vec{u})]\partial_n\delta(\vec{u}-\vec{v})\end{aligned} \tag{A15}$$

and by (A8)

$$\begin{aligned}\frac{\delta u}{\delta A_k^a(\vec{x})} &= -c^{eaf}\partial_k D^{fe}(\vec{x},\vec{x}) + \int d\vec{y}[c^{eaf}\partial_n D^{fe}(\vec{y},\vec{y})\partial_k\partial_n\Delta_0{}^{-1}(\vec{x},\vec{y}) + c^{bcd}c^{edf}\partial_k D^{ab}(\vec{x},\vec{y})\partial_l D^{fe}(\vec{y},\vec{y})A_l^c(\vec{y})] \\ &\quad - \int d\vec{y}\, d\vec{u}\, c^{bcd}c^{edf}\partial_k D^{ab}(\vec{x},\vec{y})\partial_l\partial_n\Delta_0{}^{-1}(\vec{y},\vec{u})\partial_n D^{fe}(\vec{u},\vec{u})A_l^c(\vec{y}) .\end{aligned} \tag{A16}$$

We shall simplify the expressions on the right-hand side of (A14) and (A16) employing

$$\int d\vec{v}(\nabla_n\partial_n)^{db}(\vec{u},\vec{v})D^{bf}(\vec{v},\vec{z})=\Delta_0 D^{df}(\vec{u},\vec{z})+c^{dcb}A_n^c(\vec{u})\partial_n D^{bf}(\vec{u},\vec{z})=\delta^{df}\delta(\vec{u}-\vec{z}) \ . \tag{A17}$$

Using the symmetry of D, the antisymmetry of c^{abc}, and integrating by parts we get

$$\begin{aligned}\int d\vec{y}\,d\vec{u}\,c^{bcd}c^{edf}\partial_k D^{ae}(\vec{x},\vec{u})\partial_l\partial_n\Delta_0^{-1}(\vec{y},\vec{u})\partial_n D^{fb}(\vec{u},\vec{y})A^c(\vec{y})&\\ =\int d\vec{y}\,d\vec{u}\,c^{dcb}\partial_l D^{bf}(\vec{y},\vec{u})\overleftarrow{\partial}_n A_l^c(\vec{y})\partial_n\Delta_0^{-1}(\vec{y},\vec{u})\partial_k D^{ae}(\vec{x},\vec{u})c^{edf}&\\ =-\int d\vec{y}\,d\vec{u}[\delta^{df}\partial_n\delta(\vec{y}-\vec{u})\partial_n\Delta_0^{-1}(\vec{y},\vec{u})\partial_k D^{ae}(\vec{x},\vec{u})c^{edf}&\\ +\Delta_0 D^{df}(\vec{y},\vec{u})\overleftarrow{\partial}^n\partial_n\Delta_0^{-1}(\vec{u},\vec{y})\partial_k D^{ae}(\vec{x},\vec{u})c^{edf}]&\\ =-\int d\vec{y}\,c^{edf}\partial_n D^{df}(\vec{y},\vec{y})\overleftarrow{\partial}^n\partial_k D^{ae}(\vec{x},\vec{y})=0&\end{aligned} \tag{A18}$$

and

$$\begin{aligned}-\int d\vec{y}\,d\vec{u}\,c^{bcd}c^{edf}\partial_k D^{ab}(\vec{x},\vec{y})\partial_l\partial_n\Delta_0^{-1}(\vec{y},\vec{u})\partial_n D^{fe}(\vec{u},\vec{u})A_l^c(\vec{y})&\\ =-\int d\vec{y}\,d\vec{u}\,c^{dcb}\partial_l D^{ba}(\vec{y},\vec{x})\overleftarrow{\partial}^k A_l^c(\vec{y})\partial_n\Delta_0^{-1}(\vec{y},\vec{u})\partial_n D^{fe}(\vec{u},\vec{u})c^{edf}&\\ =-\int d\vec{y}\,d\vec{u}\,\delta^{da}\partial_k\delta(\vec{x}-\vec{y})\partial_n\Delta_0^{-1}(\vec{y},\vec{u})\partial_n D^{fe}(\vec{u},\vec{u})c^{edf}&\\ -\int d\vec{y}\,d\vec{u}\,\Delta_0 D^{da}(\vec{y},\vec{x})\overleftarrow{\partial}^k\partial_n\Delta_0^{-1}(\vec{u},\vec{y})\partial_n D^{fe}(\vec{u},\vec{u})c^{edf}&\\ =-\int d\vec{y}\,c^{eaf}\partial_n D^{fe}(\vec{y},\vec{y})\partial_k\partial_n\Delta_0^{-1}(\vec{x},\vec{y})&\\ -\int d\vec{y}\,c^{edf}\partial_n D^{da}(\vec{y},\vec{x})\overleftarrow{\partial}^k\partial_n D^{fe}(\vec{y},\vec{y}) \ .&\end{aligned} \tag{A19}$$

