Asymptotically free fluids

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(Received 15 July 1982)

I study the field theory of a scalar field ω with solely cubic interactions in three dimensions—$(\omega^3)_3$. I insist that the ω field represents the density of a fluid, so ω must always be ≥ 0. If there are long-range couplings in the fluid such that the inverse ω propagator is linear rather than quadratic in momentum, then $(\omega^3)_3$ theory is asymptotically free. The asymptotic freedom of $(\omega^3)_3$ theory is closely related to the existence of a nonzero, ultraviolet-stable fixed point in large-N $(\bar{\phi}^6)_3$ theory. There are also analogies between the asymptotic freedom of fluids (in three and six dimensions) and that of non-Abelian gauge fields (in four dimensions).

I. INTRODUCTION

A profound triumph of the renormalization group is the explanation of scaling in hadronic processes at high energies. Scaling is understood to reflect the underlying asymptotic freedom of the strong interactions. In four dimensions, the restriction of asymptotic freedom leads immediately to a theory of SU(3) non-Abelian gluons with a few light fermions—quantum chromodynamics $(\mathrm{QCD})_4$.[1]

Most field theories are infrared free.[2] Four-dimensional quantum electrodynamics with any number of scalars or fermions—scalar or fermion $(\mathrm{QED})_4$ — provides a familiar example. In these models, infrared freedom can be viewed as quantum diamagnetism, as the response of fluctuations in the physical vacuum to a constant background magnetic field.[3]

A partial understanding of asymptotic freedom in $(\mathrm{QCD})_4$ can be gained from the theory of massless vector fields (having anomalous magnetic moments with $g=2$) coupled to photons—vector $(\mathrm{QED})_4$.[3] Unlike scalar $(\mathrm{QED})_4$, vector $(\mathrm{QED})_4$ is asymptotically free, where the change in sign is due solely to the fact that the vector fields carry anomalous magnetic moments. Hence the asymptotic freedom of vector $(\mathrm{QED})_4$, and so $(\mathrm{QCD})_4$,[3] can be looked upon as a result of quantum paramagnetism.

It is also helpful to develop analogies to simpler asymptotically free theories,[4] such as those involving only scalar fields with at best global, instead of local, symmetries. The classic instance of such a theory is the nonlinear $\bar{\sigma}$ model.[5] However, even though asymptotic freedom is determined by calculation in perturbation theory, the diagrams of the nonlinear $\bar{\sigma}$ model and $(\mathrm{QCD})_4$ bear no obvious relation to each other.

An asymptotically free theory of scalars for which the diagrams of perturbation theory literally look like those of $(\mathrm{QCD})_4$ is provided by $(\phi^3)_6$ theory, the field theory of a single field with cubic interactions in six dimensions.[6] Nevertheless, $(\phi^3)_6$ theory has rightly been dismissed as a toy of perturbation theory, since if ϕ can assume negative values, then even classically there is not stable ground state.

A crucial observation of this paper is that $(\phi^3)_6$ theory can be transformed into a reputable field theory, $(\omega^3)_6$, by stipulating that it characterize an effective theory of fluids in six demensions. This is accomplished simply by replacing ϕ $(\phi \equiv -\infty \rightarrow +\infty)$ by ω, where ω, as the density of the fluid, is *by definition* positive semidefinite.[7] This transmutation, albeit elementary, is by no means devoid of physical content, as the considerations of Secs. II and IV will show. There is at least no need to study perturbative $(\omega^3)_6$ theory: order by order in the cubic coupling constant, the renormalization-group functions of $(\phi^3)_6$ and $(\omega^3)_6$ theories are identical.

The model I study here will not be $(\omega^3)_6$ theory but a three-dimensional theory which is in some sense the square root of $(\omega^3)_6$, $(\omega^3)_3$. The essential difference between them is that while the inverse ω propagator in momentum space of $(\omega^3)_6$ theory is the familiar p^2, in $(\omega^3)_3$ it is p,[8] as could arise from long-range couplings in the ω fluid. Aside from detailed numerical differences as in the coefficients of perturbation theory, qualitatively $(\omega^3)_3$ and $(\omega^3)_6$ theories should be very similar.

In the next section, I derive $(\omega^3)_3$ theory from a large-N tricritical field theory, $(\bar{\phi}^6)_3$. Previously, I have shown that large-N $(\bar{\phi}^6)_3$ theory, albeit infrared free, has a nonzero ultraviolet-stable fixed point calculable in perturbation theory.[9] Large-N

$(\bar{\phi}^6)_3$ theory provides the only example of such a field theory to date.[10] It will be seen that the existence of an ultraviolet-stable fixed point in large-N $(\bar{\phi}^6)_3$ theory is an immediate consequence of asymptotic freedom in $(\omega^3)_3$ theory. This insight provided the original motivation for the present study. The explanation for the above (admittedly indirect) introduction, and the thrust of this work, lies in my belief that the renormalizable fluids $(\omega^3)_3$ and $(\omega^3)_6$ are of fundamental interest in their own right.

