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Classes of invariance and infrared divergences in scalar quantum electrodynamics

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We construct, in the simpler case of scalar QED, a manifestly gauge-invariant perturbation expansion in terms of an infinite set of completely transverse vertices which we call "classes of invariance." We use these classes to investigate the infrared divergences of the *S*-matrix elements. The mechanism of the exponentiation of the infrared divergences and the general proof of the infrared existence of the Bloch-Nordsieck cross sections, of the Kinoshita-Lee-Nauenberg probabilities, and of the transition amplitudes between coherent states is presented in this context.

I. INTRODUCTION

The purpose of our work is to analyze the organization of gauge invariance and of infrared divergences in perturbative gauge field theories. This paper is devoted to the Abelian part of the subject and expects to contribute in part to a better understanding of the physically meaningful gauge-invariant structure of the gauge field theories. In the literature, we may distinguish several types of approaches to this problem. A first approach[1] introduces new gauge-invariant variables of the type $\oint A_\mu dx_\mu$ and tries to develop the perturbation in a manifestly gauge-invariant way. Another approach has been achieved by Cvitanovic *et al.*[2] who study the group structure of the perturbation expansion in order to classify the Feynman contributions into gauge-invariant sectors. Such sectors appear to be important if, for instance, we wish to discover the cancellations which occur between Feynman graphs of the same sector, of powers, and powers of logarithms of a given asymptotic regime (for example, Regge behavior[3]). The fact that such spurious powers and spurious powers of logarithms are developed by Feynman graphs separately and canceled within gauge sectors indicate that rearrangement of Feynman graphs in each sector into more physically meaningful objects might have to be discovered. In other words, it appears to us that a deeper structure should exist inside each invariant sector and should describe more completely the gauge dependence of the amplitude and, as a consequence, the gauge independence of the *S* matrix. This implies that, in addition to the classification according to the SU(N) weight of the Feynman amplitudes, we should also investigate the Feynman expressions themselves, split them into pieces at each order of perturbation, and collect the pieces differently in a way which manifestly expresses the gauge dependence of the theory.

The first goal of this paper is to perform such a reconstruction of the perturbative expansion in the simpler case of scalar QED (generalization to spinor QED can be translated easily). The main idea which is applied to construct this new perturbation theory is that the group of invariance for the total Lagrangian which describes the evolution of the interpolating field and the group of invariance for the free Lagrangian responsible for the in and out asymptotic fields should be the same. This is achieved by performing a dressing transformation of the matter field in such a way that the Abelian transformation of the photon in the "photon cloud" induces the usual local (space-time dependent) transformation of the matter field. Such a dressing transformation involves necessarily hard, soft, and virtual photons and becomes more and more singular when the photons become softer and softer. The soft-photon part of this transformation will be used to cure the infrared troubles of the *S* matrix while the remaining-photons part is used to express the new Green's functions in a way which shows trivially the gauge dependence (or independence) of the theory; this last property is due to the structure of the vertices of the new Lagrangian which we call "classes of invariance."

The classes we introduce in this paper provide manifestly gauge-invariant expressions for the *S* matrix amplitudes. We believe that these classes will clarify the organization of asymptotic behaviors, and a first sign in this direction is that their introduction enables us to study the structure of infrared behavior. This problem which was first explored in the Abelian case by Bloch and Nordsieck[4] in 1937, and then essentially solved in QED by Yennie, Frautschi, and Suura[5] in 1961, remains relatively difficult to understand especially when we pass to non-Abelian field theory. Apart from calculations in the eikonal approximation,[6] a modern approach uses the fact that infrared behavior can, as many asymptotic behaviors, be described by differential equations.[7] In a similar way that scaling behavior of Green's functions can be obtained either from the renormalization-group equations[8] or at the level of Feynman graphs in terms of forests of divergent subgraphs,[9] the infrared behavior can be investigated in the Abelian case at least, either from differential equations or using classes of invariance; we even expect that in the non-Abelian case, where we do not know what differential equation to use for solving the infrared behavior, the classes of invariance will provide the appropriate language for a detailed description of this behavior.

Let us now describe the content of this work. We first emphasize that throughout this paper we shall not mention, except if needed for the definition of the S matrix, the difficulties related to ultraviolet renormalization and in particular we do not introduce $(\phi^\dagger\phi)^2$ counterterms. Such difficulties have been solved completely in Abelian[10] as well as non-Abelian[11] gauge field theories in the framework of the usual perturbation expansion; it is not the purpose of this paper to formulate the renormalization procedure in terms of classes of invariance.

In Sec. II, we define notations and remind the reader of the Ward-Takahashi identities in the language of a generating functional. The integration of the functional integral over the matter fields permits us to define an infinite set of new vertices[12] and we establish the transversality properties of these vertices as a consequence of gauge invariance. In Sec. III, we first decompose the low-order vertices of Sec. II into transversal and longitudinal components and we define the low-order classes of invariance. The end of this section is devoted to the general definition of classes of invariance. This is achieved by performing a matter-field transformation in the functional integral. Such a transformation has been already introduced by Bialynicki-Birula[13] some years ago so that the first part of this paper is a detailed perturbative description of the work of Ref. 13. This transformation is a photon-field-dependent transformation (photon-cloud effect) which disentangles the local and the global gauge-invariance properties of the Lagrangian. This formalism leads directly to completely transversal amplitudes. This transformation is not unique but we show that the corresponding S matrix for the new matter field is unique, manifestly gauge invariant, and formally equal to the S matrix for the usual matter field.

In Sec. IV, we give examples of manifestly gauge-invariant calculations of some S-matrix elements. The classes introduced in Sec. III are purely formal in the sense that they lead for the Green's functions to severe infrared divergences even for matter off the mass shell. In order to cure them, in Sec. V we define an infrared subtraction to the above transformation which leads to an infrared-subtracted theory with infrared-finite off-shell Green's functions. Moreover, we show that there exists a subclass of transformations which lead for the subtracted theory to an infrared-finite S matrix. It is then easy to relate the new matter-field theory to the new subtracted matter-field theory. In this way we exhibit the exponentiation of the infrared-divergent integral of Refs. 4 and 5 and we describe in detail the singularities in the soft-photon momenta. The formal equality of the S matrices between the usual theory and the new theory becomes a strict equality (it is the consequence of the equivalence theorem[14]) once the IR-divergent exponential has been understood.

Finally in Sec. VI, (1) we prove in full generality the infrared existence of the Bloch-Nordsieck[4] cross sections (summation over final soft photons). (2) We also prove a stronger theorem than the Kinoshita-Lee-Nauenberg[15] (KLN) theorem; namely, we prove that if we change the initial state by any number of incoming soft photons, the Bloch-Nordsieck probability changes by a finite amount.[16] The KLN theorem becomes here a consequence of the above theorem by summing over an infinite number of incoming soft photons with a normalizable density matrix. (3) We prove the existence of the transition amplitudes between coherent states.[17] Some speculations about the existence of an S matrix for Abelian gauge fields and some possible generalization of the classes of invariance to non-Abelian theories are discussed in the Conclusion.

II. THE FUNCTIONAL POINT OF VIEW

In this section, we wish to explore formally (without any care for the ultraviolet and infrared divergences) the organization of the Green's functions in regards to their gauge symmetry. A possible approach is to introduce the generating functional for Green's function

$$Z(J,J^\dagger,\chi_\mu)=\int D\phi^\dagger D\phi\, DA_\mu \exp\left\{i\int d^4x\left[\mathcal{L}(\phi^\dagger,\phi,A_\mu)-\frac{1}{2\lambda}(\eta_\mu A_\mu)^2+J^\dagger\phi+\phi^\dagger J+\chi_\mu A_\mu\right]\right\}, \qquad (2.1)$$

where the Lagrangian is

$$\mathcal{L}(\phi^\dagger,\phi,A_\mu)=(\partial_\mu+igA_\mu)\phi^\dagger(\partial_\mu-igA_\mu)\phi-m^2\phi^\dagger\phi-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2 . \qquad (2.2a)$$

Gauge invariance of the Lagrangian (2.2a) means that $\mathcal{L}(\phi^\dagger,\phi,A_\mu)$ is invariant in the local gauge transformation

$$\begin{aligned}&\phi(x)\rightarrow e^{ig\alpha(x)}\phi(x)\ ,\\ &A_\mu(x)\rightarrow A_\mu(x)+\partial_\mu\alpha(x)\ .\end{aligned} \qquad (2.2b)$$

Obviously if $\alpha(x)$ is a constant of space-time, we have only a global gauge transformation and the invariance of the Lagrangian is then related to the property of charge conservation. Owing to the invariance (2.2b), the quadratic form $(\partial_\mu A_\nu-\partial_\nu A_\mu)^2$, which should be the inverse of the photon propagator, is not invertible. As usual we introduce a gauge-fixing term $\eta_\mu A^\mu$ where η_μ can be understood as ∂_μ in Stuckelberg gauges and η_μ in axial gauges. The photon propagator is found to be

$$\pi_{\mu\nu}(k,\eta)=-\frac{i}{k^2}\left[g_{\mu\nu}-\frac{\eta_\mu k_\nu+\eta_\nu k_\mu}{k\cdot\eta}+\frac{\lambda k^2+\eta^2}{(k\cdot\eta)^2}k_\mu k_\nu\right], \qquad (2.3)$$

where η_μ is k_μ in the Stuckelberg gauge and η_μ in the axial gauge. Equation (2.3) may also be written

$$\pi_{\mu\nu}(k,\eta)=[g_{\mu\rho}-k_\mu G_\rho(k,\eta)]\left[-\frac{i}{k^2}\right][g_{\rho\nu}-G_\rho(k,\eta)k_\nu] \qquad (2.4)$$

with

$$G_\rho(k,\eta)=\frac{\eta_\rho\pm\sqrt{\lambda}k_\rho}{k\cdot\eta}\ . \qquad (2.5)$$

From (2.1) and the transformation (2.2b), the Ward-Takahashi[18] identities for connected Green's functions may be obtained as the functional identity:

$$\left[ig\left[J(x)\frac{\partial}{\partial J(x)}-J^\dagger(x)\frac{\partial}{\partial J^\dagger(x)}\right]-\frac{1}{\lambda}\partial^x_\mu\eta_\mu\eta_\rho\frac{\partial}{\partial\chi_\rho(x)}\right]Z_c(J,J^\dagger,\chi_\mu)=-i\partial_\mu\chi_\mu(x)\ , \qquad (2.6)$$

which gives

$$\begin{aligned}g\left[\sum_{i=1}^{n}\delta(z-x_i)-\sum_{i=1}^{m}\delta(z-y_i)\right]&\left\langle\prod_{i=1}^{n}\phi(x_i)\prod_{i=1}^{m}\phi^\dagger(y_i)\prod_{i=1}^{p}A_{\mu_i}(z_i)\right\rangle_c\\ &-\frac{\partial^z_\mu\eta_\mu\eta_\rho}{\lambda}\left\langle\prod_{i=1}^{n}\phi(x_i)\prod_{i=1}^{m}\phi^\dagger(y_i)\prod_{i=1}^{p}A_{\mu_i}(z_i)A_\rho(z)\right\rangle_c=i\delta_{n0}\delta_{m0}\delta_{p1}\partial^z_{\mu_1}\delta(z-z_1)\ .\end{aligned} \qquad (2.7)$$

The consequences of (2.7) for the perturbative expansion is that at each order in the coupling constant g,

$$k^i_{\mu_i}\langle A_{\mu_1}(k_1)\cdots A_{\mu_n}(k_n)\rangle_{c,a}=0\ ,\quad n>2\ , \qquad (2.8)$$

as well as for $n=2$ except for the Born term which satisfies

$$\eta_\mu\pi_{\mu\nu}=-\frac{i\lambda k_\nu}{k\cdot\eta} \qquad (2.9)$$

[in (2.8), c and a mean connected and amputated by the inverse of the free-photon propagator]. The consequence for matter fields are first $n=m$, and

$$k_\rho\left\langle \prod_{i=1}^{n}\phi(p_i)\prod_{i=1}^{n}\phi^\dagger(p_i')\prod_{i=1}^{p}A_{\mu_i}(k_i)A_\rho(k)\right\rangle_{c,a}$$
$$=g\left[\sum_{j=1}^{n}\left\langle \prod_{i=1}^{n}\phi(p_i)\prod_{\substack{i=1\\ i\neq j}}^{n}\phi^\dagger(p_i')\phi^\dagger(p_j'-q)\prod_{i=1}^{p}A_{\mu_i}(k_i)\right\rangle_{c,a}-\sum_{j=1}^{n}\left\langle \prod_{\substack{i=1\\ i\neq j}}^{n}\phi(p_i)\phi(p_j+q)\prod_{i=1}^{n}\phi^\dagger(p_i')\prod_{i=1}^{p}A_{\mu_i}(k_i)\right\rangle_{c,a}\right], \tag{2.10}$$

where ϕ and A_μ have ingoing momentum, $\phi^\dagger$ has outgoing momentum, and where c and a mean connected and amputated for photons only (throughout this paper we shall consider that inside the Feynman graphs the matter lines which do not form a loop go from an incoming ϕ field to an outgoing $\phi^\dagger$ field, irrespective of the fact that the fields ϕ and $\phi^\dagger$ represent a particle or an antiparticle). Equations (2.8) and (2.10) describe the transversality properties of the connected Green's functions.

Instead of treating the functional (2.1) along the usual perturbative approach (namely, by replacing the fields ϕ, $\phi^\dagger$, and A_μ by $i\delta/\delta J^\dagger$, $-i\delta/\delta J$, and $-i\delta/\delta\chi_\mu$, respectively, in the interacting part of the Lagrangian and then by integrating explicitly the free Lagrangian part as a Gaussian). We shall integrate the Gaussian integral over ϕ and $\phi^\dagger$. We define

$$Z_0(J^\dagger,J,A_\mu)=\int D\phi^\dagger D\phi\,\exp\left[i\int d^4p\,d^4p'\phi^\dagger(p')K(p',p,A)\phi(p)+i\int d^4p[\phi^\dagger(p)J(p)+J^\dagger(p)\phi(p)]\right] \tag{2.11}$$

with

$$K(p',p,A)=\delta(p-p')(p^2-m^2)-g(p+p')_\mu\int\delta^4kA_\mu(k)\delta(p'-p-k)$$
$$+g^2\int\delta^4k_1\delta^4k_2A_\mu(k_1)A_\mu(k_2)\delta(p'-p-k_1-k_2)\,, \tag{2.12}$$

which we also write

$$K(p',p,A)=(p'^2-m^2)\left[\delta(p-p')-\frac{\chi(p,p',A)}{p'^2-m^2}\right]. \tag{2.13}$$

In (2.12) and later on, by δ^4k we mean $d^4k/(2\pi)^2$.

By integrating over $\phi^\dagger$ and ϕ, we have

$$Z_0(J^\dagger,J,A)=\exp\left[-i\int dp\,dp'J^\dagger(p')K^{-1}(p',p,A)J(p)\right][\det K(p,p',A)]^{-1}\,. \tag{2.14}$$

We shall write

$$K^{-1}(p',p,A)=\frac{\delta(p-p')}{p'^2-m^2}+\sum_{n=1}^{\infty}\frac{g^n}{n!}\int\delta^4k_1\cdots\delta^4k_nW_{\mu_1\cdots\mu_n}(p,p',k_1\cdots k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[p'-p-\sum_1^n k_i\right], \tag{2.15}$$

and

$$[\det K(p',p,A)]^{-1}=\exp[-\mathrm{Tr}\ln K(p',p,A)] \tag{2.16}$$

with

$$-\mathrm{Tr}\ln K(p',p,A)=\sum_{n=1}^{\infty}\frac{g^n}{n!}\int\delta^4k_1\cdots\delta^4k_nQ_{\mu_1\cdots\mu_n}(k_1\cdots k_n)\delta\left[\sum_1^n k_i\right]\prod_{i=1}^{n}A_{\mu_i}(k_i)\,. \tag{2.17}$$

From (2.13), we obtain

$$K^{-1}(p',p,A)=\frac{\delta(p-p')}{p^2-m^2}+\frac{\chi(p,p',A)}{(p^2-m^2)(p'^2-m^2)}+\int\frac{\chi(p,p'',A)\chi(p'',p',A)}{(p^2-m^2)(p''^2-m^2)(p'^2-m^2)}d^4p''+\cdots\,, \tag{2.18}$$

which may be expanded in powers of g to obtain the functions $W_{\mu_1 \cdots \mu_n}$.