Equations (A14), (A16), (A18), and (A19) give

$$\begin{aligned}\tfrac{1}{2}\hbar^{-1}\int d\vec{y}[i\pi_l^d(\vec{y}),\partial_k D^{ab}(\vec{x},\vec{y})c^{bcd}A_l^c(\vec{y})]+\frac{1}{2}\frac{\delta u}{\delta A_k^a(\vec{x})}&\\ =-\tfrac{1}{2}c^{eaf}\partial_k D^{fe}(\vec{x},\vec{x})+\tfrac{1}{2}\int d\vec{y}(c^{bcd}c^{edf}-c^{ecd}c^{bdf})\partial_k D^{ab}(\vec{x},\vec{y})\partial_l D^{fe}(\vec{y},\vec{y})A_l^c(\vec{y})&\\ -\tfrac{1}{2}\int d\vec{y}\,c^{edf}\partial_n D^{da}(\vec{y},\vec{x})\overleftarrow{\partial}_k\partial_n D^{fe}(\vec{y},\vec{y}) \ .&\end{aligned}$$

Using the Jacobi identity for the structure constants and (A17) again we show that the second and the third term on the right-hand side of (A20) cancel. Hence finally

$$\begin{aligned}H'=\tfrac{1}{2}\int d\vec{x}&\left[\int d\vec{y}[\delta_{kl}\delta^{ad}\delta(\vec{x}-\vec{y})-\partial_k D^{ab}(\vec{x},\vec{y})c^{bcd}A^c(\vec{y})].\pi_l^d(\vec{y})-\frac{i}{2}c^{abc}\partial_k D^{bc}(\vec{x},\vec{x})\right]\\ &\times\left[\int d\vec{y}[\delta_{kn}\delta^{a\rho}\delta(\vec{x}-\vec{z})-\partial_k D^{ae}(\vec{x},\vec{z})c^{ef\rho}A_n^f(\vec{z})].\pi_n^\rho(\vec{z})+\frac{i}{2}c^{aef}\partial_k D^{ef}(\vec{x},\vec{x})\right]+\tfrac{1}{4}\int d\vec{x}\,F_{kl}^a(\vec{x})^2 \ ,\end{aligned} \tag{A21}$$

which is Schwinger's Hamiltonian.[14]

APPENDIX B

We have to prove that

$$\mathscr{T}_{a_t}^{T/a_t}\underset{a_t\to 0}{\longrightarrow}\exp\left[-\frac{1}{\hbar}T\left[-\frac{\hbar^2}{2}\Delta+V\right]\right] \tag{B1}$$

strongly, see (3.4) and compare.[24] Let U_{a_t} be the operator with the kernel

$$U_{a_t}(g',g)=\exp\left[\frac{1}{\hbar}\mathrm{Re}\,\mathrm{tr}\left[\sum_{b\subset\Lambda_s} a_t^{-1}a_s^{d-2}(g_b'g_b^{-1}-1)\right]\right]\Big/\int\exp\left[\frac{1}{\hbar}\mathrm{Re}\,\mathrm{tr}\left[\sum_{b\subset\Lambda_s} a_t^{-1}a_s^{d-2}(g_b-1)\right]\right]dg\ . \tag{B2}$$

Then

$$\mathscr{T}_{a_t}^{T/a_t}=(U_{a_t}W_{a_t})^{T/a_t}\ . \tag{B3}$$

$a_t\to W_{a_t}$ is a one-parameter self-adjoint semigroup with bounded generator

$$V=-\mathrm{Re}\,\mathrm{tr}\left[\sum_{b\subset\Lambda_s} a^{d-4}(g_p-1)\right]\ .$$

U_{a_t} is a one-parameter family of bounded self-adjoint operators. We shall show that U_{a_t} is strongly continuous, $U_{a_t}\underset{a_t\to 0}{\to}1$ strongly, and that $\hbar d/dt\,|_{t=0}U_{a_t}=(\hbar^2/2)\Delta$, where the derivative is taken in the strong sense. Equation (B1) becomes a Trotter-type formula and actually the proof of the Trotter formula given in Ref. 25, Theorem VIII.30, extends immediately, yielding (B1). Thus we are left with showing that for the one-parameter family u_{a_t}: $L^2(G,dg)\to L^2(G,dg)$, with the kernel

$$u_{a_t}(g',g)=\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(g'g^{-1}-1)\right]\Big/\int\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(g-1)\right]dg\ , \tag{B4}$$