The subject of Sec. III is perturbative $(\omega^3)_3$ theory, that of Sec. IV possible nonperturbative behavior. I suggest in Sec. IV that ω fluids provide the first examples of asymptotically free scalars that exhibit nonzero infrared-stable fixed points in strong coupling (for a dimensionless coupling constant). I conclude with an elementary discussion of asymptotic freedom in ω fluids.

II. LARGE-N $(\bar{\phi}^6)_3$ AND $(\omega^3)_3$ THEORIES

The mechanics for the following derivation of $(\omega^3)_3$ theory is straightforward and not even original,[11] so I shall be brief. Let the field $\bar{\phi}(\bar{x})$ transform as a scalar under Euclidean rotations, and as an isovector under a global symmetry of O(N) isospin. In $3-\epsilon$ dimensions, the generic renormalized Lagrangian density $\mathscr{L}_{\bar{\phi}}$ is of the form

$$\mathscr{L}_{\bar{\phi}}=\tfrac{1}{2}\bar{\phi}(Z_{\bar{\phi}}p^2+4Z_m m^2)\bar{\phi} +2Z_g g(\bar{\phi}^2)^2+\tfrac{4}{3}\pi Z_\lambda \lambda(\bar{\phi}^2)^3 . \quad (2.1)$$

λ must be positive for a stable ground state to exist, but m^2 and g can be of either sign. I regularize the renormalization constants (the Z's) by minimal subtraction in ϵ^{-1}; I also insist that the dimensional parameters m^2 and g be renormalized multiplicatively.[12] A tricritical phase diagram is obtained by varying m^2 and g. Of particular interest are the critical ray $m^2=0$, $g>0$, which is a line of second-order transitions; the tricritical point $m^2=g=0$; and a line of first-order transitions for $m^2>0$, $g<0$. Implicitly, I assume the phase is one of unbroken O(N) symmetry.

Given the partition function of $\bar{\phi}$, $P_{\bar{\phi}}$,[13]

$$P_{\bar{\phi}}(m^2,g,\lambda;N)=\int_{-\infty}^{+\infty} d\bar{\phi}(x)\exp\left[-\int d^d\bar{x}\mathscr{L}_{\bar{\phi}}\right] \quad (2.2)$$

$(d=3-\epsilon)$, I introduce unity into $P_{\bar{\phi}}$ as

$$1\sim\int_0^\infty d\omega(\bar{x})\int_{-\infty}^{+\infty} d\alpha(\bar{x})\exp\left[-\int d^d\bar{x}\mathscr{L}'\right],$$

$$\mathscr{L}'=i\alpha\left[\omega-\frac{2}{N^{1/2}}\bar{\phi}^2\right] . \quad (2.3)$$

This trick allows for the elimination of the isovector $\bar{\phi}$ in favor of two isoscalars α and ω. The resulting theory of α and ω contains a nonlocal α interaction, which generates an infinite series of local terms. In an expansion around infinite N this nonlocal interaction can be approximated by the two-point function of α. The integration over α can then be carried out to yield a partition function of ω alone, P_ω. The result is that to leading order in N^{-1} about infinite N,

$$P_{\bar{\phi}}(m^2,g,\lambda;+\infty)=P_\omega(\tilde{m}^2,\tilde{g},\tilde{\lambda}) , \quad (2.4)$$

where

$$P_\omega(\tilde{m}^2,\tilde{g},\tilde{\lambda})=\int_0^\infty d\omega(\bar{x})\exp\left[-\int d^d x\mathscr{L}_\omega\right] \quad (2.5)$$

$(d=3-\epsilon)$, with

$$\mathscr{L}_\omega=\tfrac{1}{2}\omega(Z_\omega p^{1+\epsilon}+Z_g\tilde{g})\omega+Z_m\tilde{m}^2\omega+\frac{\pi}{6}Z_\lambda\tilde{\lambda}\omega^3 , \quad (2.6)$$

$$\tilde{m}^2=N^{1/2}m^2,\ \ \tilde{g}=Ng,\ \ \tilde{\lambda}=N^{3/2}\lambda , \quad (2.7)$$

where Z_ω is the ω wave-function renormalization.[14]

The correspondence between large-N $(\bar{\phi}^6)_3$ and $(\omega^3)_3$, theories, as expressed by Eq. (2.4), holds only if the phase of $\mathscr{L}_{\bar{\phi}}$ is one of unbroken symmetry.[15] I consider the entire phase diagram of $\mathscr{L}_\omega$, with $\tilde{m}^2$ and $\tilde{g}$ of arbitrary sign, for its intrinsic interest. My discussion of $(\omega^3)_3$ theory is predicated upon the parameters $\tilde{m}^2$, $\tilde{g}$, and $\tilde{\lambda}$ approaching finite, nonzero values independent of N as $N\to+\infty$, so in particular I must take $g\sim N^{-1}$ and $\lambda\sim N^{-3/2}$. I emphasize that the identity of Eq. (2.4) holds only to leading order in N^{-1}; at higher orders in N^{-1}, fluctuations in α cannot be integrated out easily and need to be included explicitly.