For instance, we obtain

$$W_\mu(p,p',k) = \frac{(p+p')_\mu}{(p^2-m^2)(p'^2-m^2)} , \tag{2.19a}$$

$$W_{\mu\nu}(p,p',k_1,k_2) = -\frac{2g_{\mu\nu}}{(p^2-m^2)(p'^2-m^2)} + \left[\frac{(2p+k_1)_\mu(2p'-k_2)_\nu}{(p^2-m^2)[(p+k_1)^2-m^2](p'^2-m^2)} + \begin{bmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{bmatrix} \right] , \tag{2.19b}$$

etc. The quantities $-iW_{\mu_1 \cdots \mu_n}(p,p',k_1 \cdots k_n)$ may be represented by the vertices of Fig. 1.

Their expressions are the sum of all Feynman tree graphs entering in the process of Fig. 1.

Similarly,

$$-\mathrm{Tr}\ln K(p',p,A) = \int d^4p \frac{\chi(p,p,A)}{p^2-m^2} + \frac{1}{2} \int d^4p\, d^4p' \frac{\chi(p,p',A)\chi(p',p,A)}{(p^2-m^2)(p'^2-m^2)} + \cdots \tag{2.20}$$

and corresponds to functions of the type

$$Q_\mu(k) = \int d^4p \frac{2p_\mu}{(p^2-m^2)} = 0 , \tag{2.21a}$$

$$Q_{\mu\nu}(k_1,k_2) = -2g_{\mu\nu} \int \frac{d^4p}{p^2-m^2} + \frac{1}{2} \left[\int d^4p \frac{(2p+k_1)_\mu(2p-k_2)_\nu}{(p^2-m^2)[(p+k_1)^2-m^2]} + \begin{bmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{bmatrix} \right] \tag{2.21b}$$

(again, ultraviolet divergences are neglected here). The quantities $Q_{\mu_1 \cdots \mu_n}(k_1 \cdots k_n)$ may be represented by the vertices of Fig. 2. Their expression is the sum of all one-matter-loop Feynman graphs in the process of Fig. 2.

From the functional

$$\begin{aligned} Z(J,J^\dagger,\chi_\mu) &= \int DA_\mu\, Z_0(J,J^\dagger,A_\mu) \exp\left\{ i \int d^4x \left[\mathscr{L}_0(A) - \frac{1}{2\lambda}(\eta_\mu A_\mu)^2 + \chi_\mu A_\mu \right] \right\} \\ &= Z_0 \left[J,J^\dagger, -\frac{i\partial}{\partial \chi_\mu} \right] \exp\left[-\frac{1}{2} \int d^4k\, \chi_\mu(k) \pi_{\mu\nu}(k,\eta) \chi_\nu(-k) \right] , \end{aligned} \tag{2.22}$$

we may obtain a perturbation in g, from a sum of graphs built with the infinite set of vertices of Figs. 1 and 2. Each vertex of Fig. 1 introduces a factor

$$-\frac{ig^n}{(2\pi)^{2n}} W_{\mu_1 \cdots \mu_n}(p',p,k_1, \ldots, k_n) ,$$

each vertex of Fig. 2 a factor

$$\frac{g^n}{(2\pi)^{2n}} Q_{\mu_1 \cdots \mu_n}(k_1, \ldots, k_n) ,$$

each photon propagator a factor $\pi_{\mu\nu}(k,\eta)$, and we integrate over all internal photon lines (the symmetries of the graphs take into account the counting factor). It is easy to find what usual perturba-

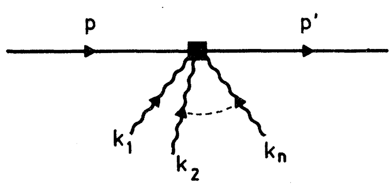

FIG. 1. The vertices $W_{\mu_1 \cdots \mu_n}(p,p',k_1,\ldots,k_n)$.

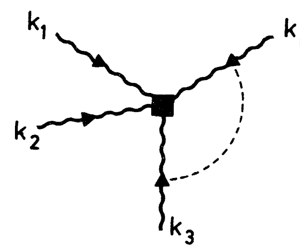

FIG. 2. The vertices $Q_{\mu_1 \cdots \mu_n}(k_1,\ldots,k_n)$.

tive Feynman graphs enter in this new set of graphs by simply shrinking all internal matter lines of the usual Feynman graphs (see, for instance, Fig. 3). Let us now concentrate on the transversality properties of the quantities $W_{\{\mu\}}$ and $Q_{\{\mu\}}$. First, we note the following symmetries: $W_{\{\mu\}}$ and $Q_{\{\mu\}}$ are defined to be entirely symmetric in the exchange of the photon variables $\{k,\mu\}$; also, on the energy-momentum conservation subspace $p'=p+\sum k_i$, the functions $W_{\{\mu\}}$ are invariant in the transformation $p\leftrightarrow p'$ and $k_i \leftrightarrow -k_i$.

In (2.11), we observe that $Z_0(0,0,A_\mu)$ is invariant in the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$; as a consequence, $\det K(p,p',A)$ is invariant and then

$$k^i_{\mu_i} Q_{\mu_1 \cdots \mu_n}(k_1, \ldots, k_n)=0 . \tag{2.23a}$$

From the fact that the vertices $Q_{\mu_1 \cdots \mu_n}$ have a Taylor expansion around all k's$=0$ (for convergent integrals or dimensionally regularized integrals), and from the transversality properties (2.23a), it is easy to prove that

$$Q_{\mu_1 \cdots \mu_n}(k_1 \cdots k_n) = k_1^{\nu_1} \cdots k_n^{\nu_n} Q'_{\{\mu\},\{\nu\}}(k_1 \cdots k_n) , \tag{2.23b}$$

where Q' has a Taylor expansion around all k's$=0$. This important property must be conserved by ultraviolet renormalization and explains why the four-photon vertex is not subtracted and the photon propagator is subtracted only once.

We also verify that

$$Z_0(J^\dagger e^{-ig\alpha}, J e^{ig\alpha}, A_\mu + \partial_\mu \alpha) = Z_0(J^\dagger, J, A_\mu) . \tag{2.24}$$

Using the fact that $\phi(x)e^{ig\alpha(x)}$ has for Fourier transform

$$\exp\left[ig \int \delta^4 k \, \alpha(k) T_{p\rightarrow p-k}\right]\phi(p) , \tag{2.25}$$

where

$$T_{p\rightarrow p-k}\phi(p)=\phi(p-k) \tag{2.26}$$

(to generate the successive convolutions), we easily obtain from (2.24)

$$k^i_{\mu_i} W_{\mu_1 \cdots \mu_n}(p,p',k_1, \ldots, k_n) = W_{\mu_1 \cdots \hat{\mu}_i \cdots \mu_n}(p,p'-k_i,k_1, \ldots, \hat{k}_i, \ldots, k_n) - W_{\mu_1 \cdots \hat{\mu}_i \cdots \mu_n}(p+k_i,p',k_1, \ldots, \hat{k}_i, \ldots, k_n) \text{ at } p'=p+\sum_{i=1}^{n} k_i . \tag{2.27}$$

The Eqs. (2.23) and (2.27) are equivalent to Eqs. (2.8) and (2.10) and describe the gauge symmetry. Of course the transversality properties of a vertex of the type (2.21b), for instance, are valid only with absolutely convergent integrals which allow change of variables and especially translation of variables.

III. CLASSES OF INVARIANCE

The sum over all Feynman graphs at a given order of perturbation satisfies transversality relations which are imposed by the gauge symmetry of the theory and which are made explicit by the Ward-Takahashi identities. On the other hand, each Feynman graph does not show this symmetry so

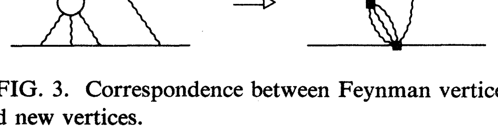

FIG. 3. Correspondence between Feynman vertices and new vertices.

that the understanding of the gauge properties is rather difficult. We wish to raise the problem whether there exists "classes of invariance," such that the sum over all classes is equal to the sum over all Feynman graphs, but each class separately shows explicitly its transversality relations. The first step to solve this problem has been to define new vertices; the advantage of these new vertices is that they collect several Feynman vertices in order to construct the transversality properties (2.23) and (2.27). When we close the photon lines between these vertices, we collect several Feynman graphs which together verify by themselves (2.8) and (2.10). The disadvantages of these new vertices are that they do not give a clear understanding of their gauge-independent and of their gauge-dependent part and especially the mass-shell gauge-independent part (to compute the S matrix) does not appear; $W_{\{\mu\}}$ is more and more singular on the mass shell with the number of one-reducible-matter lines (in Fig. 3, for instance, we have a quadruple pole on the mass shell of the upper matter line). We feel that it is necessary to develop a formalism which splits the $W_{\{\mu\}}$ into transversal and longitu-

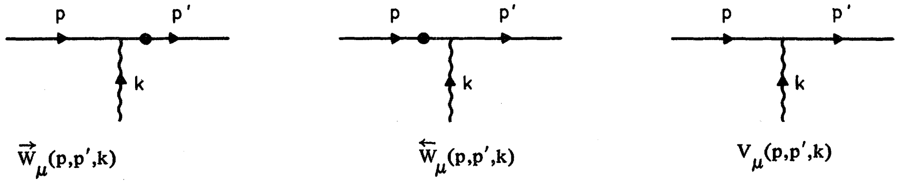

FIG. 4. The lowest-order classes of invariance.

dinal parts and which at the same time gives a clear description of the mass-shell singularities. This section is divided into three parts: (i) example of classes of invariance at low order, (ii) generating functional for classes of invariance, and (iii) formal equalities of the S matrices.

(i) Example of classes of invariance at low order. We first consider the simpler Feynman vertex

$$\overline{W}_\mu(p,p',k)=(p+p')_\mu \ . \tag{3.1}$$

Then,

$$k_\mu \overline{W}_\mu(p,p',k)=P(p')-P(p) \ , \tag{3.2}$$

where $P(p)=p^2-m^2$ is improperly called a projector on external matter lines. Then, we define three new vertices, $\overleftarrow{W}_\mu(p,p',k)$, $\overrightarrow{W}_\mu(p,p',k)$, and $V_\mu(p,p',k)$ such that

$$\overleftarrow{W}_\mu+\overrightarrow{W}_\mu+V_\mu=\overline{W}_\mu \tag{3.3}$$

and

$$\begin{aligned} &k_\mu \overrightarrow{W}_\mu=P(p') \ , \\ &k_\mu \overleftarrow{W}_\mu=-P(p) \ , \\ &k_\mu V_\mu=0 \ . \end{aligned} \tag{3.4}$$

These three vertices are represented in Fig. 4.

The values of these vertices are given later on in (3.38). These three vertices are the elementary classes of invariance and their transversality relations are given in (3.4). The class V_μ is purely transversal while $\overrightarrow{W}_\mu$ and $\overleftarrow{W}_\mu$ are longitudinal classes. Let us present another example by considering the three following Feynman graphs (Fig. 5). We denote by $\overline{W}_{\mu\nu}(p,p',k_1,k_2)$ the sum of the above Feynman graphs; it is only on the above sum that we can obtain the transversality properties:

$$k_\mu^1 \overline{W}_{\mu\nu}(p,p',k_1,k_2)=\frac{\overline{W}_\nu(p,p'-k_1,k_2)}{(p'-k_1)^2-m^2}P(p') - \frac{\overline{W}_\nu(p+k_1,p',k_2)}{(p+k_1)^2-m^2}P(p) \tag{3.5}$$

and in a similar way $k_\nu^2 \overline{W}_{\mu\nu}, \ldots$.

The first idea to construct classes of invariance according to the transversality relation (3.5) is to write three classes $\overleftarrow{W}_{\mu\nu}$, $\overrightarrow{W}_{\mu\nu}$, and $V_{\mu\nu}$ such that

$$k_\mu^1 \overrightarrow{W}_{\mu\nu}(p,p',k_1,k_2)=\frac{\overline{W}_\nu(p,p'-k_1,k_2)}{(p'-k_1)^2-m^2}P(p') \tag{3.6}$$

(and by symmetry $k_\nu^2 \overrightarrow{W}_{\mu\nu}, \ldots$), $k_\mu^1 \overleftarrow{W}_{\mu\nu}$ corresponds to the other projector, and

$$k_\mu^1 V_{\mu\nu}=k_\nu^2 V_{\mu\nu}=0 \ .$$

Unfortunately, it is easy to realize that from $k_\mu^1 \overrightarrow{W}_{\mu\nu}$ and $k_\nu^2 \overrightarrow{W}_{\mu\nu}$ we obtain two different values for $k_\mu^1 k_\nu^2 \overrightarrow{W}_{\mu\nu}$ and consequently that such a $\overrightarrow{W}_{\mu\nu}$ does not exist.

The next attempt to construct classes of invariance is to construct convolutions of the basic classes of Fig. 4. Two cases occur: first, we obtain combinations such that the internal matter line has no dot (Fig. 6).

Such classes are obtained by interaction of two elementary classes through an undotted line called "lateral interaction line" and for instance (a) has the value

$$\frac{\overleftarrow{W}_\mu(p,p+k_1,k_1)\overrightarrow{W}_\nu(p'-k_2,p',k_2)}{(p+k_1)^2-m^2}$$

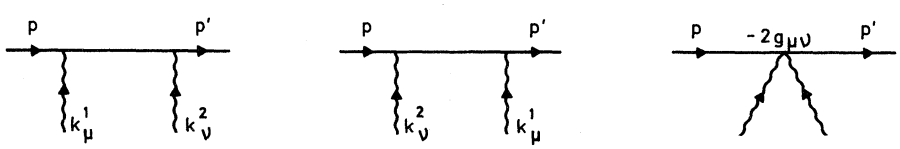

FIG. 5. Feynman tree graphs at order 2.

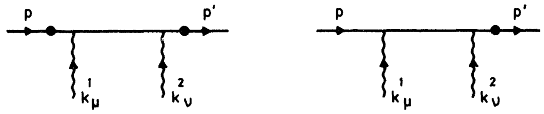

FIG. 6. Convolutions of two lowest-order classes of invariance.

and (b) is

$$\frac{V_\mu(p,p+k_1,k_1)\vec{W}_\nu(p'-k_2,p',k_2)}{(p+k_1)^2-m^2} .$$

The transversality properties of these classes are obtained by application of (3.4).

Next, we obtain convolutions with dotted internal lines; the application of (3.4) tells that such convolutions separately are not classes of invariance and only their sum (plus the extreme right graph of Fig. 5) have transversality properties in terms of projectors. We are thus led to define three new classes (Fig. 7) satisfying the relations

$$k^1_\mu \vec{W}_{\mu\nu}(p,p',k_1,k_2)=\frac{\vec{W}_\mu(p,p'-k_1,k_2)}{(p'-k_1)^2-m^2}P(p') ,$$
$$k^1_\mu \overleftarrow{W}_{\mu\nu}(p,p',k_1,k_2)=-\frac{\overleftarrow{W}_\nu(p+k_1,p',k_2)}{(p+k_1)^2-m^2}P(p) ,$$
$$k^1_\mu V(p,p',k_1,k_2)=0 , \quad (3.7)$$

and, respectively, similar results with k^2_ν. Again, the value for these classes are given later on in (3.38). We may proceed similarly with a quasi-topological description of higher-order vertices $W_{\{\mu\}}$ using classes of invariance. At all orders, we define three new classes of invariance, one purely transverse and two longitudinal; the remaining classes are obtained by convolutions of lower-order classes interacting through "lateral interaction lines."

(ii) Generating functional for classes of invariance.

We now describe a more general approach using the generating functional (2.11). In this section everything is formal in the sense that we are completely careless about ultraviolet and infrared divergences.