u_{a_t} is strongly continuous, $u\underset{a_t\to 0}{\to}1$ strongly, and $\hbar du_{a_t}/da_t\,|_0=(\hbar^2/2)\Delta$, where Δ is taken with respect to the metric $\langle A,A\rangle=a_s^{d-2}\mathrm{tr}A^2$ on $\mathscr{A}$. The Fourier transform on G diagonalizes u_{a_t}: On functions carrying irreducible representation σ it becomes multiplication by $\alpha_\sigma(a_t)$:

$$\alpha_\sigma(a_t)=\frac{1}{\dim\sigma}\int\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(g-1)\right]\bar{\chi}_\sigma(g)dg\Big/\int\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(g-1)\right]dg\ , \tag{B5}$$

where χ_σ is the trace of σ. Equation (B5) gives immediately

$$0\le\alpha_\sigma(a_t)\le 1\ , \tag{B6}$$

$$\alpha_\sigma(a_t)\text{ is continuous in }a_t\ . \tag{B7}$$

It is also clear that $\alpha_\sigma(a_t)\underset{a_t\to 0}{\to}1$ since $\mathrm{Re}\,\mathrm{tr}(g-1)$ is negative except at $g=1$. This gives

$$\|u_{a_t}\|\le 1\ ,\quad u_{a_t}\text{ is strongly continuous}\ ,\quad u_{a_t}\underset{a_t\to 0}{\to}1\text{ strongly}\ . \tag{B8}$$

Now we shall prove that

$$\hbar\frac{d\alpha_\sigma(a_t)}{da_t}\bigg|_{t=0}=\hbar\lim_{a_t\to 0}\frac{\alpha_\sigma(a_t)-1}{a_t}=\frac{1}{\dim\sigma}\frac{\hbar^2}{2}\Delta\chi_\sigma(1)\ . \tag{B9}$$

Indeed

$$\hbar\frac{\alpha_\sigma(a_t)-1}{a_t}$$

$$=\frac{\frac{\hbar}{\dim\sigma}\int\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(g-1)\right]\overline{[\chi_\sigma(g)-\chi_\sigma(1)]}dg}{\int\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(g-1)\right]dg}$$

$$=\frac{\frac{\hbar}{(\dim\sigma)a_t}\int_{a_t^{-1/4}\mathscr{O}}\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(e^{ia_t^{1/2}A}-1)\right][\overline{\chi}_\sigma(e^{ia_t^{1/2}A})-\overline{\chi}_\sigma(1)]v(a_t,A)dA+\frac{1}{a_t}e^{-\mathscr{O}(a_t^{-1/2})}}{\int_{a_t^{-1/4}\mathscr{O}}\exp\left[\frac{1}{\hbar}a_t^{-1}a_s^{d-2}\mathrm{Re}\,\mathrm{tr}(e^{ia_t^{1/2}A}-1)\right]v(a_t,A)dA+e^{-\mathscr{O}(a_t^{-1/2})}}, \tag{B10}$$

where $\mathscr{O}$ is a bounded neighborhood of zero in $\mathscr{A}$, and the invariant volume element

$$v(a_t,A)=a_t^{\dim\mathscr{A}/2}[\mathrm{const}+\mathscr{O}(a_t^{-1}\mathrm{tr}A^2)]\;. \tag{B11}$$

Since

$$\chi_\sigma(e^{ia_t^{1/2}A})-\chi_\sigma(1)=\tfrac{1}{2}a_t\chi''_{\sigma ab}(1)A^aA^b+\mathscr{O}(a_t^2\mathrm{tr}A^4)$$

where

$$\chi''_{\sigma ab}(1)=\frac{d}{ds_1}\frac{d}{ds_2}\bigg|_0\chi_\sigma(e^{i(s_1t^a+s_2t^b)})$$

and

$$\mathrm{Re}\,\mathrm{tr}(e^{ia_t^{1/2}A}-1)=-\tfrac{1}{2}a_t\mathrm{tr}A^2+\mathscr{O}(a_t^2\mathrm{tr}A^4)\;, \tag{B12}$$

it is easy to show using the Lebesgue theorem that

$$\hbar\frac{\alpha_\sigma(a_t)-1}{a_t}\underset{a_t\to 0}{\longrightarrow}\frac{\hbar}{\dim\sigma}\frac{\int\exp[-(1/2\hbar)a_s^{d-2}\mathrm{tr}A^2]\frac{1}{2}\chi''_{\sigma ab}(1)A^aA^bdA}{\int\exp[-(1/2\hbar)a_s^{d-2}\mathrm{tr}A^2]dA}$$

$$=\frac{1}{\dim\sigma}\frac{\hbar^2}{2}\Delta\chi_\sigma(1)\equiv\hbar\alpha'_\sigma(0)\;. \tag{B13}$$