$(\omega^3)_3$ represents an effective theory for some sort of fluid because of the constraint $\omega(\bar{x})\geq 0$ in the measure of P_ω, as follows immediately from the definition of $\omega\sim\bar{\phi}^2$. The restriction to $\omega\geq 0$ does of course imply that the discrete symmetry $\omega\to-\omega$ is not a symmetry of the theory. Consequently, since no symmetry prohibits it, the term of $\mathscr{L}_\omega$ linear in ω, $\sim\tilde{m}^2\omega$, does not represent a constant external source in ω, but merely another dimensional coupling, like the term for the ω "mass," $\sim\tilde{g}\omega^2$. In addition, because of the constraint $\omega \geq 0$, if $\tilde{m}^2 \neq 0$, $g > 0$, even for the "trivial" noninteracting theory ($\tilde{\lambda}=0$), there is no general method by which to calculate P_ω.[16] Fortunately, if $\tilde{m}^2=0$, $\tilde{g} \geq 0$, perturbative calculations in $\tilde{\lambda}$ can proceed by ignoring the constraint in ω. This allows for the calculation of the renormalization-group functions in the next section.

I turn now to a discussion of the phase diagram for the ω fluid. It is not obvious, but because I choose $\lambda \sim N^{-3/2}$, the ω phase diagram has no relation to the usual tricritical phase diagram for $\mathscr{L}_{\bar{\phi}}$. This will be explained at the end of Sec. III.

I begin with a discussion of $\mathscr{L}_\omega$ in the classical limit, as determined by minimizing $\mathscr{L}_\omega$ at zero momentum. In other words, this classical phase diagram is that of mean field theory, where all fluctuations in ω are suppressed.

Classically, the ω phase diagram consists of two phases. In one, there is condensation, with a vacuum expectation value $\langle \omega \rangle \neq 0$, while the other is a gaseous phase, with $\langle \omega \rangle = 0$. Separating these phases are a line of first-order transitions ($\tilde{m}^2 > 0$, $\tilde{g} < 0$) and a line of liquid-gas coexistence ($\tilde{m}^2=0$, $\tilde{g} > 0$), which meet at $\tilde{m}^2=\tilde{g}=0$ in a critical point for the ω fluid. Thus a tricritical point in $\mathscr{L}_{\bar{\phi}}$ has apparently become a critical point in $\mathscr{L}_\omega$.

Since the classical phase diagram is found from $\mathscr{L}_\omega$ at zero momentum, precisely the same diagram applies for $(\omega^3)_6$ theory.[17]

The classical diagram is quite misleading in one respect. Once fluctuations in ω are included, the true $\langle \omega \rangle$ will always be nonzero; there is always some "condensation." As for any fluid, the only place in the phase diagram where it is possible to meaningfully distinguish between the liquid and gaseous phases is right on the first-order line, where the two are simultaneously in coexistence.

What I do take from classical analysis to be a feature of the true theory is that there is a line of first-order transitions ending in some point. This is considered at length in Sec. IV.

Before considering how quantum fluctuations affect the ω fluid in general, I consider the only instance which can be treated analytically, which is perturbation theory in $\tilde{\lambda}$.

III. PERTURBATIVE $(\omega^3)_3$ THEORY

In this section I calculate the renormalization-group functions for the $\mathscr{L}_\omega$ of Eq. (2.6) to leading and next to leading order in $\tilde{\lambda}$. I avoid the complications inherent in fluids[16] by concentrating on the critical theory, $\tilde{m}^2=\tilde{g}=0$. I renormalize the theory as in Sec. II, noting that with dimensional regularization, the renormalization-group functions will change if the Laplacian in $3-\epsilon$ dimensions $p^{1+\epsilon}$ is replaced by p. I use the former because of the direct identification with large-N $(\bar{\phi}^6)_3$ theory which then follows.

To $\mathscr{L}_\omega$, I add the source term

$$\mathscr{L}_J = \tfrac{1}{2} J_{\omega^2} Z_{\omega^2} \omega^2 . \tag{3.1}$$

In three dimensions, the renormalization-group functions are given by

$$\tilde{\beta} = -2\epsilon \left[\ln \left(\frac{\tilde{\lambda} Z_\lambda}{Z_\omega^{3/2}} \right) \right]^{-1}$$

$$= -b_1 \tilde{\lambda}^3 - b_2 \tilde{\lambda}^5 + \cdots , \tag{3.2a}$$

$$\gamma_\omega = \tilde{\beta} \frac{\partial}{\partial \tilde{\lambda}} \ln Z_\omega$$

$$= c_1 \tilde{\lambda}^2 + c_2 \tilde{\lambda}^4 + \cdots , \tag{3.2b}$$

$$\gamma_{\omega^2} = \tilde{\beta} \frac{\partial}{\partial \tilde{\lambda}} \ln Z_{\omega^2}$$

$$= d_1 \tilde{\lambda}^2 + d_2 \tilde{\lambda}^4 \cdots . \tag{3.2c}$$

Calculation yields

$$\begin{aligned} &b_1 = \tfrac{1}{2}, \quad b_2 = \tfrac{1}{32}(\pi^2+6) , \\ &c_1 = 0, \quad c_2 = \tfrac{1}{4} , \\ &d_1 = \tfrac{1}{2}, \quad d_2 = \tfrac{1}{32}(\pi^2+18) . \end{aligned} \tag{3.2d}$$

The leading term in the $\tilde{\beta}$ function b_1 is given by a single diagram, Fig. 1(a). The second term b_2 receives contributions from several diagrams, Figs. 1(b)–1(e). The diagrams for γ_{ω^2} are similar to those of Figs. 1(a)–1(d). The terms in b_2 and d_2 proportional to $\sim \pi^2$ arise exclusively from diagrams like that of Fig. 1(c). The leading term in γ_ω vanishes, with the only contribution to c_2 from the diagram of Fig. 1(e). Under changes in the renormalization procedure $\tilde{\lambda} \to \tilde{\lambda}' + O((\tilde{\lambda}')^2)$, the only value of Eq. (3.2d) which is affected is d_2.