We consider that the usual matter field $\phi(x)$ is related to a new matter field $\Phi(x)$ by a nonlocal transformation $S(x,x',A)$ which is dependent of the photon field $A_\mu(x)$. In other words, $\phi(x)$ is obtained from $\Phi(x)$ by a certain "dressing" via a "photon cloud" effect. We write

$$\phi(x)=\int d^4\tilde{x}\, S(x,\tilde{x},A)\Phi(\tilde{x}) , \quad (3.8)$$

where using a Fourier transform we may write

$$S(p,\tilde{p},A)=\delta(p-\tilde{p})+g\int \delta^4k\, S_\mu(\tilde{p},k)A_\mu(k)\delta(p-\tilde{p}-k)$$
$$+\frac{g^2}{2}\int \delta^4k_1\delta^4k_2 S_{\mu\nu}(\tilde{p},k_1,k_2)A_\mu(k_1)A_\nu(k_2)\delta(p-\tilde{p}-k_1-k_2)+\cdots . \quad (3.9)$$

The condition on the transformation S is that the Lagrangian (2.2a) written in terms of $A_\mu(x)$ and of the new matter fields $\Phi(x)$ and $\Phi^\dagger(x)$ is invariant under the transformations

$$\Phi(x)\to e^{i\beta}\Phi(x) ,$$
$$A_\mu(x)\to A_\mu(x)+\partial_\mu\alpha(x) , \quad (3.10)$$

where β is a real space-time constant independent of $\alpha(x)$. The transformation S disentangles the global and the local gauge transformations; the global transformations mean charge conservation of the new matter field; the local transformation means that the new vertices are all transversal. Let us write the matter part of the Lagrangian in terms of the new matter field:

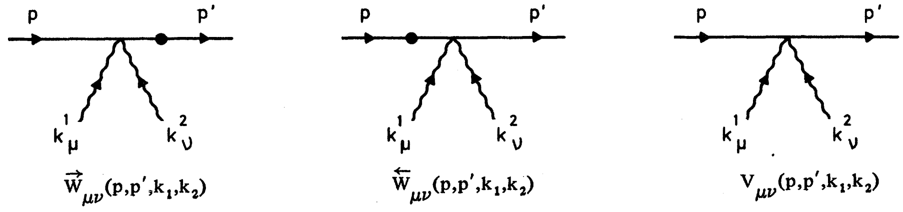

FIG. 7. Classes of invariance at order 2.

$$\int d^4x\, \mathscr{L}_m(x) = \int d^4\tilde{p}\, d^4\tilde{\tilde{p}}\, \Phi^+(\tilde{\tilde{p}})\tilde{K}(\tilde{\tilde{p}},\tilde{p},A)\Phi(\tilde{p}) \;, \tag{3.11}$$

where $\tilde{K}$ is obtained from K in (2.12) by

$$\tilde{K}(\tilde{\tilde{p}},\tilde{p},A) = S^\dagger(\tilde{\tilde{p}},p',A)K(p',p,A)S(p,\tilde{p},A) \;. \tag{3.12}$$

The infinite set of new vertices $V_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n)$ are defined by the expansion

$$\tilde{K}(p',p,A) = \delta(p-p')(p^2-m^2) - \sum_{n=1}^{\infty}\frac{g^n}{n!}\int \delta^4k_1\delta^4k_n V_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[p'-p-\sum_{i=1}^{n}k_i\right] \tag{3.13}$$

and the invariance of the Lagrangian under

$$A_\mu(k) \to A_\mu(k) + ik_\mu\alpha(k)$$

requires

$$k^i_{\mu_i}V_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n) = 0 \quad \text{for } i=1,\ldots,n \;. \tag{3.14}$$

The first few vertices can be easily computed; we find

$$V_\mu(p,p',k) = (p+p')_\mu - (p^2-m^2)S^*_\mu(p',-k) - S_\mu(p,k)(p'^2-m^2) \;, \tag{3.15}$$

$$\begin{aligned} V_{\mu\nu}(p,p',k_1,k_2) = {} & -2g_{\mu\nu} - (p^2-m^2)S^*_{\mu\nu}(p',-k_1,-k_2) - S_{\mu\nu}(p,k_1,k_2)(p'^2-m^2) \\ & + \left[(2p+k_1)_\mu S^*_\nu(p',-k_2) + S_\nu(p,k_2)(2p'-k_1)_\mu - S_\mu(p,k_1)[(p+k_1)^2-m^2]S^*_\nu(p',-k_2) \right. \\ & \left. + \begin{bmatrix} \mu\leftrightarrow\nu \\ k_1\leftrightarrow k_2 \end{bmatrix} \right] ; \end{aligned} \tag{3.16}$$

$V_{\mu\nu\rho}(p,p',k_1,k_2,k_3)$ is given in Sec. IV Eq. (4.15).

We now specify the transformation S from the transversality relations (3.14). It is found that

$$\begin{aligned} & k_\mu S_\mu(p,k) = 1 \;, \\ & k^i_{\mu_i}S_{\mu_1\cdots\mu_n}(p,k_1,\ldots,k_n) = S_{\mu_1\cdots\hat{\mu}_i\cdots\mu_n}(p,k_1,\ldots,\hat{k}_i,\ldots,k_n) \;. \end{aligned} \tag{3.17}$$

The most general transformation (3.9) satisfying the transversality relation (3.17) can be written

$$\begin{aligned} S(p,\tilde{p},A) = {} & \exp\left[g\int \delta^4k\, \psi_\mu(\tilde{p},k)A_\mu(k)T_{p\to p-k}\right] \\ & \times\left[\delta(p-\tilde{p}) + \sum_{n=1}^{\infty}\frac{g^n}{n!}\int \delta^4k_1\cdots\delta^4k_n T_{\mu_1\cdots\mu_n}(\tilde{p},k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[p-\tilde{p}-\sum_{i=1}^{n}k_i\right]\right] , \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} & k_\mu\psi_\mu(p,k) = 1 \;, \\ & k^i_{\mu_i}T_{\mu_1\cdots\mu_n}(\tilde{p},k_1,\ldots,k_n) = 0 \;, \end{aligned} \tag{3.19}$$

and $T_{p\to p-k}$ is the shift operator defined in (2.26). Under the gauge transformation $A_\mu \to A_\mu + ik_\mu\alpha(k)$,

$$S(p,\tilde{p},A) \to \exp\left[ig\int \delta^4k\, \alpha(k)T_{p\to p-k}\right]S(p,\tilde{p},A) \;, \tag{3.20}$$

so that the local gauge transformation of the usual matter field $\phi(x)$ is by (3.8) and (3.20) induced from the gauge transformation of $A_\mu(x)$ in the "cloud effect."

As we realize from (3.18), the transformation S such that the Lagrangian (3.11) is invariant under the gauge transformations (3.10) is not unique. As a first simplification, we may take all tensors $T_{\{\mu\}}$ to be zero, and in practice we shall do so in many occasions (see Sec. IV). Unfortunately, the transformation

$$S(p,\widetilde{p},A)=\exp\left[g\int\delta^4k\,\psi_\mu(\widetilde{p},k)A_\mu(k)T_{p\to p\to -k}\right]\delta(p-\widetilde{p}) \tag{3.21}$$

is not unitary if ψ_μ is $\widetilde{p}$ dependent and its inverse S^{-1} which may be written in the form (3.18) (with ψ in $-\psi$) has an infinite set of nonzero tensors $T_{\{\mu\}}$. We may also determine the tensors $T_{\{\mu\}}$ such that S is a unitary transformation, but these tensors become quickly complicated. On the other hand, if S is not unitary, we know from the equivalence theorem[14] on Green's functions and on the S matrix that the transformation S does not change the physical system only if we introduce the Jacobian of the transformation (or equivalently Faddeev-Popov ghosts) to restore the value of the matter loops (it is only in the case of a transformation S which is a local power series in the fields and its derivatives that such Jacobian contributions cancel by ultraviolet renormalization). We shall come back to this problem of Jacobian later on in this section.

We now calculate the generating functional Z_0 from (2.11):

$$\begin{aligned}Z_0(J^\dagger,J,A_\mu)=\int D\Phi^\dagger D\Phi\,\mathrm{Jac}(A)\exp\left[i\int d^4p\,d^4p'\Phi^\dagger(p')\widetilde{K}(p',p,A)\Phi(p)\right]\\ \times\exp\left[i\int d^4p\,d^4p'[J^\dagger(p')S(p',p,A)\Phi(p)+\Phi^\dagger(p')S^\dagger(p',p,A)J(p)]\right],\end{aligned} \tag{3.22}$$

where

$$\mathrm{Jac}A=\det(SS^\dagger)\,. \tag{3.23}$$

After integration upon the new matter field, we obtain

$$Z_0(J^\dagger,J,A_\mu)=\frac{\mathrm{Jac}(A)}{\det\widetilde{K}(A)}\exp\left[-i\int d^4p\,d^4p'J^\dagger(p')S(p',\widetilde{\widetilde{p}},A)\widetilde{K}^{-1}(\widetilde{\widetilde{p}}\widetilde{p},,A)S(\widetilde{p},p,A)J(p)\right], \tag{3.24}$$

which by comparing with (2.14) shows that

$$\begin{aligned}&\det K(A)=\det\widetilde{K}(A)/\mathrm{Jac}(A)\,,\\ &K^{-1}(p',p,A)=S(p',\widetilde{\widetilde{p}},,A)\widetilde{K}^{-1}(\widetilde{\widetilde{p}},\widetilde{p},A)S^\dagger(\widetilde{p},p,A)\,.\end{aligned} \tag{3.25}$$

We write the expansion of $\widetilde{K}^{-1}(p',p,A)$ as

$$\widetilde{K}^{-1}(p',p,A)=\frac{\delta(p'-p)}{p'^2-m^2}+\sum_{n=1}^{\infty}\frac{g^n}{n!}\int\delta^4k_1\cdots\delta^4k_n\,V^{-1}_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n)\prod_{i=1}^{n}A_{\mu_i}(k_i)\delta\left[p'-p-\sum_{i=1}^{n}k_i\right]. \tag{3.26}$$

We now interpret the result (3.24) in terms of classes of invariance. First, the vertices $V_{\{\mu\}}$ defined in (3.13) and which are the elementary vertices of the new $\Phi(x)$ theory are called transversal classes of invariance because of the relation (3.14). They represent the elementary classes from which all transversal classes are built. By convolution of these elementary classes through lateral interaction we construct other transversal classes. For instance

$$V^{-1}_{\mu\nu}(p,p',k_1,k_2)=\frac{V_{\mu\nu}(p,p',k_1,k_2)}{(p^2-m^2)(p'^2-m^2)}+\left[\frac{V_\mu(p,p+k_1,k_1)V_\nu(p+k_1,p',k_2)}{(p^2-m^2)[(p+k_1)^2-m^2](p'^2-m^2)}+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right], \tag{3.27}$$

which can be represented graphically by Fig. 8. This result is very similar to Feynman graphs except that each vertex is purely transverse (and that we have an infinite number of such vertices). Similarly, Φ matter loops are obtained from $[\det\widetilde{K}]^{-1}=\exp[-\mathrm{Tr}\ln\widetilde{K}]$.

If we write

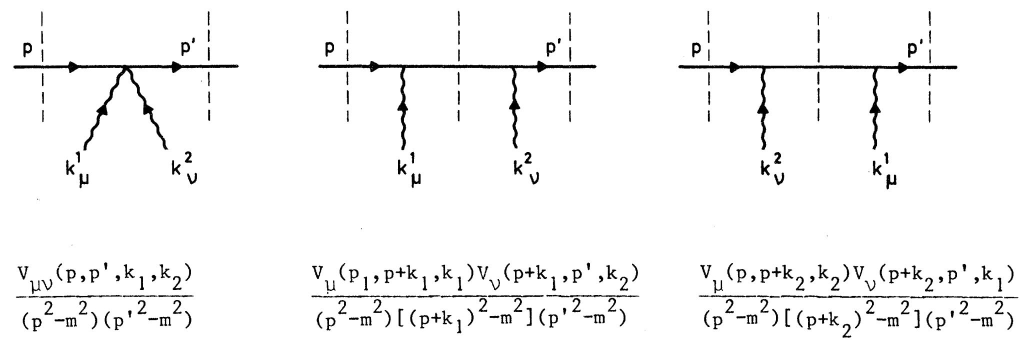

FIG. 8. Convolutions and lateral interaction lines.

$$-\mathrm{Tr}\ln\widetilde{K}=\sum_{n=1}^{\infty}\frac{g^n}{n!}\int\delta^4k_1\cdots\delta^4k_n\widetilde{Q}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)\delta\left[\sum_{i=1}^{n}k_i\right]\prod_{i=1}^{n}A_{\mu_i}(k_i)\ , \tag{3.28}$$

we obtain the completely transverse classes

$$\widetilde{Q}_\mu(k)=\int d^4p\frac{V_\mu(p,p,k)}{p^2-m^2}\ , \tag{3.29}$$

$$\widetilde{Q}_{\mu\nu}(k_1,k_2)=\int d^4p\frac{V_{\mu\nu}(p,p,k_1,k_2)}{p^2-m^2}+\tfrac{1}{2}\int d^4p\left[\frac{V_\mu(p,p+k_1,k_1)V_\nu(p+k_1,p,k_2)}{(p^2-m^2)[(p+k_1)^2-m^2]}+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right], \tag{3.30}$$

etc., which again may be interpreted as convolutions of elementary transversal classes of invariance through "lateral interaction lines" along a matter loop.

As we already mentioned, the physical system described by the Lagrangian (2.2a) is unchanged under the transformation S only if we consider the contribution of Jac(A) which is different from $\mathbb{1}$ when S is not unitary. Of course, this Jacobian is completely transversal; we write

$$\mathrm{Tr}\ln(SS^\dagger)=\sum_{n=1}^{\infty}\frac{g^n}{n!}\int\delta^4k_1\cdots\delta^4k_n\widetilde{\widetilde{Q}}_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)\delta\left[\sum_{i=1}^{n}k_i\right]\prod_{i=1}^{n}A_{\mu_i}(k_i)\ . \tag{3.31}$$

We may thus define the elementary "unitarity classes of invariance" as

$$C_\mu(p,p',k)=S_\mu(p,k)+S^*_\mu(p',-k)\ , \tag{3.32}$$

$$C_{\mu\nu}(p,p',k_1,k_2)=S_{\mu\nu}(p,k_1,k_2)+S_\mu(p,k_1)S^*_\nu(p',-k_2)+S_\nu(p,k_2)S^*_\mu(p',-k_2)+S^*_{\mu\nu}(p',-k_1,-k_2)\ , \tag{3.33}$$

etc., which are null when S is unitarity. The "unitary classes of invariance" $\widetilde{\widetilde{Q}}$ may be obtained from the above classes by their convolutions through "lateral interaction lines" with propagator equal to one; the trace in (3.31) requires that these lines form a loop which may be interpreted as a Faddeev-Popov ghost loop[19] (no convolutions between V classes and C classes are allowed):

$$\widetilde{\widetilde{Q}}_\mu(k)=\int d^4p\,C_\mu(p,p,k)\ , \tag{3.34}$$

$$\widetilde{\widetilde{Q}}_{\mu\nu}(k_1,k_2)=\int d^4p\,C_{\mu\nu}(p,p,k_1,k_2)+\frac{1}{2}\int d^4p\left[C_\mu(p,p+k_1,k_1)C_\nu(p+k_1,p,k_2)+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right]\ . \tag{3.35}$$

It may be checked from (2.21) that

$$\widetilde{Q}_{\{\mu\}}+\widetilde{\widetilde{Q}}_{\{\mu\}}=Q_{\{\mu\}}\ . \tag{3.36}$$

The new matter field theory $[\Phi(x)]$ may be constructed completely in terms of convolutions of elementary transversal and unitarity classes of invariance. To obtain the Green's functions corresponding to the new field $\Phi(x)$, we must close the photon lines using the propagator $\pi_{\mu\nu}(k)$ defined in (2.3) and (2.4). It is clear from the transversality properties of the classes that the Φ Green's functions are completely independent of the gauge introduced in the Lagrangian to quantize the photon field.