Since $(\hbar^2/2)\Delta$ is also diagonalized by the Fourier transform of $L^2(G)$ and in the σ subspace is multiplication by $(1/\dim\sigma)(\hbar^2/2)\Delta\chi_\sigma(1)$ we see that on $D\subset L^2(G)$, $D=\mathrm{span}\{\chi_\sigma\}$, $\hbar du/da_t|_0\equiv A$ and $(\hbar^2/2)\Delta$ coincide. A is symmetric, and D is an essential domain for $(\hbar^2/2)\Delta$, hence showing that $A=\overline{A|_D}$ will yield the equality of both operators. Let us notice that if $f\in D(A)$ and $f=\sum_\sigma f_\sigma$, where f_σ are the Fourier components of f, then $(Af)_\sigma=\hbar\alpha'_\sigma(0)f_\sigma$. $\overline{A|_D}$ is just defined by this expression with the domain consisting of all

$$f\in L^2(G)\text{ such that }\sum_\sigma\alpha'_\sigma(0)^2\|f_\sigma\|^2<\infty\;.$$

Thus we have to show that any such f is in $D(A)$. It is enough to show that there exist C, ϵ, such that for all $a_t<\epsilon$ and all σ

$$-\hbar\frac{\alpha_\sigma(a_t)-1}{a_t}\le C(1+|\alpha'_\sigma(0)|)\;. \tag{B14}$$

Indeed if (B14) holds, $f\in D(\overline{A|_D})$ and I is a finite set of irreducible representations of G, then

$$\hbar\frac{1}{a_t}(u_{a_t}-1)\sum_{\sigma\in I}f_\sigma=\hbar\sum_{\sigma\in I}\frac{\alpha_\sigma(a_t)-1}{a_t}f_\sigma$$

converges uniformly in $a_t<\epsilon$ over the net of I's. Hence

$$\lim_{a_t\to 0}\frac{1}{a_t}(u_{a_t}-1)f$$

exists and $f\in D(A)$. Equation (B14) follows from (B10) and

$$\begin{aligned}|\chi_\sigma(e^{ia_t{}^{1/2}A})-\chi_\sigma(1)| &\le \left|\int_0^1 ds\,(1-s)\frac{d^2}{ds^2}\chi_\sigma(e^{ia_t{}^{1/2}sA})\right|\\ &\le \int_0^1 ds\,(1-s)\left|\mathrm{tr}\frac{d^2}{ds^2}e^{ia_t{}^{1/2}sdu_\sigma(A)}\right|\\ &= \int_0^1 ds\,(1-s)a_t\,|\,\mathrm{tr}\,du_\sigma(A)^2e^{ia_t{}^{1/2}sdu_\sigma(A)}\,|\\ &\le \int_0^1 ds\,(1-s)a_t\mathrm{tr}\,du_\sigma(A)^2\le -\tfrac{1}{2}a_t\Delta\chi_\sigma(1)\mathrm{tr}A^2\\ &=\frac{1}{\hbar}a_t\dim\sigma\,|\,\alpha'(0)\,|\,\mathrm{tr}A^2\,.\end{aligned}$$

APPENDIX C

Consider a general case of a Riemannian manifold Q with metric g on which a compact group $\mathscr{G}$ of isometries acts freely. To any vector $\widetilde{Y}$ on the quotient space $\widetilde{Q}$ we may assign a unique vector field Y on Q projecting to $\widetilde{Y}$ and normal to the $\mathscr{G}$ orbits. Of course, Y is $\mathscr{G}$ invariant. The quotient Riemannian structure on $\widetilde{Q}$ is given by $\widetilde{g}(\widetilde{Y}_1,\widetilde{Y}_2)=g(Y_1,Y_2)$. Let ∇ and $\widetilde{\nabla}$ denote the corresponding metric covariant derivatives. It is easily seen that if $\widetilde{\nabla}_{\widetilde{Y}_1}\widetilde{Y}_2=\widetilde{Y}_3$ then $(\nabla_{Y_1}Y_2)^{\perp}\equiv\nabla^{\perp}_{Y_1}Y_2=Y_3$, where by $(\ )^{\perp}$ and $(\ )^{\parallel}$ we denote the normal and tangential components of vector field on Q with respect to the $\mathscr{G}$ orbits. Let X be an element of the Lie algebra Q of G. Let $\bar{X}$ be the vector field on Q corresponding to the infinitesimal action of X on Q. We have

$$g(\nabla_{Y_1}Y_2\,|\,\bar{X})=-g(Y_2\,|\,\nabla_{Y_1}\bar{X})=-g(Y_2\,|\,\nabla_{\bar{X}}Y_1)=g(\nabla_{\bar{X}}Y_2\,|\,Y_1)=-g(\nabla_{Y_2}Y_1\,|\,\bar{X})\,, \tag{C1}$$

where we have used the $\mathscr{G}$ symmetry of Y_i which yields

$$[X,Y_i]=0\,. \tag{C2}$$

Choose a basis $\bar{X}_i$ of tangential vector fields and a basis Y_α of (local) normal fields to the $\mathscr{G}$ orbits on Q. Let