The diagrams of Fig. 1 are almost identical to those of $(\omega^3)_6$ theory. The greatest difference is that fewer diagrams contribute in $(\omega^3)_3$ theory to Z_ω; e.g., $c_1 \neq 0$ in $(\omega^3)_6$ theory.[6(b)]

Since the $\tilde{\beta}$ function is negative to leading order, $(\omega^3)_3$ theory is asymptotically free. That is, the effective coupling $\tilde{\lambda}(p)$ vanishes in the ultraviolet limit

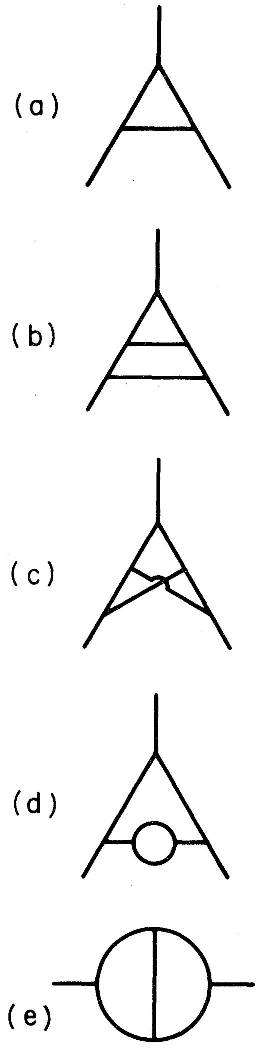

FIG. 1. Contributions to Z_λ at $\sim O(\widetilde{\lambda}^2)$ [Fig. 1(a)] and $\sim O(\widetilde{\lambda}^4)$ [Figs. 1(b)–1(d)]. The first nonzero contribution to Z_ω occurs at $\sim O(\widetilde{\lambda}^4)$ [Fig. 1(e)].

$$\widetilde{\lambda}^2(p) \underset{p\gg\mu}{\simeq} \frac{1}{2b_1 \ln(p/\mu)} - \frac{b_2}{4b_1{}^3}\frac{\ln[\ln(p/\mu)]}{\ln^2(p/\mu)} + O(\ln^{-2}(p/\mu)) , \qquad (3.3)$$

with μ the mass scale at which the value of $\widetilde{\lambda}(\mu)$ is fixed. Similarly, the scaling behavior of the ω field and insertions of the operator ω^2 in the ultraviolet limit is determined by γ_ω and γ_{ω^2}.

Equation (3.3) exemplifies a general property of ω fluids which holds in both three and six dimensions. Classically, $\widetilde{\lambda}$ must be positive to ensure stability of the ground state. This is true for the quantum theory as well, as can be established by using asymptotic freedom. Perturbation theory, however, is an expansion in $\widetilde{\lambda}^2$, and so is insensitive to the sign of $\widetilde{\lambda}$.

Replacing $\widetilde{\lambda}^2$ by $N^3\lambda^2$ [Eq. (2.7)] in $\widetilde{\beta}/\widetilde{\lambda}$ and the γ yields part of the renormalization-group functions for large-N $(\bar{\phi}^6)_3$ theory. This allows the values of b_1, c_1, and d_1 to be checked by comparison with known results in $(\bar{\phi}^6)_3$ theory[9], where $\widetilde{\beta}/\widetilde{\lambda} \to \beta/\lambda$, $\gamma_\omega \to 16\gamma_{\bar{\phi}^2}$, and $\gamma_{\omega^2} \to \gamma_{(\bar{\phi}^2)^2}$.[18] To a given order in λ^2, for arbitrary N a term $\sim N^3\lambda^2$ is the first term of a polynomial in N. For example, $c_1 = 0$ implies that there are no terms $\sim N^3\lambda^2$ in $\gamma_{\bar{\phi}^2}$, although there are terms $\sim N^2\lambda^2$, $\sim N\lambda^2$, and $\sim \lambda^2$.