If we want to construct the usual matter field $[\phi(x)]$ theory, we must perform for each matter line the derivatives $(-i\delta/\delta J)\,(-i\delta/\delta J^{\dagger})$ upon the functional (3.24). The term $S\tilde{K}^{-1}S^{\dagger}$ shows that for each matter line, in addition to the convolutions of transversal classes, we must introduce convolutions with longitudinal classes of "invariance" defined by $S^{*}_{\{\mu\}}$ on the ingoing part of the matter line and by $S_{\{\mu\}}$ on the outgoing part. A typical matter line for the usual $\phi(x)$ field is shown in Fig. 9, and is described by

$$S^{*}_{\mu\nu\rho}(p+k_1+k_2+k_3,-k_1,-k_2,-k_3)[(p+k_1+k_2+k_3)^2-m^2]^{-1}V_{\sigma\lambda}(p+k_1+k_2+k_3,p'-k_6,k_4,k_5)$$
$$\times[(p'-k_6)^2-m^2]^{-1}S_{\tau}(p'-k_6,k_6)\,. \quad (3.37)$$

Of course, several (one or zero) convolutions of V'_S may occur while we have only one or zero convolution with S^* at one end and one or zero with S at the other end of the matter line. When we close the photon lines with the propagator $\pi_{\mu\nu}(k)$ defined in (2.3) and (2.4) all the dependence in the gauge of the Lagrangian is taken by the longitudinal classes of "invariance"; it is the purpose of the last part of this section to show that the longitudinal classes contribute nothing to the S matrix.

It is clear from the above description that the quantities introduced in the first part of this section, (3.4) and (3.7), may be identified. The transversal classes V_μ and $V_{\mu\nu}$ are given in (3.15) and (3.16) while

$$\begin{aligned}
&\overleftarrow{W}_\mu(p,p',k)=S^*_\mu(p',-k)P(p)\,,\\
&\overrightarrow{W}_\mu(p,p',k)=S_\mu(p,k)P(p')\,,\\
&\overleftarrow{W}_{\mu\nu}(p,p',k_1,k_2)=S^*_{\mu\nu}(p',-k_1,-k_2)P(p)\,,\\
&\overrightarrow{W}_{\mu\nu}(p,p',k_1,k_2)=S_{\mu\nu}(p,k_1,k_2)P(p')\,.
\end{aligned} \quad (3.38)$$

(iii) Formal equalities of the S matrices.

The equivalence theorem[14] implies that the S matrices for the new matter field $\Phi(x)$ and for the usual matter field $\phi(x)$ are equal (if we keep the contribution of the Jacobian of the transformation, that is, of the "unitarity classes of invariance"). This theorem is here formally valid in the sense that we close our eyes to ultraviolet divergences and infrared divergences of the original $\phi(x)$ field theory and of the transformation S. The formal equality of the S matrices proves formally the

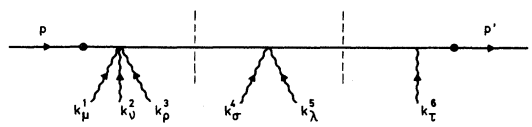

FIG. 9. Convolutions of longitudinal and transversal classes.

gauge independence of the S matrix of the usual $\phi(x)$ theory and the independence of the S matrix of the new $\Phi(x)$ theories with regard to the transformations S. This theorem can be made rigorous by ultraviolet renormalization but this is not the purpose of this paper; on the other hand, the infrared problem is entirely solved in Sec. V.

First, let us construct the usual matter-field complete propagator $\langle\phi(x)\phi(y)\rangle$ from the classes of invariance by closing the photon lines. We wish to classify the graphs according to the number of one-matter reducible lines which are, as shown in Fig. 9, lateral interaction lines and which carry the propagator $(p^2-m^2)^{-1}$. First we have a set of graphs, necessarily dotted at both ends (with longitudinal classes) which have no propagators $(p^2-m^2)^{-1}$. The sum of these contributions are denoted by $\overleftrightarrow{F}(p^2)$. Then, we consider the graphs with one propagator $(p^2-m^2)^{-1}$; they have at least one longitudinal class of invariance except for the Born term $(p^2-m^2)^{-1}$ itself. Such graphs are summed under the form

$$\frac{[1+\overleftarrow{F}(p^2)][1+\overrightarrow{F}(p^2)]}{(p^2-m^2)}\,.$$

The functions $\overleftrightarrow{F}$, $\overleftarrow{F}$, and $\overrightarrow{F}$ are all gauge dependent. Then comes the possibility of having several propagators $(p^2-m^2)^{-1}$. Between two consecutive such propagators, we sum the contributions in a gauge-independent function $\Sigma(p^2)$. We also sum over the usual geometrical series of ratio $\Sigma(p^2)/(p^2-m^2)$ and we obtain the complete propagator as

$$-iG(p^2)=\overleftrightarrow{F}(p^2)+\frac{[1+\overleftarrow{F}(p^2)][1+\overrightarrow{F}(p^2)]}{p^2-m^2-\Sigma(p^2)}\,. \quad (3.39)$$

Clearly, the complete propagator for the new field theory $\Phi(x)$ is

$$i[p^2-m^2-\Sigma(p^2)]^{-1}\,.$$

Equation (3.39) shows (formally at least) that the position of the poles of the propagators are gauge independent in the usual $\phi(x)$ theory and of course independent of the transformation S in the new $\Phi(x)$ theories. The functions $\overleftarrow{F}(p^2)$, $\overleftrightarrow{F}(p^2)$, $\overrightarrow{F}(p^2)$ and $\Sigma(p^2)$, are ultraviolet divergent. Although we do not want to enter into the details of the renormalization procedure, let us simply say that we may always subtract these functions (by usual power-counting arguments) in such a way that

$$\Sigma(m^2)=\frac{d\Sigma}{dp^2}(p^2=m^2)$$
$$=\overleftarrow{F}(m^2)=\overrightarrow{F}(m^2)=0\ , \tag{3.40}$$

so that the pole of the propagator is at $p^2=m^2$ and the residue is equal to i.

Next, we concentrate on the vertex functions of the usual $\phi(x)$ field. Similar analysis leads to

$$-iG_\mu(p,p',q)=\overleftrightarrow{F}_\mu(p,p',q)+\frac{\overleftarrow{F}_\mu(p,p',q)[1+\overrightarrow{F}(p'^2)]}{p'^2-m^2-\Sigma(p'^2)}+\frac{[1+\overleftarrow{F}(p^2)]\overrightarrow{F}_\mu(p,p',q)}{p^2-m^2-\Sigma(p^2)}$$
$$+\frac{[1+\overleftarrow{F}(p^2)]}{p^2-m^2-\Sigma(p^2)}\Sigma_\mu(p,p',q)\frac{[1+\overrightarrow{F}(p'^2)]}{p'^2-m^2-\Sigma(p'^2)}\ . \tag{3.41}$$

In (3.41), the functions F, F_μ, Σ, and Σ_μ contain neither the propagator $(p^2-m^2)^{-1}$ nor $(p'^2-m^2)^{-1}$; the functions F and F_μ are gauge dependent while the functions Σ and Σ_μ are gauge independent.

Ward identities imply

$$\begin{aligned}
&q_\mu\Sigma_\mu(p,p',q)=0\ ,\\
&q_\mu\overrightarrow{F}_\mu(p,p',q)=[1+\overrightarrow{F}(p^2)]\ ,\\
&q_\mu\overleftarrow{F}_\mu(p,p',q)=-[1+\overleftarrow{F}(p'^2)]\ ,\\
&q_\mu\overleftrightarrow{F}_\mu(p,p',q)=\overleftrightarrow{F}(p^2)-\overleftrightarrow{F}(p'^2)\ .
\end{aligned} \tag{3.42}$$

We could proceed to higher-order (in external legs) Green's functions but the rule is easy to understand. We obtain many terms which do not have on every external matter line a pole of the type

$$[p^2-m^2-\Sigma(p^2)]^{-1};$$

then, we always have a term of the form

$$\prod_{\substack{\text{incoming}\\ \text{matter line}}}\left[\frac{1+\overleftarrow{F}(p^2)}{p^2-m^2-\Sigma(p^2)}\right]\prod_{\substack{\text{outgoing}\\ \text{matter lines}}}\left[\frac{1+\overrightarrow{F}(p'^2)}{p'^2-m^2-\Sigma(p'^2)}\right]\Sigma_{\{\mu\}}(p_i,p_i',q_s')\ . \tag{3.43}$$

Using the renormalization conditions (3.40) which define the matter mass shell, we get after amputation (a),

$$-iG^{(a)}_{\{\mu\}}(p,p',q_s')\Big|_{\substack{\text{matter}\\ \text{mass shell}}}=\Sigma^{(a)}_{\{\mu\}}(p_i,p_i',q_s')\Big|_{\substack{\text{matter}\\ \text{mass shell}}}\ . \tag{3.44}$$

This proves formally the equality of the S matrices, the gauge independence of the usual S matrix, and the equality of the S matrices obtained from the various new fields $\Phi(x)$ defined in (3.8).

Let us close this section by discussing the difference between gauge dependence of the usual $\phi(x)$ theory and transformation S dependence of the new field theories $\Phi(x)$. The gauge introduced in the Lagrangian (2.2a) in order to quantize the photon field appears in the photon propagator (2.3) with $\lambda\neq 0$. The form (2.4) of this propagator shows that it is equivalent to consider (i/k^2) as the photon propagator while the vertices of the theory are dressed into

$$\prod_{\substack{\text{photon}\\ \text{lines}}}\left[g_{\mu\nu}-\frac{(\eta_\mu\pm\sqrt{\lambda}k_\mu)}{k\cdot n}k_\nu\right]W_{\nu\dots}(p,k_s')\ . \tag{3.45}$$

There exists no transformation S which corresponds to such a dressing ($\lambda\neq 0$). If we take the limit $\lambda\to 0$ of the Lagrangian field theory we obtain the so-called Landau gauges which are purely transverse; to this dressing corresponds the transformations S where all T's in (3.18) are null and

$$\psi_\mu(\tilde{p},k)=\eta_\mu(k)/k\cdot\eta(k) .$$

Such S transformations are unitary. But the set of S transformations is much larger than the set of Landau gauges since the transformation S is in general dependent on the matter momentum p. In Sec. V, it is a very specific p dependence which will be chosen for S in order to take care of infrared divergences.

IV. PRACTICAL CALCULATIONS OF S-MATRIX ELEMENTS (INFRARED DIVERGENT)

Although the expressions for the new vertices look rather complicated, several simplifications occur by the use of transversality and by the choice of convenient transformation S (since the S matrix is independent of this transformation). As a general rule, we shall, for practical calculations, take all tensors $T_{\{\mu\}}$ in (3.18) equal to zero. Then, from (3.21), all quantities $S_{\mu_1\cdots\mu_n}(p,k_1,\ldots,k_n)$ factorize into

$$\prod_{i=1}^{n}\psi_{\mu_i}(p,k_i) .$$

The transversality of the vertices are such that we may multiply them by

$$[g_{\mu\chi}-k_\mu\psi_\chi(p,k)]$$

or

$$[g_{\mu\chi}+k_\mu\psi^*_\chi(p',-k)]$$

and obtain for instance

$$\begin{aligned}V_\mu(p,p',k)&=\overleftarrow{\pi}_\mu(p,k)-(p^2-m^2)\xi_\mu(p,p'k)\\&=\vec{\pi}_\mu(p',k)-(p'^2-m^2)\xi_\mu(p,p',k) ,\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}&\xi_\mu(p,p',k)=\psi_\mu(p,k)+\psi^*_\mu(p',-k) ,\\&\overleftarrow{\pi}_\mu(p,k)=(2p+k)_\mu-k(2p+k)\psi_\mu(p,k) ,\\&\vec{\pi}_\mu(p',k)=(2p'-k)_\mu+k(2p'-k)\psi^*_\mu(p',-k) ,\end{aligned} \tag{4.2}$$

$$\begin{aligned}V_{\mu\nu}(p,p',k_1,k_2)=&-2\overleftarrow{g}_{\mu\nu}(p,k_1,k_2)-(p^2-m^2)\xi_\mu(p,p',k_1)\xi_\nu(p,p',k_2)\\&+\overleftarrow{\pi}_\mu(p,k_1)\xi_\nu(p,p',k_2)+\overleftarrow{\pi}_\nu(p,k_2)\xi_\mu(p,p',k_1)\\=&-2\vec{g}_{\mu\nu}(p',k_1,k_2)-(p'^2-m^2)\xi_\mu(p,p',k_1)\xi_\nu(p,p',k_2)\\&+\xi_\mu(p,p',k_1)\vec{\pi}_\nu(p',k_2)+\xi_\nu(p,p',k_2)\vec{\pi}_\mu(p',k_1) ,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}&\overleftarrow{g}_{\mu\nu}(p,k_1,k_2)=[g_{\mu\lambda}-k^1_\lambda\psi_\mu(p,k_1)][g_{\lambda\nu}-k^2_\lambda\psi_\nu(p,k_2)] ,\\&\vec{g}_{\mu\nu}(p',k_1,k_2)=[g_{\mu\lambda}+k^1_\lambda\psi^*_\mu(p',-k_1)][g_{\lambda\nu}+k^2_\lambda\psi^*_\nu(p',-k_2)] .\end{aligned} \tag{4.4}$$

If moreover we choose

$$\psi_\mu(p,k)=\frac{(2p+k)_\mu}{k\cdot(2p+k)} , \tag{4.5}$$

which of course is possible only if $k\cdot(2p+k)\neq 0$, we have

$$\overleftarrow{\pi}_\mu(p,k)=0 \text{ if } k\cdot(2p+k)\neq 0 ,$$
$$\overrightarrow{\pi}_\mu(p',k)=0 \text{ if } k\cdot(2p'-k)\neq 0 , \qquad (4.6)$$
$$\xi_\mu(p,p',k)=0 \text{ if } p'=p+k \text{ but } p'^2\neq p^2 .$$

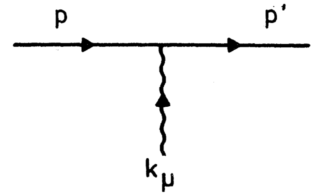

FIG. 10. Matter interacting with one photon at lowest order.

Such simplifications give for the classes of invariance

$$V_\mu(p,p',k)=0 \quad \text{if } p^2\neq p'^2 ,$$
$$V_{\mu\nu}(p,p',k_1,k_2)=-2\overleftarrow{g}_{\mu\nu}(p,k_1,k_2)-(p^2-m^2)\xi_\mu(p,p',k_1)\xi_\nu(p,p',k_2)$$
$$=-2\overrightarrow{g}_{\mu\nu}(p',k_1,k_2)-(p'^2-m^2)\xi_\mu(p,p',k_1)\xi_\nu(p,p',k_2) , \qquad (4.7)$$

etc.

We may now calculate the following S-matrix elements:

(a) The lowest-order vertex function of Fig. 10:

$$V_\mu^{-1(a)}(p,p',k)\,|_{p^2=p'^2=m^2}=V_\mu(p,p',k)\,|_{p^2=p'^2=m^2} . \qquad (4.8)$$

Since we are on the mass shell, the choice (4.5) is not allowed but any nonsingular choice for ψ gives

$$\overleftarrow{\pi}_\mu(p,k)=(2p+k)_\mu , \qquad (4.9)$$

$$\overrightarrow{\pi}_\mu(p',k)=(2p'-k)_\mu , \qquad (4.10)$$

so that from (4.1) and (4.2)

$$V_\mu^{-1(a)}(p,p',k)\,|_{p^2=p'^2=m^2}=(p+p')_\mu \qquad (4.11)$$

as expected.

(b) The two-photon amplitude at lowest order (Fig. 11).

Here, we have

$$V_{\mu\nu}^{-1(a)}(p,p',k_1,k_2)\,|_{p^2=p'^2=m^2}=V_{\mu\nu}(p,p',k_1,k_2)\,|_{p^2=p'^2=m^2}+\left[\frac{V_\mu(p,p+k_1,k_1)V_\nu(p+k_1,p',k_2)}{k_1\cdot(2p+k_1)}+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right]_{p^2=p'^2=m^2} . \qquad (4.12)$$

With the choice (4.5) and from (4.7), we have

$$V_{\mu\nu}^{-1(a)}(p,p',k_1,k_2)\,|_{p^2=p'^2=m^2}=-2\overleftarrow{g}_{\mu\nu}(p',k_1,k_2)=-2\overrightarrow{g}_{\mu\nu}(p',k_1,k_2) . \qquad (4.13)$$

(c) The three-photon amplitude at lowest order (Fig. 12).