$$g(\nabla_{Y_\alpha}Y_\beta\,|\,\bar{X}_i)\equiv B_{\alpha\beta i}=-B_{\beta\alpha i}\,, \tag{C3}$$

and let

$$g(Y_\alpha\,|\,\nabla_{\bar{X}_i}\bar{X}_j)=g(Y_\alpha\,|\,\nabla_{\bar{X}_j}\bar{X}_i)=-g(\nabla_{\bar{X}_i}Y_\alpha\,|\,\bar{X}_j)=-g(\nabla_{Y_\alpha}\bar{X}_i\,|\,\bar{X}_j)\equiv A_{\alpha ij}=A_{\alpha ji} \tag{C4}$$

be the second fundamental from of the $\mathscr{G}$ orbits.

Introduce now the $\mathscr{G}$-invariant function

$$h=\tfrac{1}{2}\,\mathrm{Tr}\ln(g_{ij}) \tag{C5}$$

and the function $\tilde{h}$ it defined on $\tilde{Q}$, where $g_{ij}=g(\bar{X}_i,\bar{X}_j)$. e^h is proportional to the volume of the $\mathcal{G}$ orbits. Let us compute

$$\langle \tilde{Y}_1 \mid \nabla_{\tilde{Y}_2} d\tilde{h} \rangle = [Y_2(\langle Y_1 \mid dh \rangle) - \langle \nabla_{Y_2} Y_1 \mid dh \rangle]^{\sim} = \langle Y_1 \mid \nabla_{Y_2} dh \rangle^{\sim}, \tag{C6}$$

$$\langle Y_1 \mid dh \rangle = \tfrac{1}{2} g^{ji} [g(\nabla_{Y_1} \bar{X}_i \mid \bar{X}_j) + g(\bar{X}_i \mid \nabla_{Y_1} \bar{X}_j)] . \tag{C7}$$

But

$$g(\nabla_{Y_1} \bar{X}_i \mid \bar{X}_j) = g(\nabla_{\bar{X}_i} Y_1 \mid \bar{X}_j) = -g(Y_1 \mid \nabla_{\bar{X}_i} \bar{X}_j) = -g(Y_1 \mid \nabla_{\bar{X}_j} \bar{X}_i) = g(\nabla_{Y_1} \bar{X}_j \mid \bar{X}_i) . \tag{C8}$$

Hence

$$\langle Y_1 \mid dh \rangle = g^{ji} g(\nabla_{Y_1} \bar{X}_i \mid \bar{X}_j) \tag{C9}$$

and

$$Y_2(\langle Y_1 \mid dh \rangle) = -2 g^{jk} g(\bar{X}_k \mid \nabla_{Y_2} \bar{X}_l) g^{li} g(\nabla_{Y_1} \bar{X}_i \mid \bar{X}_j) + g^{ji} g(\nabla_{Y_2} \nabla_{Y_1} \bar{X}_i \mid \bar{X}_j) + g^{ji} g(\nabla_{Y_1} \bar{X}_i \mid \nabla_{Y_2} \bar{X}_j) , \tag{C10}$$

$$\begin{aligned} \langle \nabla_{Y_2} Y_1 \mid dh \rangle &= g^{ji} g(\nabla_{\nabla_{Y_2} Y_1} \bar{X}_i \mid \bar{X}_j) = g^{ji} g(\nabla_{\bar{X}_i} \nabla_{Y_2} Y_1 \mid \bar{X}_j) \\ &= g^{ji} g(\nabla_{Y_2} \nabla_{\bar{X}_i} Y_1 \mid \bar{X}_j) + g^{ji} g(R(\bar{X}_i, Y_2) Y_1 \mid \bar{X}_j) \\ &= g^{ji} g(\nabla_{Y_2} \nabla_{Y_1} \bar{X}_i \mid \bar{X}_j) + g^{ji} g(R(\bar{X}_i, Y_2) Y_1 \mid \bar{X}_j) , \end{aligned} \tag{C11}$$

where R is the curvature tensor on Q and we have used $[\bar{X}_i, \nabla_{Y_2} Y_1]=0$ following from the $\mathcal{G}$ invariance of $\nabla_{Y_2} Y_1$. Moreover,

$$g^{jk} g(\bar{X}_k \mid \nabla_{Y_2} \bar{X}_l) \bar{X}_j = (\nabla_{Y_2} \bar{X}_l)^{||} . \tag{C12}$$

Equations (C6) and (C10)–(C12) give

$$\langle Y_1 \mid \nabla_{Y_2} dh \rangle = -g^{ji} g(R(\bar{X}_i, Y_2) Y_1 \mid \bar{X}_j) - g^{ji} g((\nabla_{Y_1} \bar{X}_i)^{||} \mid (\nabla_{Y_2} \bar{X}_j)^{||}) + g^{ji} g((\nabla_{Y_1} \bar{X}_i)^{\perp} \mid (\nabla_{Y_2} \bar{X}_j)^{\perp}) . \tag{C13}$$