Unlike the ω fluid, perturbative $(\bar{\phi}^6)_3$ theory is an expansion in λ, not λ^2. This explains the otherwise mysterious property of large-N $(\bar{\phi}^6)_3$ theory that the dominant diagrams occur only at every other order in λ.[9]

More interesting observations are also attendant. About zero coupling, $(\bar{\phi}^6)_3$ theory is infrared free, where the leading term in β is $\sim +N\lambda^2$ for large N. The leading term and the term represented by b_1, $\sim -N^3\lambda^3$, balance to produce an ultraviolet-stable fixed point $\lambda^*_{\rm uv} \sim N^{-2}$.[9]

In short, the only way in which the asymptotically free theory $(\omega^3)_3$ can be derived from large-N $(\bar{\phi}^6)_3$ theory, which starts out in small coupling as infrared free, is if there is a fixed point $\lambda^*_{\rm uv}$ which changes the slope of the β function. $(\omega^3)_3$ theory is asymptotically free because the fixed point $\lambda^*_{\rm uv}$ becomes on the scale of λ $(\sim \lambda N^{+3/2})$ of order $N^{-1/2}$, which is zero at infinite N.

The existence of $\lambda^*_{\rm uv}$ explains why the phase diagram of $(\omega^3)_3$ theory has no relation to the usual tricritical phase diagram of large-N $(\bar{\phi}^6)_3$ theory. There are at least two distinct theories of large-N $(\phi^6)_3$ theory: one where $0 < \lambda < \lambda^*_{\rm uv}$ and one with $\lambda^*_{\rm uv} < \lambda$. These are distinct theories since $\lambda^*_{\rm uv}$, as a fixed point with finite slope, can only be reached in the limit of infinitely large momentum. The theory probed by previous N^{-1} expansions of $(\bar{\phi}^6)_3$,[11] with $\lambda \sim N^{-2} < \lambda^*_{\rm uv}$, is infrared free, while the theory probed by $(\omega^3)_3$, with $\lambda \sim N^{-3/2} > \lambda^*_{\rm uv}$, is not.

IV. NONPERTURBATIVE BEHAVIOR

The question of nonperturbative structure in ω fluids can be succinctly stated. Following the classical analysis, I assume the ω phase diagram consists of a line of first-order transitions ending in a point P. The question is then, is the point P a true critical point: e.g., is the correlation length at P

really infinite? Solely because of asymptotic freedom, if P is a critical point, there must be an infrared-stable fixed point $\tilde{\lambda}^*_{\rm ir}$ in the $\tilde{\beta}$ function at strong coupling. This occurs in strong coupling because the value of $\tilde{\lambda}^*_{\rm ir}$, like the values of the anomalous dimensions at $\tilde{\lambda}^*_{\rm ir}$, are *a priori* of order one.

The existence of $\tilde{\lambda}^*_{\rm ir}$ in ω fluids would be remarkable on two counts. First of all, all other theories with infrared-stable fixed points, such as $(\phi^4)_3$, are at dimensions below their critical dimensionality d_c[2]. When $d=d_c$, these theories are invariably infrared free, so there are at least calculable infrared-stable fixed points in $d_c-\epsilon$ dimensions, $\sim O(\epsilon)$ for $\epsilon \ll 1$. As ϵ increases, the fixed point moves smoothly into strong coupling. Secondly, in all other examples of asymptotically free theories of scalars[5(a),5(b)] and fermions[5(c)] (at $d=d_c$), there is never anything like $\tilde{\lambda}^*_{\rm ir}$, and correlation lengths are always finite.[19] For example, consider fermions with four-point interactions in two dimensions.[5(c)] As the "bare" mass is adjusted to vanish, a critical point is not approached, contrary to a classical analysis. Rather, the theory conspires to generate a mass, and so a finite correlation length, dynamically.[5(c)]

Why should ω fluids provide the exception to the rule? Because they are the only model with two-dimensional parameters such that there is a line of first-order transitions ending in a point. As P is approached, it is expected that the latent heat across the first-order line will vanish, with the free energy at P scaling in a manner determined by $\lambda^*_{\rm ir}$.

To emphasize the naturalness of P as a critical point, I note that it does not appear possible for the line of first-order transitions to end in a point which is itself first order. Purely on topological grounds, for this to happen the value of $\langle \omega \rangle$ at P, $\langle \omega_P \rangle$, would have to depend on θ, where θ is an angle characterizing the direction by which P is approached. Since P is a point of phase transition, this infinity of values in $\langle \omega_P \rangle$ must all have equal free energy. By continuity, the θ dependence of $\langle \omega \rangle$ would have to hold not only at P, but along the entire line of first-order transitions. Putting aside how unphysical this all is, it is not even obvious that there exists *any* function with the desired properties which would be required of the free energy.

One possibility is that the theory is simply ill-defined in some region of the phase diagram which includes point P. However, I see no reason why or how this could occur.

The study of the complete phase diagram in ω fluids appears possible only by such means as Monte Carlo simulations. In this, the study of $(\omega^3)_6$ theory may be preferable to that of $(\omega^3)_3$ theory, since although there are twice as many dimensions, it is easier to transcribe the short-range interactions of $(\omega^3)_6$ theory onto a lattice. Even so, the numerical study of ω fluids will be tedious, as two parameters will need to be varied in order to map out the phase diagram.

If $\lambda^*_{\rm ir}$ does exist, there will be signs of it in the ultraviolet as well as in the infrared limit. Consider starting at an arbitrary point in the phase diagram. If $\tilde{\lambda} < \tilde{\lambda}^*_{\rm ir}$, $\tilde{\lambda}$ will asymptote to zero in the ultraviolet limit; if $\tilde{\lambda} > \tilde{\lambda}^*_{\rm ir}$, then whatever form the ultraviolet limit takes, it will not look like $\tilde{\lambda} \to 0$. This suggestion is a bit glib, since the object of interest is not strong-coupling $(\omega^3)_3$ theory at a finite lattice spacing, but in the continuum limit.