If we choose ψ to be (4.5) such that the classes $V_\mu=0$, then

$$V_{\mu\nu\rho}^{-1(a)}(p,p',k_1,k_2,k_3)\,|_{p^2=p'^2=m^2}=V_{\mu\nu\rho}(p,p',k_1,k_2,k_3)\,|_{p^2=p'^2=m^2} . \qquad (4.14)$$

We have by generalization of (3.15) and (3.16)

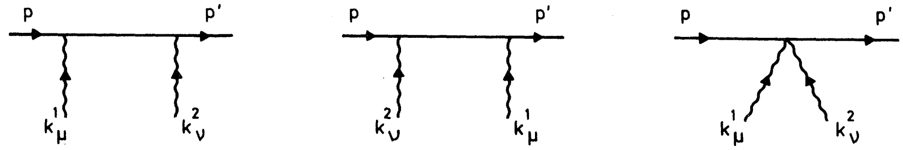

FIG. 11. Matter interacting with two photons at lowest order.

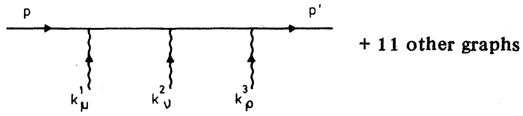

FIG. 12. Matter interacting with three photons at lowest order.

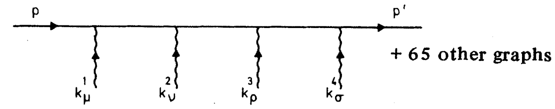

FIG. 13. Matter interacting with four photons at lowest order.

$$
\begin{aligned}
V_{\mu\nu\rho}(p,p',k_1,k_2,k_3) = & -(p^2-m^2)S^*_{\mu\nu\rho}(p',-k_1,-k_2,-k_3)-(p'^2-m^2)S_{\mu\nu\rho}(p,k_1,k_2,k_3) \\
& - \sum_{\{1,2,3\}} S_{\mu\nu}(p,k_1,k_2)[(p'-k_3)^2-m^2]S^*_\rho(p',-k_3) \\
& - \sum_{\{1,2,3\}} S_\mu(p,k_1)[(p+k_1)^2-m^2]S^*_{\nu\rho}(p',-k_2,-k_3) \\
& + \sum_{\{1,2,3\}} [S_{\mu\nu}(p,k_1,k_2)(2p'-k_3)_\rho + S_\mu(p,k_1)(p+p'+k_1-k_3)_\nu S^*_\rho(p',-k_3) \\
& \qquad + (2p+k_1)_\mu S^*_{\nu\rho}(p',-k_2,-k_3)] \\
& -2 \sum_{\{1,2,3,\}} \{g_{\mu\nu}S^*_\rho(p',-k_3)+g_{\mu\nu}S_\rho(p,k_3)\} \ .
\end{aligned} \tag{4.15}
$$

Using transversality properties and (4.5), we get

$$
\begin{aligned}
V^{-1(a)}_{\mu\nu\rho}(p,p',k_1,k_2,k_3)\,|_{p^2=p'^2=m^2} & = -2 \sum_{\{1,2,3,\}} \overleftarrow{g}_{\mu\nu}(p,k_1,k_2)\xi_\rho(p,p',k_3) \\
& = -2 \sum_{\{1,2,3\}} \vec{g}_{\mu\nu}(p',k_1,k_2)\xi_\rho(p,p',k_3) \ .
\end{aligned} \tag{4.16}
$$

(d) The four-photon amplitude at lowest order (Fig. 13).

Without giving the details, we obtain

$$
\begin{aligned}
V^{-1(a)}_{\mu\nu\rho\sigma}(p,p',k_1,k_2,k_3,k_4)\,|_{p^2=p'^2=m^2} = & \ V_{\mu\nu\rho\sigma}(p,p',k_1,k_2,k_3,k_4)\,|_{p^2=p'^2=m^2} \\
& + \sum_{\text{perm}} \left[\frac{V_{\mu\nu}(p,p+k_1+k_2,k_1,k_2)V_{\rho\sigma}(p'-k_3-k_4,p',k_3,k_4)}{(k_1+k_2)(2p+k_1+k_2)} \right]_{p^2=p'^2=m^2} ,
\end{aligned} \tag{4.17}
$$

that is,

$$
\begin{aligned}
V^{-1(a)}_{\mu\nu\rho\sigma}(p,p',k_1,k_2,k_3,k_4)\,|_{p^2=p'^2=m^2} = & -2\sum_{\text{perm}} \overleftarrow{g}_{\mu\nu}(p,k_1,k_2)\xi_\rho(p,p',k_3)\xi_\sigma(p,p',k_4) \\
& + 4\sum_{\text{perm}} \frac{\overleftarrow{g}_{\mu\nu}(p,k_1,k_2)\vec{g}_{\rho\sigma}(p',k_3,k_4)}{(k_1+k_2)(2p+k_1+k_2)} \ .
\end{aligned} \tag{4.18}
$$

(e) Matter-matter scattering (the Born term in Fig. 14).

From (a), we get trivially the gauge-independent result

$$
A(p,q,p',q') = \frac{+i}{(2\pi)^4}\frac{(p+p')\cdot(q+q')}{(p'-p)^2} \ . \tag{4.19}
$$

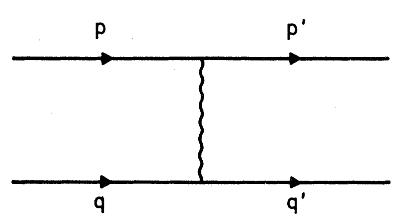

FIG. 14. Matter-matter scattering at order 2.

(f) Matter-matter scattering (order 4 partially → Fig. 15):

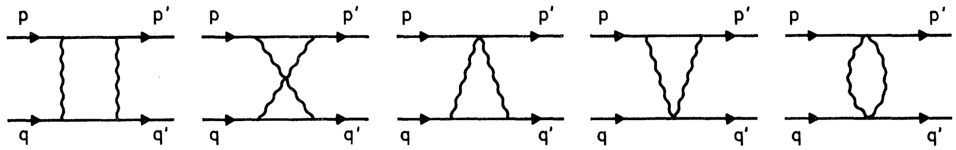

FIG. 15. Matter-matter scattering at order 4.

$$A(p,q,p',q')=\frac{4}{(2\pi)^8}\int \frac{d^4k_1}{k_1^2}\frac{d^4k_2}{k_2^2}\delta(p'-p-k_1-k_2)\overleftarrow{g}_{\mu\nu}(p,k_1,k_2)\overleftarrow{g}_{\mu\nu}(q,-k_1,-k_2)\ . \tag{4.20}$$

(g) The vertex function at order 3$\rightarrow$ Fig. 16. Here, as in (a), the choice (4.5) is not allowed since

$$q(2p+q)=q(2p'-q)=0\ .$$

We find it convenient to choose

$$\psi_\mu=\frac{(2p+k+N)_\mu}{k(2p+k+N)} \tag{4.21}$$

with the possibility of calculating the $N\rightarrow 0$ limit (for simplicity) since the result is N independent.

It must be said that the result (4.16) cannot be used because it becomes singular at $k_1+k_2=0$; the reason is that closing the photon line $(^{\mu}_{k_1}\ ^{\nu}_{k_2})$ on Fig. 12 generates self-energy graphs which are not in Fig. 16 and are infinite on the mass shell.

We then calculate

$$-\frac{1}{2(2\pi)^6}\int\frac{d^4k}{k^2}\left[V_{\mu\mu\rho}(p,p',k,-k,q)+\left[\frac{V_{\mu\rho}(p,p'+k,k,q)V_\mu(p'+k,p',-k)}{k\cdot(2p'+k)}+\frac{V_\mu(p,p+k,k)V_{\mu\rho}(p+k,p',-k,q)}{k\cdot(2p+k)}+\frac{V_\mu(p,p+k,k)V_\rho(p+k,p'+k,q)V_\mu(p'+k,p',-k)}{k\cdot(2p+k)k\cdot(2p'+k)}+(k\leftrightarrow -k)\right]\right]. \tag{4.22}$$

In (4.22), we have eliminated the terms $V_{\mu\mu}V_\rho$ or $V_\mu V_\mu V_\rho$ which correspond to self-energy parts. In the $N\rightarrow 0$ limit,

$$V_{\mu\rho}(N)\rightarrow V_{\mu\rho}(0)\neq 0$$

while V_μ and

$$V_\rho(p+k,p'+k,q)\sim N\rightarrow 0\ .$$

Consequently, only the limit $N\rightarrow 0$ of $V_{\mu\mu\rho}$ has to be calculated.

With the choice (4.21), we obtain

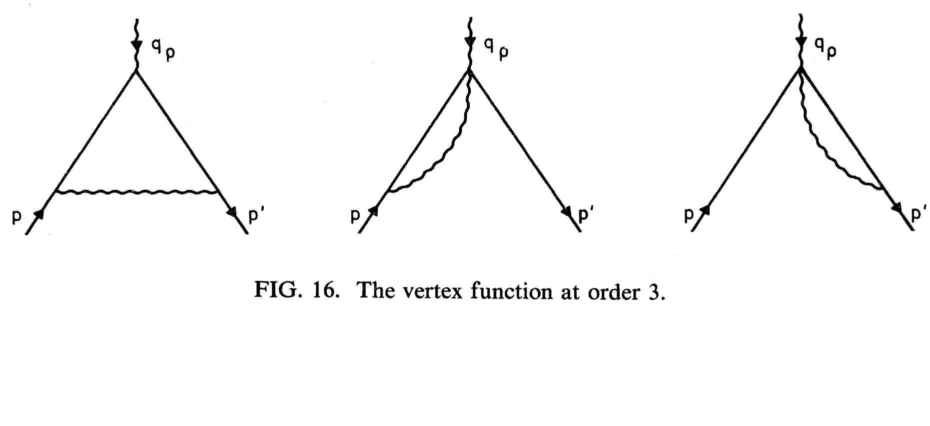

FIG. 16. The vertex function at order 3.

$$V_{\mu\mu\rho}(p,p',k,-k,q)_N=\xi_\mu(p,p',k,N)\xi_\mu(p,p'-k,N)(p+p')_\rho$$
$$-2\left[\left[\frac{g_{\sigma\mu}-k_\sigma(2p+k+N)_\mu}{k(2p+k+N)}\right]\left[\frac{g_{\sigma\rho}-q_\sigma(2p+q+N)_\rho}{qN}\right]\xi_\mu(p,p'-k,N)+(k\leftrightarrow -k)\right], \tag{4.23}$$

where we use the fact that $\xi_\rho(p,p',q,N)$ is null.

Before making the limit $N\to 0$, we note that

$$\left[\left[\frac{g_{\sigma\mu}-k_\sigma(2p+k)_\mu}{k(2p+k)}\right]\xi_\mu(p,p',-k)+(k\leftrightarrow -k)\right]$$
$$=\left[\frac{(2p+k)_\sigma}{k(2p+k)}+\frac{(2p'+k)_\sigma}{k(2p'+k)}-\frac{k_\sigma(2p+k)(2p'+k)}{[k(2p+k)][k(2p'+k)]}+(k\leftrightarrow -k)\right], \tag{4.24}$$

so that

$$\int\frac{d^4k}{k^2}q_\sigma\times(4.24)=0\,. \tag{4.25}$$

The expression for the graphs of Fig. 16 is then

$$-\frac{1}{2}\frac{(p+p')_\rho}{(2\pi)^6}\int\frac{d^4k}{k^2}\xi_\mu(p,p',k)\xi_\mu(p,p',-k)$$
$$+\frac{2}{(2\pi)^6}\int\frac{d^4k}{k^2}\left[\frac{(2p+k)_\rho}{k(2p+k)}+\frac{(2p'+k)_\rho}{k(2p'+k)}-\frac{k_\rho(2p+k)(2p'+k)}{k(2p+k)k(2p'+k)}\right]. \tag{4.26}$$

The expression (4.26) is transverse in q_ρ by (4.25) and is infrared singular. Naive power counting gives an infrared divergence for the first line of (4.26); it is this first line which contributes to the exponentiation of infrared divergences (Sec. V).

(h) One-photon-matter loop Fig. 17.

This example is very formal since the following integrals do not exist. For matter loops, we must calculate the transversal classes $Q_{\{\mu\}}$ in (3.29) and (3.30) and the unitarity classes $Q_{\{\mu\}}$ in (3.34) and (3.35). With the choice (4.5) $\tilde{Q}_\mu(k)$ is zero. Then

$$Q_\mu(k)=\tilde{\tilde{Q}}_\mu(k)=\int d^4p\left[\frac{2p+k}{k(2p+k)}-\frac{2p-k}{k(2p-k)}\right]=0 \tag{4.27}$$

by formal change p in $-p$. This is a consequence of Furry's theorem.

(i) Two-photon-matter loop Fig. 18.

With the choice (4.5),

$$\tilde{Q}_{\mu\nu}(k_1,k_2)=\int\frac{d^4p}{p^2-m^2}[-2\overleftarrow{g}_{\mu\nu}(p,k_1,k_2)-(p^2-m^2)\xi_\mu(p,p,k_1)\xi_\nu(p,p,k_2)]\,, \tag{4.28}$$

$$\tilde{\tilde{Q}}_{\mu\nu}(k_1,k_2)=\int d^4p\,\xi_\mu(p,p,k_1)\xi_\nu(p,p,k_2)+\frac{1}{2}\int d^4p\left[\xi_\mu(p,p+k_1,k_1)\xi_\nu(p+k_1,p,k_2)+\left[\begin{matrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{matrix}\right]\right], \tag{4.29}$$

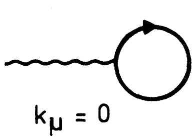

FIG. 17. One-photon matter loop.

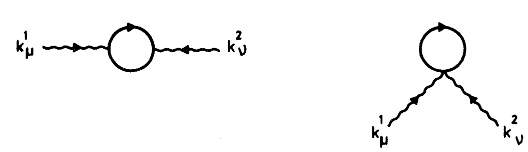

FIG. 18. Two-photon matter loop.

but the choice (4.5) makes $\xi_\mu(p,p+k_1,k_1)$ to be zero. Then

$$Q_{\mu\nu}(k_1,k_2)=-2\int \frac{d^4p}{p^2-m^2}\overleftarrow{g}_{\mu\nu}(p,k_1,k_2) \quad \text{at } (k_1+k_2)=0. \tag{4.30}$$

This result, which is manifestly transverse, can be shown to be equal to (2.21b) after formal manipulations.

These few examples show clearly that classes of invariance are not pure theoretical entities but can be used in practical calculations to simplify greatly the expressions and to rewrite them in a way which manifestly exhibits the properties of transversality and of gauge invariance. The formal aspect of this section will be made rigorous from the infrared point of view at the end of this paper. We believe that ultraviolet subtractions are easy to understand.

V. AN INFRARED-SUBTRACTED THEORY

The approach developed in the preceding sections is essentially formal in the sense that we did not care about the existence (ultraviolet and infrared divergences) of the integrals involved in our field theory. The equivalence theorem which tells that the S matrix is the same in the old ϕ and the new Φ theory is valid only if the old ϕ theory exists and if the transformation from ϕ to Φ is nonsingular. Apart from the ultraviolet divergences, the usual $\phi(x)$ field theory is infrared divergent for matter on the mass shell.

Moreover, the transformation $S(p,\tilde{p})$ defined in (3.9) is singular on certain characteristic manifolds due to the transversality relations (3.17). For instance, if

$$S_\mu(p,k)=\frac{p_\mu}{p\cdot k} \quad \text{or} \quad \frac{(2p+k)_\mu}{k(2p+k)}, \tag{5.1}$$

the hyperplane $p\cdot k=0$ or the hypersurface $k(2p+k)=0$ are singularity surfaces for the transformation $S(p,p')$. Of course, whatever choice is made for $S_\mu(p,k)$, such a singularity surface exists and the origin point $k_\mu=0$ is always on this surface.