We may write (C13) as

$$\langle Y_\alpha \mid \nabla_{Y_\beta} dh \rangle = -R^{i}{}_{i\beta\alpha} - A_{\alpha ij} A_{\beta}{}^{ij} + B_{\alpha\gamma i} B_{\beta}{}^{\gamma i} , \tag{C14}$$

and contracting α and β

$$\tilde{\Delta}_{\mathrm{LB}} \tilde{h} = (-R_{i\alpha}{}^{\alpha i} - A_{\alpha ij} A^{\alpha ij} + B_{\alpha\beta i} B^{\alpha\beta i})^{\sim} . \tag{C15}$$

Finally, for $\tilde{g}(d\tilde{h} \mid d\tilde{h})$ we get from (C4) and (C9)

$$\tilde{g}(d\tilde{h} \mid d\tilde{h}) = (A_{\alpha i}{}^{i} A^{\alpha}{}_{j}{}^{j})^{\sim} . \tag{C16}$$

For continuum theory Q is flat. We shall compute formally $\tilde{\Delta}_{\mathrm{LB}} \tilde{h}$ and $\tilde{g}(d\tilde{h} \mid d\tilde{h})$ for this case. In fact, these expressions diverge and need regularization (e.g., the lattice one). The vectors $Y \equiv (Y_k^a(\vec{x}))$ are normal to the $\mathcal{G}$ orbit at $\vec{A} \equiv (A_k^a(\vec{x}))$, if $\nabla_k Y_k = 0$ where $\nabla_k \equiv \nabla_k(\vec{A})$ is the covariant derivative. For $\Lambda = (\Lambda^a(\vec{x}))$ the vector field $\bar{X}$ is $\nabla\Lambda$ if $X \equiv \Lambda$. Notice that

$$\nabla_Y(\nabla\Lambda) = -i[Y, \Lambda] \tag{C17}$$

and

$$g(\nabla_{Y_\alpha}(\nabla\Lambda_j) \mid Y_\beta) = -i\langle [Y_{\alpha k}, \Lambda_j] \mid Y_{\beta k} \rangle = \int d\vec{x}\, c^{abc} Y_{\alpha k}^b(\vec{x}) \Lambda_j^c(\vec{x}) Y_{\beta k}^a(\vec{x}) = B_{\alpha\beta j} , \tag{C18}$$

$$B_{\alpha\beta j} B_{\gamma\delta n} = \int d\vec{x}\, c^{abc} Y_{\alpha k}^b(\vec{x}) \Lambda_j^c(\vec{x}) Y_{\beta k}^a(\vec{x}) \int d\vec{y}\, c^{def} Y_{\gamma l}^e(\vec{y}) \Lambda_n^f(\vec{y}) Y_{\delta l}^d(\vec{y}) . \tag{C19}$$

Similarly,

$$g(\nabla_{\nabla\Lambda_i}(\nabla\Lambda_j)\,|\,Y_\beta)=\int d\vec{x}\,c^{abc}(\nabla_k\Lambda_i)^b(\vec{x})\Lambda_j^c(\vec{x})Y^a_{\beta k}(\vec{x})=A_{\beta ij}\ , \tag{C20}$$

$$A_{\beta ij}A_{\delta mn}=\int d\vec{x}\,c^{abc}(\nabla_k\Lambda_i)^b(\vec{x})\Lambda_j^c(\vec{x})Y^a_{\beta k}(\vec{x})\int d\vec{y}\,c^{def}(\nabla_l\Lambda_m)^e(\vec{y})\Lambda_n^f(\vec{y})Y_l^d(\vec{y})\ . \tag{C21}$$

We shall first compute

$$B_{\alpha\beta j}B^{\alpha}{}_{\delta n}+A_{\beta ij}A_{\delta}{}^{i}{}_{n}\ . \tag{C22}$$

To this end we have simply to treat (C19) as a quadratic form in Y_α and Y_γ and compute its total trace (in both the tangential and normal directions). Hence

$$(\text{C22})=\int d\vec{x}\,c^{abc}c^{dbf}\Lambda_j^c(\vec{x})\Lambda_n^f(\vec{x})Y^a_{\beta k}(\vec{x})Y^d_{\delta k}(\vec{x})\ .$$