The implications of $\lambda^*_{\rm ir}$ for large-N $(\bar{\phi}^6)_3$ theory are clear. Besides the infrared-stable origin and $\lambda^*_{\rm uv} \sim N^{-2}$, $\tilde{\lambda}^*_{\rm ir}$ becomes an infrared-stable fixed point $\lambda^*_{\rm ir} \sim N^{-3/2}$. There would then be three distinct theories of large-N $(\bar{\phi}^6)_3$: $0<\lambda<\lambda^*_{\rm uv}$, $\lambda^*_{\rm uv}<\lambda<\lambda^*_{\rm ir}$, and $\lambda^*_{\rm ir}<\lambda$, where the first two have well-defined infrared and ultraviolet limits.

The existence of a nonzero $\lambda^*_{\rm ir}$ also has important implications for the fixed-point structure of large-N $(\bar{\phi}^6)_d$ theory below three dimensions. When $d=3-\epsilon$, $\epsilon \ll 1$, there are at least three fixed points: the origin is ultraviolet stable on dimensional grounds, an infrared-stable fixed point $\lambda^*_{\rm ir'}(\epsilon) \sim \epsilon$, and an ultraviolet-stable fixed point $\lambda^*_{\rm uv}(\epsilon) \sim \lambda^*_{\rm uv} - \lambda^*_{\rm ir'}(\epsilon)$. I noticed in Ref. 9 that there is an $\epsilon_c \sim N^{-2}$ at which $\lambda^*_{\rm ir'}(\epsilon_c) = \lambda^*_{\rm uv}(\epsilon_c)$, while for $\epsilon > \epsilon_c$, both disappear. Thus, if $\lambda^*_{\rm ir} \sim N^{-3/2}$ does exist in three dimensions, then below $3-\epsilon_c$ dimensions only the infrared-stable fixed point related to $\lambda^*_{\rm ir}$ would control the infrared limit. Since ϵ_c may very well be <1 for all $N \ge 0$,[9] when $N=0$ and 1 the existence of nonclassical tricritical points in two dimensions may merely be the shadow of a nonzero $\lambda^*_{\rm ir}$ (for $N=0$ and 1) in three dimensions.

V. ASYMPTOTIC FREEDOM IN ω FLUIDS

Why is it that ω fluids are asymptotically free, while other linear field theories of scalars — like $(\bar{\phi}^6)_3$ and $(\bar{\phi}^4)_4$ — are not? Let λ be the dimensionless coupling for either $(\bar{\phi}^6)_3$ or $(\bar{\phi}^4)_4$ theories. Bearing in mind that perturbation theory derives from expanding $\exp(-\mathscr{L})$, the perturbative expansion is actually one in $-\lambda$. The concomitant expectation that quantities calculated in perturbation theory alternate in sign is usually found to be true at lowest orders (no claims are made at high orders in λ). In particular, for all N the β function is positive at $\sim\lambda^2$, negative at $\sim\lambda^3$, and so on. Heuristically, then, $(\bar{\phi}^6)_3$ and $(\bar{\phi}^4)_4$ theories are infrared free as they involve expansions in $-\lambda(\beta\sim+\lambda^2+\cdots)$; ω fluids are asymptotically free as expansions in $+\tilde{\lambda}^2(\beta\sim-\lambda^3+\cdots)$.

The analogy between perturbation theory in ω fluids and $(\mathrm{QCD})_4$ is evident, and shall not be belabored — the cubic ω coupling directly mimics the coupling of three gluons. There is no analogy to four-gluon interactions, but these "merely" result from the local gauge symmetry anyway.

In light of the above remarks, what is surprising is not the asymptotic freedom of $(\mathrm{QCD})_4$, but the infrared freedom of scalar and fermion $(\mathrm{QED})_4$. After all, with gauge coupling g, each involves expansions in g^2, with β functions that begin at $\sim g^3$. To answer this, the example of vector $(\mathrm{QED})_4$ must be appealed to. In this way ω fluids and vector $(\mathrm{QED})_4$ provide a complementary understanding of asymptotic freedom in $(\mathrm{QCD})_4$.

ACKNOWLEDGMENTS

I would like to thank F. Alcaraz, P. Hohenberg, D. Nicole, D. Toussaint, and F. Wilczek for their comments and criticism. This research was supported in part through the National Science Foundation, Grant No. PHY77-27084.

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[1]W. Caswell, Phys. Rev. Lett. 33, 244 (1974); D. R. T. Jones, Nucl. Phys. B75, 531 (1974), and references therein.