How much of a problem do we get from these singularity surfaces? The answer to this problem depends whether we look at the off (matter) mass-shell Green's functions or at the on (matter) mass-shell Green's functions. In this section, we first study the infrared singularities of the $(\Phi,\Phi^\dagger,A_\mu)$ theory. We prove that logarithmic singularities exist even for matter off the mass shell [the usual $(\phi,\phi^\dagger,A_\mu)$ theory cancels these singularities only for matter off the mass shell], and that these infrared singularities exponentiate. In order to achieve this goal, we define an infrared-subtracted theory which is infrared convergent for matter off and on the mass-shell, and we construct the relation between the subtracted and the nonsubtracted theories. All the proofs are based on naive power counting although we know that more rigorous proofs would require more sophisticated power countings.[20]

A. Off (matter) mass-shell Green's functions of the $\Phi(x)$ theory

The presence of a singularity surface for $S_\mu(p,k)$ may destroy the existence of the convolution integrals. Let us try to analyze this difficulty on the following off (matter) mass-shell example: if we look at Fig. 19 and at the structure of the vertices $V_\mu(p,p-k,-k)$ and $V_\mu(q,q+k,k)$ from (3.15), we certainly have to integrate

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}S_\mu(p,-k)S_\mu(q,k)f(p-k,q+k), \tag{5.2}$$

which diverges logarithmically as $k\to 0$. If we make one of the choices (5.1), the divergence is of the type

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{p_\mu}{p\cdot k}\frac{q_\mu}{q\cdot k}f(p,q). \tag{5.3}$$

More generally, the vertices $V_{\mu_1\cdots\mu_n}(p,p',k_1,\ldots,k_n)\sim k_1^{-1}\cdots k_n^{-1}$ when the k's$\to 0$. Such simple arguments show that the transformation S generates infrared divergences for

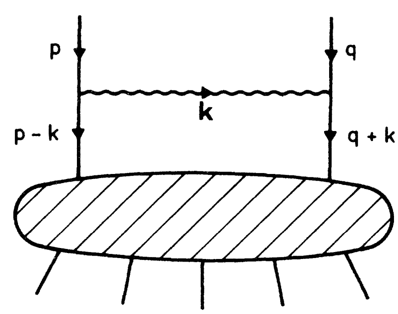

FIG. 19. The elementary infrared divergence.

the $\Phi(x)$ theory which were absent in the $\phi(x)$ theory. We now construct an infrared-subtracted theory.

First, we must take a choice of integration around the singularity surfaces. To follow the standard rule of quantum field theory, we choose the $+i\epsilon$ rule although any other choice might have been considered (this is simply a choice for the new Green's functions and if the equivalence theorem can be made rigorous, the S matrix is independent of this choice); this rule defines properly the Feynman integrals away from the point $k=0$.

Next, the infrared divergences around $k=0$ have to be subtracted. Let us consider the transformation $S(p,\widetilde{p})$ as written in (3.18). If the tensors $T_{\{\mu\}}$ are chosen to exist when $k_i \to 0$, the only singular factor at $k=0$ comes from the exponential term.

It is absurd to make a subtraction over $\psi_\mu(p,k)$; indeed, $\psi_\mu(p,k)$ is necessarily of the form

$$\frac{f_\mu(p,k)}{k\cdot f(p,k)} .$$

Then a possible subtraction at $k=0$ would be to introduce

$$\left[\frac{f_\mu(p,k)}{k\cdot f(p,k)}-\frac{f_\mu(p,0)}{k\cdot f(p,0)}\right]$$

but the transversality properties and the classes of invariance would be completely destroyed by such a choice [the bracketed quantity might even be zero for instance if $f_\mu(p,k)=p_\mu$]. Another absurd choice would be to introduce $[A_\mu(k)-A_\mu(0)]$ since the photon propagator is highly singular at $k=0$ when the photon is massless. The only possibility left to subtract the amplitude at $k=0$ is a subtraction over the shift operator T. We define

$$S_\gamma(p,\widetilde{p},A) =\exp\left[-g\int \delta^4k\, \psi_\mu(\widetilde{p},k)A_\mu(k)\gamma(k)\right]S(p,\widetilde{p},A) , \quad (5.4)$$

where $\gamma(k)$ is any function satisfying $\gamma(0)=1$ [in position space we ask $\gamma(x)$ to be real]. Under a gauge transformation $A_\mu \to A_\mu + ik_\mu\alpha(k)$,

$$S_\gamma(p,\widetilde{p},A)\to e^{-ig\alpha(\gamma)}\exp\left[ig\int\delta^4k\,\alpha(k)T_{p\to p-k}\right] \times S_\gamma(p,\widetilde{p},A) , \quad (5.5)$$

where $\alpha(\gamma)$ is a real space-time constant defined by

$$\alpha(\gamma)=\int \delta^4k\,\alpha(k)\gamma(k) . \quad (5.6)$$

The usual matter field $\phi(x)$ is obtained from a "subtracted" new matter field $\Phi_\gamma(x)$ by

$$\phi(x)=\int d^4\widetilde{x}\, S_\gamma(x,\widetilde{x},A)\Phi_\gamma(\widetilde{x}) , \quad (5.7)$$

so that local gauge transformations of $\phi(x)$ are now induced by the gauge transformation of A_μ in the subtracted "cloud effect" and by a global gauge transformation $e^{ig\alpha(\gamma)}$ of the subtracted new matter field. We do not want to interpret further this property.

The matter part of the Lagrangian can be written in terms of subtracted new matter fields as

$$\int d^4x\,\mathscr{L}_m(x)=\int d^4\widetilde{p}\,d^4\widetilde{\widetilde{p}},\ \Phi_\gamma^\dagger(\widetilde{\widetilde{p}})\widetilde{K}_\gamma(\widetilde{\widetilde{p}},\widetilde{p},A)\Phi_\gamma(\widetilde{p}) \quad (5.8)$$

with

$$\widetilde{K}_\gamma(\widetilde{\widetilde{p}},\widetilde{p},A)=\exp\left[-g\int\delta^4k\,[\psi_\mu^*(\widetilde{\widetilde{p}},-k)+\psi_\mu(\widetilde{p},k)]A_\mu(k)\gamma(k)\right]\widetilde{K}(\widetilde{\widetilde{p}},\widetilde{p},A) . \quad (5.9)$$

The fact that the bracketed quantity is purely transverse is such that the transversality properties of the classes of invariance are unchanged while their expressions will be subtracted at $k=0$. The Lagrangian (5.8) written in terms of subtracted new matter fields is still invariant under the transformations

$$\begin{aligned}&\Phi_\gamma(x)\to e^{i\beta}\Phi_\gamma(x) ,\\ &A_\mu(x)\to A_\mu(x)+\partial_\mu\alpha(x) ,\end{aligned} \quad (5.10)$$

where β and $\alpha(x)$ are defined in (3.10).

Let us now calculate the low-order subtracted classes of invariance

$$V_\mu^\gamma(p,p',k)=V_\mu(p,p',k)\delta(p'-p-k)+\xi_\mu(p,p',k)\gamma(k)(p^2-m^2)\delta(p'-p) , \quad (5.11)$$

where $\xi_\mu(p,p',k)$ is defined in (4.2):

$$V^{\gamma}_{\mu\nu}(p,p',k_1,k_2)=V_{\mu\nu}(p,p',k_1,k_2)\delta(p'-p-k_1-k_2)-\left[\xi_\mu(p,p',k_1)\gamma(k_1)V_\nu(p,p',k_2)\delta(p'-p-k_2)+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right]$$

$$-\xi_\mu(p,p',k_1)\xi_\nu(p,p',k_2)\gamma(k_1)\gamma(k_2)(p^2-m^2)\delta(p'-p)\ , \tag{5.12}$$

etc. Similarly, the subtracted unitarity classes become

$$C^{\gamma}_{\mu}(p,p',k)=C_\mu(p,p',k)\delta(p'-p-k)-\xi_\mu(p,p',k)\gamma(k)\delta(p'-p)\ ,$$

$$C^{\gamma}_{\mu\nu}(p,p',k_1,k_2)=C_{\mu\nu}(p,p',k_1,k_2)\delta(p'-p-k_1-k_2) \tag{5.13}$$

$$-\left[\xi_\mu(p,p',k_1)\gamma(k_1)C_\nu(p,p',k_2)\delta(p'-p-k_2)+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right]$$

$$+\xi_\mu(p,p',k_1)\xi_\nu(p,p',k_2)\gamma(k_1)\gamma(k_2)\delta(p'-p)\ . \tag{5.14}$$

The originality of these subtracted vertices is that their support is not only the usual energy-momentum conservation manifold but also all manifolds obtained from the original ones when any subset of photons momenta are set equal to zero.

The subtracted classes are distributions in the set of external momenta. It is clear that when any set of photons momenta tend to zero, these classes have a limit as a distribution in the remaining external momenta.

The convolutions between subtracted classes of invariance through lateral matter lines can be described from the generating functional:

$$\widetilde{K}_\gamma{}^{-1}(p',p,A)=\exp\left[g\int \delta^4k\ \xi_\mu(p,p',k)\gamma(k)A_\mu(k)\right]\widetilde{K}^{-1}(p',p,A) \tag{5.15a}$$

for each in and out matter line, and from

$$\frac{\det(S_\gamma S_\gamma^\dagger)}{\det\widetilde{K}_\gamma}=\frac{\det(SS^\dagger)}{\det\widetilde{K}}=[\det K]^{-1} \tag{5.15b}$$

for matter and ghost loops. For instance, (3.27) becomes after subtraction

$$[V^{-1}_{\mu\nu}]^{\gamma}(p,p',k_1,k_2)=\frac{V^{\gamma}_{\mu\nu}(p,p',k_1,k_2)}{(p^2-m^2)(p'^2-m^2)}+\int d^4Q\left[\frac{V^{\gamma}_{\mu}(p,Q,k_1)V^{\gamma}_{\nu}(Q,p',k_2)}{(p^2-m^2)(Q^2-m^2)(p'^2-m^2)}+\begin{bmatrix}\mu\leftrightarrow\nu\\ k_1\leftrightarrow k_2\end{bmatrix}\right]. \tag{5.16}$$

The convolutions $[V^{-1}]^{\gamma}$ are distributions in the external momenta with the same properties as those stated above for the elementary classes. At $p=p'$ (forward amplitudes or convolutions along matter or ghost loops), some $\delta^4(0)$ appears in the expressions of the subtracted classes; these contributions disappear for matter on the mass shell or by (5.15b) where we sum over all convolutions around a matter and a ghost loop. These contributions may also disappear for matter off the mass shell and class by class if we choose $\xi_\mu(p,p,k)=0$, that is, for instance $\psi_\mu(p,k)=p_\mu/k\cdot p$. These special choices for $\psi_\mu(p,k)$ are responsible for several simplifications; for instance

$$V^{\gamma}_{\{\mu\}}(p,p,\{k\})=V_{\{\mu\}}(p,p,\{k\})$$

and

$$C^{\gamma}_{\{\mu\}}(p,p,\{k\})=0\ .$$

In that case all convolutions around matter or ghost loops are distributions of the external photons momenta (each δ contains at least two-photon momenta) and have a limit as a distribution in the remaining photon momenta when any subset of photon momenta tend to zero [provided this remaining is not empty to avoid another $\delta^4(0)$]. Anyhow, these complications are completely eliminated when we sum over all classes around a matter and a ghost loop since by (5.15b) we must refind the vertices $Q_{\{\mu\}}(k_i)$ defined in (2.17) which have for only support the surface $(\sum k_i)=0$, and which satisfy the property (2.23b).

When we perform the convolutions of the subtracted classes through lateral photons lines which join them, the above properties ensure the infrared convergence at nonexceptional momenta (no partial sum of external momenta equal to zero). The resulting subtracted Green's functions are infrared convergent and are distributions in the external momenta. Again their support is not only the surface of energy-momentum conservation. These Green's functions have a limit as distributions when the photon momenta tends to zero at nonexceptional matter momenta. It is convenient to relate the subtracted Green's functions to the nonsubtracted Green's functions of Sec. III which have for support only the surface of energy-momentum conservation; of course the Green's functions of Sec. III are infrared divergent and to establish this relation, we find it more rigorous to introduce a mass μ to the photon propagator (2.3). We use

$$\pi_{\mu\nu}(k,\eta)=\frac{-i}{k^2-\mu^2+i\epsilon}\left[g_{\mu\nu}-\frac{\eta_\mu k_\nu+\eta_\nu k_\mu}{k\cdot\eta}+\frac{\lambda k^2+\eta^2}{(k\cdot\eta)^2}k_\mu k_\nu\right], \tag{5.17}$$

which is not a propagator obtained from a Lagrangian, so that μ plays the role of an infrared cutoff; it is important to note that such a cutoff does not destroy the classes of invariance since the photon lines are always lateral interaction lines. Let us note that from the Lagrangian for massive photons we would have obtained a propagator of the type

$$\frac{-i}{(k^2-\mu^2)}\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{\mu^2}\right];$$

such a propagator generates a nonrenormalizable field theory with the usual matter field but we obtain a renormalizable one with the new matter field and we may define an S matrix from it.

From the functional

$$\prod_{\substack{\text{external}\\\text{photons}}}\left[-i\frac{\partial}{\partial\chi_\mu}\right]\prod_{\substack{\text{in-out}\\\text{matter}\\\text{lines}}}\left[i\widetilde{K}_\gamma^{-1}\left[p',p,-i\frac{\partial}{\partial\chi}\right]\right]$$
$$\times\left[\det\widetilde{K}_\gamma\left[p,p',-i\frac{\partial}{\partial\chi}\right]\right]^{-1}\text{Jac}_\gamma\left[-i\frac{\partial}{\partial\chi}\right]\exp\left[-\frac{1}{2}\int d^4k\,\chi_\mu(k)\pi_{\mu\nu}(k)\chi_\nu(-k)\right]\Bigg|_{\chi=0}, \tag{5.18}$$

which generates the subtracted Green's functions $\widetilde{G}^{\gamma}_{\mu_1\cdots\mu_N}(k_1,\ldots,k_N)$ (where we distinguish in the notation only the external photons), and from (5.15) we obtain

$$\begin{aligned}\widetilde{G}^{\gamma}_{\mu_1\cdots\mu_N}(k_1,\ldots,k_N)=\widetilde{G}_{\mu_1\cdots\mu_N}(k_1,\ldots,k_N)&+g\int\delta^4k\,\theta_\mu(\{p,p'\},k)\gamma(k)\widetilde{G}_{\mu\mu_1\cdots\mu_N}(k,k_1,\ldots,k_N)\\&+\frac{g^2}{2}\int\delta^4k\,\delta^4k'\theta_\mu(\{p,p'\},k)\theta_{\mu'}(\{p,p'\},k')\gamma(k)\gamma(k')\\&\times\widetilde{G}_{\mu\mu'\mu_1\cdots\mu_N}(k,k',k_1,\ldots,k_N)+\cdots\,.\end{aligned} \tag{5.19}$$

In (5.19), the Green's functions $\widetilde{G}$ contain, of course, disconnected photon lines and external propagators (not amputated); the quantity θ_μ is defined as

$$\theta_\mu(\{p,p'\},k)=\sum_{\substack{\text{in-out}\\\text{matter}\\\text{lines}}}\xi_\mu(p_i,p_i',k) \tag{5.20}$$

and θ_μ is zero if there is no in-out matter line.

The subtracted Green's functions appear as linear combinations of the nonsubtracted ones with a larger or equal number of external photons. If we consider $\gamma(k)$ $[\gamma(0)=1]$ as a distribution decreasing rapidly around $k=0$, Eq. (5.19) can be interpreted as the introduction of virtual soft photons attached to the external matter [how soft being described by $\gamma(k)$]. Let us give a graphical description of Eq. (5.19). If we describe a Green's function $\tilde{G}$ in terms of a sum of convolutions of vertices $V^{-1}_{\mu_1\cdots\mu_N}(p,p',k_1,\ldots,k_n)$ and $Q_{\mu_1\cdots\mu_n}(k_1,\ldots,k_n)$, in addition to the usual graphs (Fig. 20), we must add all the graphs obtained from the previous ones by taking away from the vertice V^{-1} (and not Q) successively all the photon lines and by attaching them to all external matter lines (Fig. 21).