Now we shall contract β and δ in (C22). To do this we have to treat (C22) as a quadratic form in Y_β, Y_δ, compute its total trace, and subtract the trace in tangential direction. This gives

$$B_{\alpha\beta j}B^{\alpha\beta}{}_{n}+A_{\beta ij}A^{\beta i}{}_{n}=\int d\vec{x}\,c^{abc}c^{abf}\Lambda_j^c(\vec{x})\Lambda_n^f(\vec{x})\delta(0)\delta_{kk}-\int d\vec{x}\,c^{abc}c^{dbf}\Lambda_j^c(\vec{x})\Lambda_n^f(\vec{x})(\nabla_k\Delta^{-1}\nabla_k)^{ad}(\vec{x},\vec{x})\ . \tag{C23}$$

Next we contract j and n:

$$\begin{aligned}B_{\alpha\beta j}B^{\alpha\beta j}+A_{\beta ij}A^{\beta ij}&=-\int d\vec{x}\,c^{abc}c^{abf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})\delta(0)\delta_{kk}+\int d\vec{x}\,c^{abc}c^{dbf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})(\nabla_k\Delta^{-1}\nabla_k)^{ad}(\vec{x},\vec{x})\\&=\int d\vec{x}(c^{abc}c^{dbf}c^{abf}c^{dbc})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})(\nabla_k\delta^{-1})^{ad}(\vec{x},\vec{x})\\&\quad-\int d\vec{x}\,c^{abc}c^{abf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})\delta(0)(\delta_{kk}-1)\\&=\int d\vec{x}(2c^{abf}c^{dbc}+c^{abd}c^{cbf})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\quad-\int d\vec{x}\,c^{abc}c^{abf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})\delta(0)(\delta_{kk}-1)\ ,\end{aligned} \tag{C24}$$

where we have used integration by parts and Jacobi identity for the structure constants to remove the second ∇_k from $\nabla_k\Delta^{-1}\nabla_k$. Similarly we compute

$$\begin{aligned}A_{\beta ij}A^{\beta}{}_{mn}&=\int d\vec{x}\,c^{abc}c^{aef}(\nabla_k\Lambda_i)^b(\vec{x})\Lambda_j^c(\vec{x})(\nabla_k\Lambda_m)^e(\vec{x})\Lambda_n^f(\vec{x})\\&\quad-\int d\vec{x}\,d\vec{y}\,c^{abc}c^{def}(\nabla_k\Lambda_i)^b(\vec{x})\Lambda_j^c(\vec{x})(\nabla_l\Lambda_m)^e(\vec{y})\Lambda_n^f(\vec{y})(\nabla_k\Delta^{-1}\nabla_l)^{ad}(\vec{x},\vec{y})\ ,\end{aligned} \tag{C25}$$

$$\begin{aligned}A_{\beta ij}A^{\beta}{}_{m}{}^{j}&=-\int d\vec{x}\,c^{abc}c^{aef}(\nabla_k\Lambda_i)^b(\vec{x})(\nabla_k\Lambda_m)^e(\vec{x})(\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\quad+\int d\vec{x}\,d\vec{y}\,c^{abc}c^{def}(\nabla_k\Lambda_i)^b(\vec{x})(\nabla_l\Lambda_m)^e(\vec{y})(\Delta^{-1})^{cf}(\vec{x},\vec{y})(\nabla_k\Delta^{-1}\nabla_l)^{ad}(\vec{x},\vec{y})\\&=\int d\vec{x}\,c^{abc}c^{aef}\Lambda_i^b(\vec{x})(\Delta\Lambda_m)^e(\vec{x})(\Delta^{-1})^{cf}(\vec{x},\vec{x})+\int d\vec{x}\,c^{abc}c^{aef}\Lambda_i^b(\vec{x})(\nabla_k\Lambda_m)^e(\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\quad+\int d\vec{x}\,c^{abc}c^{aef}\Lambda_i^b(\vec{x})(\nabla_k\Lambda_m)^e(\vec{x})(\nabla_k\Delta^{-1})^{fc}(\vec{x},\vec{x})\\&\quad-\int d\vec{x}\,d\vec{y}\,c^{abc}c^{def}\Lambda_i^b(\vec{x})(\Delta\Lambda_m)^e(\vec{y})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{y})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{y})\\&\quad-\int d\vec{x}\,c^{abc}c^{aef}\Lambda_i^b(\vec{x})(\Delta\Lambda_m)^e(\vec{x})(\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\quad+\int d\vec{x}\,d\vec{y}\,c^{abc}c^{def}(\nabla_k\Lambda_i)^b(\vec{x})(\nabla_l\Lambda_m)^e(\vec{y})(\nabla_l\Delta^{-1})^{fc}(\vec{y},\vec{x})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{y})\ ,\end{aligned} \tag{C26}$$

$$\begin{aligned}A_{\beta ij}A^{\beta ij}&=-\int d\vec{x}(c^{abc}c^{aef}+c^{abf}c^{aec})(\nabla_k\Delta^{-1})^{eb}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\quad+\int d\vec{x}\,c^{abc}c^{dbf}(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\quad+\int d\vec{x}\,d\vec{y}\,c^{abc}c^{def}(\nabla_k\Delta^{-1}\nabla_l)^{be}(\vec{x},\vec{y})(\nabla_l\Delta^{-1})^{fc}(\vec{y},\vec{x})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{y})\ .\end{aligned} \tag{C27}$$