[2]In four dimensions, see, e.g., S. Coleman and D. J. Gross, Phys. Rev. Lett. 31, 861 (1973). In the present work unless explicitly stated otherwise I shall only consider field theories at their critical dimensionality d_c. I define d_c as the dimension at which the "typical" coupling constant is dimensionless. In addition, if there are as well any dimensional couplings in the theory, I assume these are adjusted to vanish; for scalar field theories, this corresponds to sitting precisely on the critical or tricritical point, etc. With these conventions, if about zero coupling the origin is infrared stable, the theory is infrared free; if the origin is ultraviolet stable, the theory is termed (by historical convention) asymptotically free.

[3]I do not describe infrared freedom in scalar or fermion $(\mathrm{QED})_4$ as the screening of electric charge because of the difficulty in explaining asymptotic freedom by means of "antiscreening." For a lucid discussion of "quantum" diamagnetism and paramagnetism, see R. J. Hughes, Nucl. Phys. B186, 376 (1981); K. A. Johnson, CTP Report No. 977, 1981 (unpublished). These authors demonstrate how a precise analogy can be established between the $(\mathrm{QCD})_4$ of SU(2) gluons and vector $(\mathrm{QED})_4$.

[4]An important example of a theory more complex than $(\mathrm{QCD})_4$ is provided by renormalizable theories of gravity in four dimensions. If the classical action of pure gravity respects conformal symmetry, so that it can only consist of the Weyl tensor squared, then the quantum theory is both asymptotically free and devoid of tachyons. See A. Zee, Phys. Lett. 109B, 183 (1982), and references therein.

[5](a) For the nonlinear $\bar{\sigma}$ model, $d_c=2$: S. Hikami and E. Brézin, J. Phys. A 11, 1141 (1978), and references therein. In these studies, $\bar{\sigma}$ is an N-component isovector; only for $N\geq 3$ is the theory asymptotically free. Generalizations to higher representations are also of interest; e.g., E. Brézin, S. Hikami, and J. Zinn-Justin, Nucl. Phys. B165, 528 (1980). Other asymptotically free theories include (b) capillary-wave treatments of surface fluctuations, with $d_c=1$: D. J. Wallace and R. K. P. Zia, Phys. Rev. Lett. 43, 808 (1979); (c) fermions with four-point interactions, $d_c=2$: D. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).

[6](a) With imaginary coupling, $(\phi^3)_d$ $(d\leq 6)$ is related to the problem of Yang-Lee zeros in a ferromagnet: M. E. Fisher, Phys. Rev. Lett. 40, 1610 (1978). Reference 11 of this work includes a summary to the literature of $(\phi^3)_d$ theory with real coupling. Most of these works consider generalized $(\phi^3)_d$ theories, where ϕ is an N-component field transforming under a global symmetry group G. For example, $N=0$ theories are applicable to percolation and spin glasses. In the present work $N=1$, there is no G, and the replacement of ϕ by ω is direct. In contrast, if $N\neq 1$ and G is a continuous group [e.g., A. J. McKane, D. J. Wallace, and R. K. P. Zia, Phys. Lett. 65B, 171 (1976)], there is no consistent way in which to define a positive field ω transforming under G. If $N\neq 1$ and G is a discrete group, from $\bar{\phi}$ a field $\bar{\omega}$ can be defined, but the resulting theory does not appear to admit any natural physical interpretation. For a treatment of a $(\phi^3)_d$ theory with a discrete G and the same regularization procedure as the present work, see (b) D. J. Amit, J. Phys. A 9, 1441 (1976).

[7]Only the equilibrium and macroscopic properties of a fluid are probed by $(\omega^3)_6$ and $(\omega^3)_3$ theories: in the ω fluids there is no vorticity, no underlying substructure, etc. ω fluids are distinct from other field theories used to model the critical properties of fluids, which invariably use a field ϕ instead of ω. Typically, if $\langle\omega\rangle$ is the average density, ϕ is defined as $\phi=\omega-\langle\omega\rangle$, but instead of the range of ϕ being $\phi:-\langle\omega\rangle\rightarrow+\infty$, in such "$\phi$ fluids" its range is taken to be $\phi:-\infty\rightarrow+\infty$. For

example, such theories have been used to study three-dimensional tricritical points in multicomponent fluid mixtures: M. Kaufman, K. K. Bardhan, and R. B. Griffiths, Phys. Rev. Lett. 44, 77 (1980), and references therein, especially R. B. Griffiths, J. Chem. Phys. 60, 195 (1974). For a discussion of why ϕ fluids are adequate in three dimensions, see Ref. 17.

[8]Throughout, p^α is to be read as $(\bar{p}^2)^{\alpha/2}$. In Sec. II, I shall also use the shorthand of p^α for $(-\bar{\partial}^2)^{\alpha/2}$.

[9]R. D. Pisarski, Phys. Rev. Lett. 48, 574 (1982).

[10]This is in stark contrast to $(\phi^4)_4$ theory, where it has apparently been rigorously proven (for one- and two-component fields, under certain technical assumptions) that the only fixed point of the β function is the infrared-stable origin: J. Frohlich, Nucl. Phys. B (to be published). This implies that in four dimensions a ϕ^4 theory is like that above four dimensions, where the only sensible field theory in the continuum limit is free field theory [above, and M. Aizenman, Phys. Rev. Lett. 47, 1 (1981)]. Thus, unlike $(\phi^4)_4$ theory, large-N $(\bar{\phi}^6)_3$ theory appears to contradict the long-standing prejudice [see, e.g., G. 't Hooft, Phys. Lett. 109B, 474 (1982)] that all renormalizable field theories must also be asymptotically free.