Figure 20 is part of $\tilde{G}_\rho(q)$; Fig. 21(a) is part of

$$\int \delta^4k_1\delta^4k_2\psi_{\mu_1}(p_1,k_1)\gamma(k_1)\psi_{\mu_2}(p_1',-k_2)\gamma(k_2)G_{\mu_1\mu_2\rho}(k_1,k_2,q)\;;$$

Fig. 21(b) is part of

$$\int \delta^4k_1\cdots\delta^4k_4\psi_{\mu_1}(p_1,k_1)\gamma(k_1)\psi_{\mu_2}(p_1',-k_2)\gamma(k_2)\psi_{\mu_3}(p_1',-k_3)\gamma(k_3)$$
$$\times\psi_{\mu_4}(p_2',-k_4)\gamma(k_4)\tilde{G}_{\mu_1\mu_2\mu_3\mu_4\rho}(k_1,k_2,k_3,k_4,q)\;,$$

etc. When we sum over all possible ways of attaching the photon lines to the external matter lines, the various functions ψ_μ recombine into the transversal quantities ξ_μ and θ_μ.

Conversely, if we wish to express the nonsubtracted Green's functions in terms of the subtracted ones, we use the functional (5.18) without the letter γ, and (5.15) in the inverse way. We obtain a relation of the type (5.19) with the interchange $\tilde{G}\leftrightarrow\tilde{G}^\gamma$ and $\gamma(k)\leftrightarrow-\gamma(k)$. It is then interesting to investigate where the infrared divergences of $\tilde{G}$ are when the photon mass $\mu^2\to0$. Since all $\tilde{G}^\gamma$ are infrared convergent and have a finite limit in the sense of distribution when the momenta of external photons tend to zero, the only infrared divergences come from the integrals $\int d^4k\,\theta_\mu(\{p,p'\},k)\,\tilde{G}^\gamma_\mu(k_1\ldots)$ and more exactly from the photon disconnected parts of the Green's function. It is important to note that these disconnected parts have the same expressions in the subtracted and the nonsubtracted Green's functions. In fact, the photon disconnected parts with matter loops are infrared convergent by (2.23b) so that the only infrared-divergent contributions at $\mu^2=0$ come from graphs an example of which is given in Fig. 22. It is easy to see that in the corresponding sum (5.19) the bare photon propagators exponentiate [see also (5.15) and (5.18)].

Consequently we have

$$\tilde{G}'_{\mu_1\cdots\mu_N}(k_1,\ldots,k_N)=\Bigg[\tilde{G}'^{\gamma}_{\mu_1\cdots\mu_N}(k_1,\ldots,k_N)-g\int\delta^4k\,\theta_\mu(\{p,p'\},k)\gamma(k)\tilde{G}'^{\gamma}_{\mu\mu_1\cdots\mu_N}(k,k_1,\ldots,k_N)$$
$$-\frac{g}{(2\pi)^2}\sum_{i=1}^{n}\theta_{\mu_i}(\{p,p'\},-k_i)\gamma(-k_i)\tilde{G}'^{\gamma}_{\mu_1\cdots\hat{\mu}_i\cdots\mu_N}(k_1,\ldots,\hat{k}_i,\ldots,k_N)+\cdots\Bigg]$$
$$\times\exp\left[-\frac{ig^2}{2(2\pi)^4}\int\frac{d^4k}{k^2-\mu^2+i\epsilon}\theta_\mu(\{p,p'\},k)\theta_\mu(\{p,p'\},-k)\,|\gamma(k)|^2\right],\qquad(5.21)$$

where the Green's functions G' simply means that the bare photon propagators are omitted. In (5.21), we see explicitly under what form will come the infrared divergences of $\tilde{G}'$ when $\mu^2\to0$, and how $\tilde{G}'$ becomes large when some external momenta $k_i\to0$. The combination of $\tilde{G}'^\gamma$ in the square brackets above has for support the usual energy-momentum conservation surface.

B. On (matter) mass-shell Green's functions

The purpose of this subsection is not to define a S matrix for the subtracted matter field Φ_γ (some speculations about such a definition are stated in the Conclusion). Here we wish to explain the organization of the infrared divergences for the S matrix of the theory $(\Phi,\Phi^\dagger,A_\mu)$, that is, for the usual S matrix of the theory $(\phi,\phi^\dagger,A_\mu)$.

As long as the photon mass μ^2 is different from

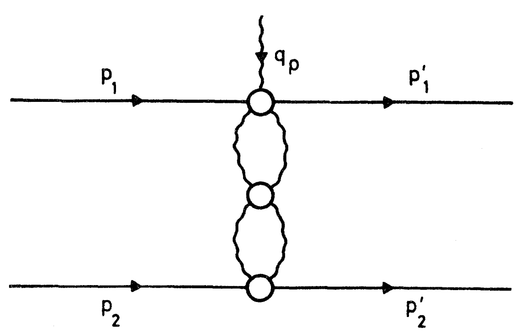

FIG. 20. A graph contributing to $\tilde{G}_\rho(q)$.

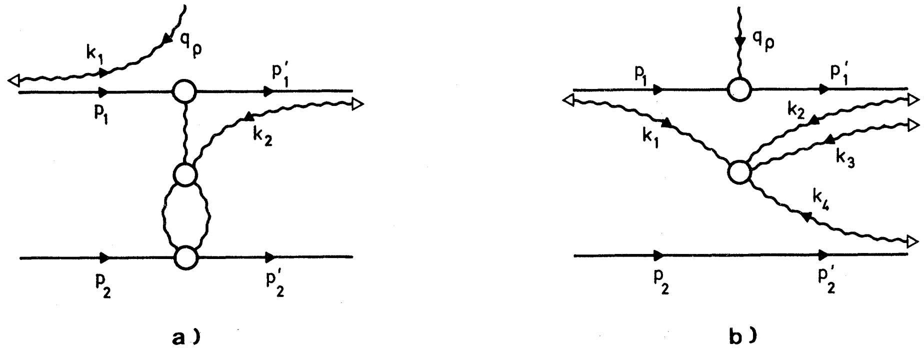

FIG. 21. Two graphs contributing to $\tilde{G}^{\gamma}_{\rho}(q)$.

zero, the equivalence theorem stated in Sec. III is rigorously valid (provided that the ultraviolet divergences are subtracted away). Equations (3.39) and (3.41) and their generalization to all Green's functions can be used. It could be proved that the subtracted longitudinal classes of invariance summed up into quantities which contain and destroy the exponential term in (5.21) so that the usual $\phi(x)$ Green's functions are infrared finite at $\mu^2=0$ when the matter is off the mass shell.

When the matter is on the mass shell, two different things happen.

(a) At $p_i^2=p_i'^2=m^2$, the longitudinal classes (and their exponential infrared part) cancel by ultraviolet renormalization and amputation (see Sec. III). We are left with a structure of the type (5.21).

(b) At $p_i^2=p_i'^2=m^2$ the quantity in square brackets in (5.21), which is a combination of $\tilde{G}'^{\gamma}$ Green's functions, develops new infrared singularities at $\mu^2=0$. We now study these new singularities and show their exponentiation.

The argument which proves that the classes of invariance have a finite limit in the sense of distribution when the photon momenta tend to zero is still valid at $p^2=p'^2=m^2$. On the other hand, the convolutions of classes through lateral matter lines destroy this argument since at $p^2=p'^2=m^2$ the matter propagators are singular when $k\to 0$. From naive power counting, it may be shown that the only convolution of classes which develop infrared problems are the convolutions of one-external-photon class V^{γ}_{μ} (Fig. 23): The propagators 1, 2, and 3 behave successively at small k_i as $(k_1\cdot p)^{-1}$, $[(k_1+k_2)\cdot p]^{-1}$, and $[(k_1+k_2+k_3)\cdot p]^{-1}$; when we perform the convolutions of two matter lines (Fig. 23) through lateral massless photons we develop a logarithmic infrared divergence when $k_1\to 0$, then another such divergence when $k_1=0$, $k_2\to 0$, etc.

We now prove the exponentiation of these divergences and show that they do not cancel the exponential of (5.21) but rearrange to give the well-known Bloch-Nordsieck (BN) and Yennie-Frautschi-Suura[4,5] (YFS) infrared-divergent exponential.

The S matrix for the usual $\phi(x)$ matter field is of course independent of the transformation $S(p,\tilde{p},A)$ defined in (3.18). It turns out that there exists a subclass of these transformations which satisfy the following properties:

$$\lim_{k\to 0} V_\mu(p,p',k)\Big|_{p^2=m^2} \sim 2p_\mu - S_\mu(p,k)2p\cdot k + O(k) \sim O(k) \,,$$
$$\lim_{k\to 0} V_\mu(p,p',k)\Big|_{p'^2=m^2} \sim 2p'_\mu + S^*_\mu(p',-k)2p'\cdot k + O(k) \sim O(k) \,, \tag{5.22}$$

which requires

$$\lim_{k\to 0} S_\mu(p,k) = \lim_{k\to 0} \psi_\mu(p,k) \sim \frac{P_\mu}{p\cdot k} \,. \tag{5.23}$$

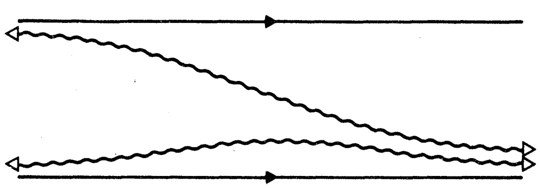

FIG. 22. The infrared-divergent contributions.

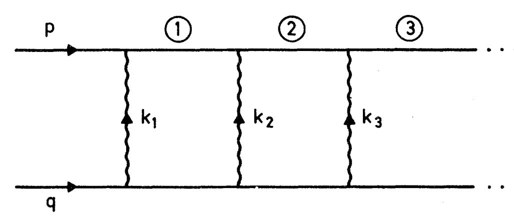

FIG. 23. The on-matter-shell infrared divergence.

With this special choice for $\psi_\mu(p,k)$, $V_\mu(p,p',k)$ as well as $V^\gamma_\mu(p,p',k)$ tend not only towards a finite limit but towards zero when $k \to 0$ and only one matter momentum is on the mass shell. This choice makes the quantity in square brackets in (5.21) finite, so that for this choice all the infrared divergences at $\mu^2=0$ are in the exponential term with, in that case, $\theta_\mu(\{p,p'\},k)$ constrained by the condition (5.23).

Our conclusion is that the S matrix for the usual $\phi(x)$ theory may be written as

$$S_{\mu_1 \cdots \mu_N}(k_1,\ldots,k_N) = \Bigg[S^\gamma_{\mu_1 \cdots \mu_N}(k_1,\ldots,k_N) - \frac{g}{(2\pi)^2} \sum_{i=1}^{n} \theta_{\mu_i}(\{p,p'\},-k_i)\gamma(-k_i) \times S^\gamma_{\mu_1 \cdots \hat{\mu}_i \cdots \mu_N}(k_1,\ldots,\hat{k}_i,\ldots,k_N) + \cdots \Bigg] \times \exp\left[-\frac{ig^2}{2(2\pi)^4} \int \frac{d^4k}{k^2-\mu^2+i\epsilon} \theta_\mu(\{p,p'\},k)\theta_\mu(\{p,p'\}-k)\,|\gamma(k)|^2 \right], \tag{5.24}$$

where

$$S^\gamma_{\mu_1 \cdots \mu_N}(k_1,\ldots,k_N) = \left[\tilde{G}'^{\gamma}_{\mu_1 \cdots \mu_N}(k_1,\ldots,k_N) - g \int \delta^4 k\, \theta_\mu(\{p,p'\},k)\gamma(k) \tilde{G}'^{\gamma}_{\mu\mu_1 \cdots \mu_N}(k,k_1,\ldots,k_N) + \cdots \right]_* . \tag{5.25}$$

In (5.25), the asterisk means ultraviolet renormalized on the mass shell and amputated. The exponential in (5.24) is not unique since the only constraint is (5.23) and $\gamma(0)=1$; at small k, they all coincide with the BN and YFS exponential. All quantities S^γ are infrared finite at $\mu^2=0$ and exist in the sense of distribution when the external photons have their momenta $\to 0$. The expansion (5.24) has a finite number of terms and the expansion (5.25) has an infinite number of terms but only a finite number of terms at each order of perturbation.

VI. INFRARED FINITE QUANTITIES

In this section, outgoing particles have momentum defined in the outgoing direction. Given a physical process with n and r incoming matter and photon particles, and n' and r' outgoing matter and photon particles, the differential cross section corresponding to this process with outgoing photons at a given energy ϵ'_i is given by

$$\frac{\delta^{r'}\sigma}{\delta\epsilon'_1 \cdots \delta\epsilon'_{r'}} = S_{\{\mu\}\{\mu'\}}(p_i,p'_i;q_j,q'_j) \otimes S^*_{\{\mu\}\{\mu'\}}(p_i,p'_i,q_j,q'_j) , \tag{6.1}$$

where the direct product means summation over polarization and integration over phase space restricted to the energies $\epsilon'_1 \cdots \epsilon'_{r'}$. The transversality properties of S makes the sum over 2, 3, or 4 polarizations equal. Because of the presence of the exponential term in the amplitude $S, S \otimes S^*$ contains the following factor:

$$B(p,p') = \exp\left[-\frac{g^2}{(2\pi)^3} \int \frac{d^3k}{2k_0} \theta_\mu(\{p,p'\},k)\theta_\mu(\{p,p'\},-k)\,|\gamma(k)|^2 \right] \underset{\mu\to 0}{\sim} \exp\left\{ -\frac{g^2 \ln\mu^2}{(2\pi)^2} \left[1 - \frac{u}{\tanh u} \right] \right\}, \tag{6.2}$$

which is zero by infrared divergence, while the remaining part of $S \otimes S^*$ is infrared finite for any family of positive ϵ'_i; in (6.2), $p^2=p'^2=m^2$ and $p\cdot p'=m^2\cosh u$. The effect of the infrared divergence is that the probabilities for the above physical processes are simply zero.[4]

A. Differential cross sections with summation over outgoing soft photons (Bloch-Nordsieck[4])

The usual interpretation of the above result is the following: experimentally, the detectors of particles have a certain energy resolution $\Delta\epsilon$ which is characteristic of each detector and which makes them unable to distinguish particles of energy ϵ from particles of energy $\epsilon+\Delta\epsilon$. This experimental fact meets no theoretical difficulty when the limit $\Delta\epsilon\to 0$ exists, which is usually the case for massive particles. For zero-mass particles, since ϵ can be null, we may have an infinite number of final particles of total energy $\leq\Delta\epsilon$ which are not detected. We note that in (6.1) the infrared finite part of $S\otimes S^*$ becomes large when the $\epsilon_j'\to 0$. In addition to the presence of r' photons of energy $\epsilon_j'\pm(\Delta\epsilon/2)$ (this $\Delta\epsilon$ gives no problem), we may have an infinite number of undetected "soft" photons of total energy $\leq\Delta\epsilon$. Consequently, the expression (6.1) should include an infinite sum over outgoing soft photons.

From the structure of the S matrix as described in (5.24), it is easy to see that the quantity in square brackets, which represents the infrared-finite part of the perturbation, can be expanded according to the function θ relative to any subset of external photons. For instance, this quantity can be expanded relative to s external soft photons as

$$S_{0s}\{k_j\}=S'_{0s}\{k_j\}-\frac{g}{(2\pi)^2}\sum_{i=1}^{s}\theta_{\mu_i}(\{p,p'\},+k_i)\gamma(k_i)S'_{0s-1}\{k_{j\neq i}\}+\cdots\ , \tag{6.3}$$

where $S_{0s}\{k_j\}$ is the square brackets of (5.24) for the process (6.1) with s outgoing soft photons of momentum k_j, while $S'_{0s}\{k_j\}$ in the corresponding quantity which has a limit when the $k_s'\to 0$. The above expansion is finite for a given finite number of soft photons. If we calculate (6.1) for any finite number of soft photons, from (6.3), we obtain a polynomial in the logarithmic infrared divergence so that, as already mentioned, the exponential (6.2) makes the probability of such a process equal to zero. But, if we sum the probabilities to have one, two, n soft photons, up to an infinite number of photons with total energy $\leq\Delta\epsilon$, we may obtain a finite result. We now prove this result. We first define a generating functional for soft photons of total energy $\leq\Delta\epsilon$,

$$S_0(J,\Delta\epsilon)=\sum_{n=0}^{\infty}\frac{1}{n!}\int d^3k_i d^3k_n\theta\left[\Delta\epsilon-\sum_{i=1}^{n}k_i^0\right]S_{0n}(k_1,\ldots,k_n)\prod_{i=1}^{n}J(k_i)\ , \tag{6.4}$$

which we write

$$S_0(J,\Delta\epsilon)=\frac{1}{2i\pi}\int_{-\infty}^{+\infty}dy\frac{e^{iy\Delta\epsilon}}{y-i\eta}S_0(Je^{-iyk_0})\ . \tag{6.5}$$

$S_0(Je^{-iyk_0})$ is a short notation for (6.4) where $J(k)\to J(k)e^{-iyk_0}$ and $\Delta\epsilon=+\infty$. The summation over final soft photons should be described by the so-called efficiency operator[21] which tells the ability of the apparatus to absorb the various final states. To compensate the infrared divergences, we must admit that the efficiency operator tends to the unit operator when applied over states of soft photons with energy tending to zero (the apparatus absorb totally the photons of energy zero). For simplicity, we shall here suppose that the efficiency operator is equal to $\mathbb{1}$ over the soft-photon subspace.