The last term is equal to

$$\begin{aligned}&\int d\vec{x}\,c^{abc}c^{dec}(\nabla_k\Delta^{-1})^{be}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})\\&\qquad-\int d\vec{x}\,d\vec{y}\,c^{abc}c^{def}(\nabla_k\Delta^{-1})^{be}(\vec{x},\vec{y})(\nabla_l\Delta^{-1})^{fc}(\vec{y},\vec{x})(\nabla_k\Delta^{-1}\nabla_l)^{ad}(\vec{x},\vec{y})\ ,\end{aligned} \tag{C28}$$

where the second term is again the same except for the sign. Hence the last term in (C27) is equal to

$$\tfrac{1}{2}\int d\vec{x}\, c^{abc}c^{dec}(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{be}(\vec{x},\vec{x})$$

and

$$A_{\beta ij}A^{\beta ij}=-\tfrac{1}{2}\int d\vec{x}(c^{abf}c^{dbc}-c^{abd}c^{cbf})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\ . \tag{C29}$$

Finally

$$\langle Y_\beta\,|\,dh\rangle=\mathrm{tr}\nabla\Delta^{-1}[Y_\beta,\cdot]=\int d\vec{x}(\nabla_k\Delta^{-1})^{ab}(\vec{x},\vec{x})c^{bda}Y^{d}_{\beta k}(\vec{x})\ . \tag{C30}$$

Hence

$$g(dh\,|\,dh)=A_{\beta i}{}^{i}A^{\beta}{}_{j}{}^{j}=\int d\vec{x}\,c^{abd}c^{cbf}(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\ . \tag{C31}$$

Gathering (C15), (C24), and (C29) we get

$$\begin{aligned}\widetilde{\Delta}_{\rm LB}\widetilde{h}=3\Big[&\int d\vec{x}\,c^{abf}c^{dbc}(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&-\int d\vec{x}\,c^{abc}c^{abf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})\delta(0)(\delta_{kk}-1)\Big]^{\sim}\end{aligned} \tag{C32}$$

and

$$\begin{aligned}\delta\widetilde{V}&=\frac{\hbar^2}{4}[\widetilde{\Delta}_{\rm LB}\widetilde{h}+\tfrac{1}{2}\widetilde{g}(d\widetilde{h}\,|\,d\,\widetilde{h})]\\&=\frac{\hbar^2}{2}\Big[\int d\vec{x}(3c^{abf}c^{dbc}+\tfrac{1}{2}c^{abd}c^{cbf})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&\qquad-\int d\vec{x}\,c^{abc}c^{abf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})\delta(0)(\delta_{kk}-1)\Big]^{\sim}\ .\end{aligned} \tag{C33}$$

$\delta\widetilde{V}$ may be compared to the geometric objects like the scalar curvatures $\widetilde{r}$ and $\hat{r}$ of $\widetilde{Q}$ and of the $\mathscr{G}$ orbits, respectively. For the general situation we consider here

$$\hat{r}=R_{ij}{}^{ji}+A_{\alpha i}{}^{i}A^{\alpha}{}_{j}{}^{j}-A_{\alpha ij}A^{\alpha ij}\ , \tag{C34}$$

$$\widetilde{r}=R_{\alpha\beta}{}^{\beta\alpha}+3B_{\alpha\beta i}B^{\alpha\beta i}\ . \tag{C35}$$

(See Ref. 26, Theorems 1 and 2.)

In our case

$$\hat{r}=\tfrac{1}{2}\Big[\int d\vec{x}(c^{abf}c^{dbc}+c^{abd}c^{cbf})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\Big]^{\sim} \tag{C36}$$

and

$$\begin{aligned}\widetilde{r}=\frac{3}{2}\Big[&\int d\vec{x}(5c^{abf}c^{dbc}+c^{abd}c^{cbf})(\nabla_k\Delta^{-1})^{ad}(\vec{x},\vec{x})(\nabla_k\Delta^{-1})^{cf}(\vec{x},\vec{x})\\&-3\int d\vec{x}\,c^{abc}c^{abf}(\Delta^{-1})^{cf}(\vec{x},\vec{x})\delta(0)(\delta_{kk}-1)\Big]^{\sim}\ .\end{aligned} \tag{C37}$$

Hence

$$\delta\widetilde{V}=\frac{\hbar^2}{4}(\tfrac{1}{3}\widetilde{r}+\hat{r}-A_{\alpha i}{}^{i}A^{\alpha}{}_{i}{}^{i})\ . \tag{C38}$$

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