[11]P. K. Townsend, Phys. Rev. D 12, 2269 (1975). Related studies of large-N $(\bar{\phi}^6)_3$ theory, especially of the effective potential, include P. K. Townsend, Phys. Rev. D 14, 1715 (1976); Nucl. Phys. B118, 199 (1977); T. Appelquist and U. Heinz, Phys. Rev. D 24, 2169 (1981); 24, 2620 (1982); U. Heinz, *ibid.* 25, 2717 (1982). In all these works, as $N\rightarrow\infty$, $\lambda\sim N^{-2}$, while I consider $\lambda\sim N^{-3/2}$. It is necessary to take $\lambda\sim N^{-2}$ in order to study the tricritical phase diagram in the infrared-free phase: see my remarks at the end of Sec. III. I would like to thank E. Brézin and L. Brown (private communications) for reminding me of the interest in choosing $\lambda\sim N^{-3/2}$.

[12]For a discussion of the virtues of this procedure, and references to the $d=3$ tricritical literature, see R. D. Pisarski, Phys. Lett. 85A, 356 (1981); 86A, 497(E) (1981).

[13]The extension of the results in Sec. II from the vacuum to vacuum functional $P_{\bar{\phi}}$ to the analogous quantity in the presence of external sources is direct, and taken for granted in Sec. III.

[14]Formally, $Z_\omega=(Z_{\bar{\phi}})^2$ is obtained. However, if a source term $\sim Z_{\bar{\phi}^2}J_{\bar{\phi}^2}\phi^2$ is introduced into $\mathscr{L}_{\bar{\phi}}$, Z_ω should be related to $Z_{\bar{\phi}^2}$, not $Z_{\bar{\phi}}$. This is confirmed by direct calculation (see Ref. 18).

[15]When the phase of $\mathscr{L}_{\bar{\phi}}$ is of broken symmetry, Eq. (2.3) is useless. With $\bar{\phi}^2=\phi_1{}^2+\bar{\pi}^2(\langle\phi_1\rangle\neq 0,\langle\pi\rangle=0)$, instead of to $\bar{\phi}^2$ the appropriate transformation is to the two scalar variables $\phi_1-\langle\phi_1\rangle$ and $\bar{\pi}^2$. This transformation has no direct relevance to the ω fluid.

[16]In zero space-time dimensions, where the measure of Eq. (2.5) is over a single point, P_ω for arbitrary $\tilde{m}^2$, $\tilde{g}>0$, and $\tilde{\lambda}=0$ is related to the integral representation of the error function. Thus, even for $\tilde{\lambda}=0$, $\tilde{g}>0$, the P_ω of Eq. (2.5) can be taken as a functional generalization of the error function.

[17]The phase diagram for fluids below six dimensions $[\tilde{\lambda}(\omega^3)_d$, with the Laplacian equal to p^2 for all $d]$ should be analogous to that for $d=6$ down to four dimensions. Most notably, if there is a critical point in six dimensions, it probably persists for $4<d<6$. Everything changes dramatically in four dimensions, since for $d\leq 4$, a (positive) ω^4 coupling must be included as a renormalizable interaction; likewise, ω^5 and ω^6 couplings must be added when $d\leq 3$. Although the possible dimensionality of the phase diagram increases (e.g., unlike $d>4$, when $d\leq 4$ $\tilde{\lambda}$ can be of either sign), it still has the same general form—if two parameters are varied, there is a line of first-order transitions ending in a critical point. For $d\leq 4$, however, the critical point is controlled by the ω^4 instead of the ω^3 coupling, while the existence of the critical point is hardly exceptional. Indeed, when $d\leq 4$, as the ω^4 term stabilizes the ground state regardless of the positivity of ω, it is probably sheer perversity to use a field ω instead of ϕ. This is not only due to the technical difficulties in treating ω fluids versus ϕ fluids (Ref. 16). More to the point, for real fluids in three dimensions the effects of fluctuations are almost always small. The only exceptions are near critical or tricritical points. In both instances, whether or not an ω or a ϕ field is used, the universality class of both theories, including corrections to scaling, are surely the same. This compromise provides a natural unification in the uses of ω and ϕ fluids: only when $d>4$ is it necessary to use an ω field.

[18]Graphically, it can be proven that with $\tilde{\lambda}$ fixed as $N\rightarrow\infty$, to all orders in $\tilde{\lambda}$—but only leading order in $N^{-1}-Z_{\bar{\phi}}=1$ and $\gamma_\omega=16\gamma_{\bar{\phi}^2}$. For the definitions of $\gamma_{\bar{\phi}^2}$ and $\gamma_{(\bar{\phi}^2)^2}$, see Ref. 9.

[19]This is usually but not universally true in $(\mathrm{QCD})_4$: there are atypical examples which are asymptotically free with a g^*_{ir} in weak coupling (W. Caswell, Ref. 1). However, there is surely no $(\mathrm{QCD})_4$ with a g^*_{ir} in strong coupling.