Consequently, the quantity (6.1) after summation over the final soft photons can be obtained from

$$S_0\left[\frac{\partial}{\partial z_1},\Delta\epsilon\right]\otimes S_0^*\left[\frac{\partial}{\partial z_2},\Delta\epsilon\right]\exp\left[-2\pi\int\frac{d^3k}{2k_0}z_1(k)z_2(k)\right]\Bigg|_{z_1=z_2=0}\ . \tag{6.6}$$

In this section δ^3k means $d^3k/(2\pi)^3$; we have taken into account the summation over the polarizations $(\sum_i\epsilon_\mu^i\epsilon_\nu^i=-g_{\mu\nu})$. Now, the decomposition (6.3) shows that

$$S_0(Je^{-iyk_0})=\exp\left[-\frac{g}{(2\pi)^2}\int d^3k\,\theta_\mu(\{p,p'\},k)\gamma(+k)J(k)e^{-iyk_0}\right]S_0'(Je^{-iyk_0})\ , \tag{6.7}$$

where the generating functional S_0' has no more singularity in the momentum of the soft final photons.

Finally, taking into account (6.1), (6.2), and (6.5)–(6.7), we find the infrared-finite probability

$$\left[\frac{\delta^{r'}\sigma}{\delta\epsilon_1'\cdots\delta\epsilon_{r'}'}\right]_{\Delta\epsilon}=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dy\,dy'\frac{e^{i(y-y')\Delta\epsilon}}{(y-i\eta)(y'+i\eta)}S_0'\left[\frac{\partial}{\partial z_1}e^{-iyk_0}\right]\otimes S_0'\left[\frac{\partial}{\partial z_2}e^{iy'k_0}\right]$$
$$\times\exp\left\{-2\pi\int\frac{d^3k}{2k_0}\left[z_1(k)\cdot z_2(k)-\frac{g}{(2\pi)^2}\theta(\{p,p'\},+k)\gamma^*(+k)e^{iy'k_0}z_1(k)\right.\right.$$
$$\left.\left.-\frac{g}{(2\pi)^2}\theta(\{p,p'\},k)\gamma(+k)e^{-iyk_0}z_2(k)\right]\right\}_{z_1=z_2=0}$$
$$\times\exp\left\{-g^2\int\frac{\delta^3k}{2k_0}\theta_\mu(\{p,p'\},+k)\,|\,\gamma(k)\,|^2\right.$$
$$\left.\times[\theta_\mu(\{p,p'\},-k)+\theta_\mu(\{p,p'\}+k)e^{-ik_0(y-y')}]\right\}. \tag{6.8}$$

The derivatives $\partial/\partial z_i$ generate an infrared-convergent expansion; the integrations over dy and dy' are easily performed in perturbation. This completes the proof of the infrared existence of the Bloch-Nordsieck probabilities.

B. The Kinoshita-Lee-Nauenberg probabilities[15]

The first question we want to solve here is the following: it is clear that the initial state is also experimentally known up to some soft photons; suppose that in the infrared-convergent Bloch-Nordsieck formalism presented in A, we change the initial state into the same initial state + one soft photon: Is the probability going to diverge when the momentum of the soft photon tends to zero?[16] Of course such a result would not be understandable. We are going to prove that this is not the case and, more generally, if the initial state contains any number of soft photons (with total energy $\leq\Delta\epsilon$ to be nondetectable), then the probability remains finite when these initial photons become softer and softer.

As a consequence we may sum in the initial state over an infinite number of nondetectable soft photons according to any normalized density matrix and we do obtain an infrared-convergent probability usually called the Kinoshita-Lee-Nauenberg probability. This statement is the generalization to quantum electrodynamics of the Lee-Nauenberg theorem of quantum mechanics (in the case of massive matter and massless photons).

Let us consider the functional $S_0'(J)$ which is introduced in (6.7) and which has no singularity with the soft final photons; we denoted by S_{0n}' the coefficient of this functional which describes n final soft photons and no initial soft photons. We suppose now that the initial state contains one soft photon of momentum s; depending whether this soft photon interacts in a connected way with the system or not, we may write

$$S_{1n}'(s,k_1,\ldots,k_n)=S_{1n}'^{c}(s;k_1,\ldots,k_n)-\sum_{i=1}^{n}S_{0n-1}'^{c}(k_1,\ldots,k_2,\ldots,k_n)2s_0/2\pi\delta^3(s-k_i) \tag{6.9}$$

and, if we wish to introduce the quantities with no singularity in the initial soft photon,

$$S_{1n}'^{c}(s;k_1,\ldots,k_n)=S_{1n}''(s;k_1,\ldots,k_n)-\frac{g}{(2\pi)^2}\theta(\{p,p'\}-s)\gamma(-s)S_{0n}''(k_1,\ldots,k_n)\,. \tag{6.10}$$

The minus sign in (6.9) is due to the negative norm of the polarization vectors. From (6.9) and (6.10), we obtain for the functional $S_1'(s,J)$:

$$S_1'(S,J)=S_1''(S,J)-\left[\frac{g}{(2\pi)^2}\theta(\{p,p'\},-s)\gamma(-s)+2s_0\frac{J(+s)}{2\pi}\right]S_0'(J)\,. \tag{6.11}$$

After insertion of (6.11) in (6.8), we obtain an infrared-finite result of the type (6.8) where $S_0' \otimes S_0'$ has been changed into

$$S_1''\left[s,\frac{\partial}{\partial z_1}e^{-iyk_0}\right]\otimes S_1''^{*}\left[s,\frac{\partial}{\partial z_2}e^{iy'k_0}\right]+S_1''\left[s,\frac{\partial}{\partial z_1}e^{-iyk_0}\right]\otimes S_0'\left[\frac{\partial}{\partial z_2}e^{iy'k_0}\right]A(p,p',y,y',s,z_1)$$

$$+S_0'\left[\frac{\partial}{\partial z_1}e^{-iyk_0}\right]\otimes S_1''\left[s,\frac{\partial}{\partial z_2}e^{iy'k_0}\right]A(p,p',y,y',s,z_2)$$

$$+S_0'\left[\frac{\partial}{\partial z_1}e^{-iyk_0}\right]\otimes S_0'\left[\frac{\partial}{\partial z_2}e^{iy'k_0}\right] \tag{6.12a}$$

$$\times\left[-\frac{2s_0\delta^3(0)}{2\pi}e^{-i(y-y')s_0}+A(p,p',y,y',s,z_1)A^*(p,p',y',y,s,z_2)\right], \tag{6.12b}$$

where

$$A(p,p',y,y',s,z)=z(s)e^{iy's_0}-\frac{g}{(2\pi)^2}\left[\theta(\{p,p'\},-s)+\theta(\{p,p'\},s)e^{i(y'-y)s_0}\right]\gamma(s)\,. \tag{6.12c}$$

In (6.12), the small-s limit exists and this proves the finite amount of the one-initial-soft-photon correction. The quantity $\delta^3(0)$ is due to the way we have evaluated the probabilities in (6.6); for disconnected single photons we should introduce normalized wave packets and replace $[2s_0\delta^3(0)/2\pi]e^{i(y-y')s_0}$ by a finite function of $(y-y')$ equal to one at $y=y'$. This result can be extended to any number of initial soft photons.

Equation (6.11) becomes

$$S_n'(s_1,\ldots,s_n,J)=\sum_{p=0}^{\infty}\sum_{sym}(-)^p\prod_{i=1}^{p}\left[\frac{g}{(2\pi)^2}\theta(\{p,p'\},-s_i)\gamma(-s_i)+2s_i^0\frac{J(+s_i)}{2\pi}\right]S_{n-p}''(s_{p+1},\cdots,s_n,J)\,. \tag{6.13}$$

The $1/s_i$ factor developed by the function θ is always canceled when $J(+s_i)$ is transformed into $(\partial/\partial z)(+s_i)$ and applied to the functional (6.8).

Consequently, there is no need to sum over initial soft photons to obtain infrared convergence in QED although such a sum with a normalized density matrix may be performed without developing any divergences. Recent results seem to indicate that the sum over initial soft gluons is necessary in QCD.[22]

C. Amplitudes between coherent states

It is not the purpose of this subsection to develop a theory of coherent states; this has been done largely in the literature and we refer to it.[17] It is known that the coherent states which lead to infrared-finite amplitudes do not belong to the Fock space constructed from free matter and photon fields because the norm of these states in this space is infrared singular. In a rigorous way, we should introduce coherent states in a Fock space of massive photon fields so that their norm remain finite; we should calculate amplitudes between such coherent states via the interactions of massive photons and matter, and at the end of all calculations let the photon mass tend to zero and verify the existence of such amplitudes. Here, we intend simply to explain the combinatoric which leads to infrared-finite amplitudes.

Coherent states may be described in the usual Fock space by

$$N\prod_{\substack{\text{final}\\ \text{matter}\\ \text{particles}}}\left[\exp\left[\frac{g}{2\pi}\int_\Omega\frac{d^3k}{2k_0}\frac{p\cdot\epsilon^{(i)}(k)}{p\cdot k}a^{\dagger(i)}(k)\right]\phi^\dagger(p)\right]\prod_{\substack{\text{hard}\\ \text{photon}}}a^{\dagger(j)}(k_i)\,|\,0\rangle\,. \tag{6.14}$$

In (6.14), N is a normalization factor which is finite as long as the photons are massive. We could have used coherent states of photons (without matter) since the states with a finite number of photons can be written as an expansion of coherent states of photons[17]; for simplicity we prefer to split the momentum space of photons into a soft part Ω containing $k=0$ and a hard part. Let us calculate N:

$$N=\exp\left[\frac{g^2}{2}\int_\Omega \frac{\delta^3 k}{2k_0}\left[\sum_i \frac{p_i}{p_i k}\right]\cdot\left[\sum_i \frac{p_i}{p_i k}\right]\right]. \tag{6.15}$$

We now compute the S matrix between two states (6.14). The S matrix between matter and hard photons will be denoted by $B^{1/2}(p,p')S_{00}$ while the S matrix between matter, hard photons, n_1 initial soft photons, and n_2 final soft photons is denoted by $B^{1/2}(p,p')S_{n_1 n_2}(k_1,\ldots,k_{n_1};k_1',\ldots,k_{n_2}')$. Of course S_{n_1,n_2} contains the possibility of having soft disconnected photons and must be decomposed into connected parts according to soft photons. So,

$$\sum_{n,m}^{\infty}(-)^m\frac{(g/2\pi)^{n+m}}{n!m!}\int_\Omega \frac{d^3k_1}{2k_0^1}\cdots\frac{d^3k_n}{2k_0^n}\frac{d^3k_1'}{2k_0'^1}\cdots\frac{d^3k_m'}{2k_0'^m}\prod_{j=1}^{n}\left[\sum\left[\frac{p_i}{p_i k_j}\right]\right]\prod_{l=1}^{m}\left[\sum\left[\frac{p_i'}{p_i'\cdot k_l}\right]\right]S_{nm}(k_j,k_l')$$
$$=\exp\left\{-g^2\int_\Omega\frac{\delta^3k}{2k_0}\left[\sum\left[\frac{p_i}{p_i\cdot k}\right]\right]\cdot\left[\sum\left[\frac{p_i'}{p_i'\cdot k}\right]\right]\right\}\sum_{n,m=1}^{\infty}\frac{g^{n+m}}{n!m!}\cdots S_{nm}^c(k_j,k_l'). \tag{6.16}$$

Moreover, S_{nm}^c still contains singularity in the soft-photon momentum which can be exponentiated: (6.16) becomes

$$\exp\left[-g^2\int_\Omega\frac{\delta^3k}{2k_0}\left\{\left[\sum\left[\frac{p_i}{p_i\cdot k}\right]\right]\cdot\left[\sum\left[\frac{p_i'}{p_i'\cdot k}\right]\right]+\left[\sum\left[\frac{p_i}{p_i k}\right]\right]\theta(\{p,p'\},+k)\gamma(+k)\right.\right.$$
$$\left.\left.+\left[\sum\left[\frac{p_i'}{p_i'\cdot k}\right]\right]\theta(\{p,p'\},-k)\gamma(-k)\right\}\right]\sum_{n,m=1}^{\infty}(-)^m\frac{g^{n+m}}{n!m!}\cdots S_{nm}'(k_j,k_l'), \tag{6.17}$$

where $S_{nm}'(k_j,k_l')$ has no singularity in the soft-photon momentum. When we collect the exponential terms in N_i and N_f, in $B(p,p')$, and in (6.17), we see that the point $k=0$ does not generate anymore infrared singularities for massless photons. This result remains valid if the coherent states are defined from any function $f_\mu(p,k)$ such that $\lim_{k\to 0} f_\mu(p,k)\sim p_\mu/p\cdot k$.

VII. CONCLUSION

In this work, we have presented a new organization of the perturbation expansion for Abelian gauge fields interacting with scalar matter. The introduction of classes of invariance helped us to understand the symmetry properties, the gauge-dependent and -independent parts of the theory, in a detailed manner which is not transparent from the Ward-Takahashi identities over Green's functions. Moreover, as an application of this structure, we have been able to understand better the organization of the infrared divergence and its connection to gauge invariance. The above considerations permit us easily to prove the infrared existence for arbitrary processes, of the Bloch-Nordsieck and the Kinoshita-Lee-Nauenberg probabilities, and of the transition amplitudes between coherent states.

Apart from the problem of defining the ultraviolet renormalization and of the problem of describing spinor electrodynamics in terms of classes of invariance, it seems to us that more investigation should be performed on the interpretation of the new matter field $\Phi_\gamma(x)$. The infrared existence of the corresponding S matrix might indicate that physical processes should be described in terms of the new matter field $\Phi_\gamma(x)$. The physical system which is usually described by a Lagrangian invariant under local gauge transformations of matter and photon fields (in a coupled way) cannot asymptotically reduce itself to an "in" or "out" system of free matter fields which are known to exhibit only global gauge invariance, and of free photon fields which still exhibit local gauge invariance; the infrared divergences reflect this impossibility. When the system is described in terms of the new subtracted matter fields each interacting field exhibits already its asymptotic symmetry in a completely decoupled way. Once the Green's functions have been shown to exist (subtracted S

transformation), the absence of infrared divergences in the corresponding S matrix indicate that the Lehmann-Symanzik-Zimmermann[23] formalism for the reduction formulas can be applied. Consequently, the matter part of the Fock space should be built not from the in and out asymptotic $\phi(x)$ field but from the in and out asymptotic new subtracted $\Phi_\gamma(x)$ field. Finally, coherent states in the usual matter-field Fock space is an attempt of mapping the new matter states of the new Fock space upon the usual Fock space (as expected this mapping is singular).

It is in this spirit that we now consider the case of non-Abelian field theory. The main difference there is that the gluon field plays simultaneously the role of zero-mass matter field as well as the role of radiative-photon field. As a consequence the S transformations which define new matter fields must certainly act upon the gluon field as well. We expect (and this is supported by the construction of lowest-order classes of invariance) that the generalization of the present approach to QCD can be performed and this will be the subject of our next investigation.

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