Classical and quantal Liouville field theory

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The canonical structure of the Liouville theory is investigated. We present two canonical transformations which map the theory onto a free field theory. The first makes use of conformal invariance and relies on a Yang-Feldman solution to the field equation. The second employs the inverse scattering method, which is uncommonly intricate, owing to the conformal invariance. We also analyze the quantized theory. Semiclassical arguments, supplemented by a study of the exact effective potential, suggest that the theory has a conformally invariant, continuous energy spectrum, bounded from below, but no translationally invariant ground state.

I. INTRODUCTION

The Liouville model is a two-dimensional relativistic field theory, governed by the following (Minkowski space) Lagrangian:

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi-\frac{m^2}{\beta^2}e^{\beta\Phi}\,. \tag{1.1}$$

The field equation

$$\Box\Phi+\frac{m^2}{\beta}e^{\beta\Phi}=0 \tag{1.2}$$

possesses a Bäcklund transformation connecting every solution Φ to a solution of the two-dimensional wave equation:

$$\begin{aligned}\partial_+(\Phi-\phi_0)&=\frac{m}{\beta}\alpha e^{\beta(\Phi+\phi_0)/2}\,,\\ \partial_-(\Phi+\phi_0)&=-\frac{m}{\beta}\frac{1}{\alpha}e^{\beta(\Phi-\phi_0)/2}\,.\end{aligned} \tag{1.3}$$

Here $\Box\phi_0=0$ and α is an arbitrary parameter. The general solution to (1.2) can be constructed with the help of (1.3); it was given by Liouville in terms of two arbitrary functions F and G (Ref. 1):

$$\begin{aligned}\Phi(x)&=\frac{1}{\beta}\ln\frac{F'(x^+)G'(x^-)}{[1+\frac{1}{4}m^2F(x^+)G(x^-)]^2}\,,\\ x^{\pm}&=(x^0\pm x^1)/\sqrt{2}\,.\end{aligned} \tag{1.4}$$

We use units in which the velocity of light is 1, and the action $\int d^2x\,\mathscr{L}$ is dimensionless. Then the field Φ also is dimensionless, as is β. Since it can be removed from the classical theory by redefining Φ, we set $\beta=1$, in the first three sections of our paper. The quantity m^2, with dimensions of (mass)^2, is taken to be positive, so the energy density

$$\mathscr{E}=\tfrac{1}{2}\dot{\Phi}^2+\tfrac{1}{2}\Phi'^2+m^2e^{\Phi} \tag{1.5}$$

is manifestly positive. The solution (1.4) is form invariant against the replacement of the functions F and G by their fractional transforms:

$$\frac{m}{2}F\to\frac{\gamma\frac{m}{2}F-\delta}{\epsilon\frac{m}{2}F+\eta}\,,\quad \frac{m}{2}G\to\frac{\eta\frac{m}{2}G-\epsilon}{\delta\frac{m}{2}G+\gamma}\,,\qquad \gamma\eta+\delta\epsilon\neq0\,. \tag{1.6}$$

The Liouville model has arisen in many areas of mathematics and physics.[1] For particle physicists the theory was important in the study of instantons and solitons,[2] and more recently in reformulations of the dual string model.[3] Although an explicit solution of the Liouville equation is in hand, there has been considerable study of its mathematical structure.[4,5] We contribute here to this literature by presenting a canonical analysis of the model: we exhibit two canonical transformations which map the Liouville theory into a free field theory.

Our first transformation, which is discussed in Sec. II, uses conformal invariance of the Liouville dynamics. We establish this symmetry and find an infinite number of constants of motion. Because they are of course expressible in terms of F and G, their Poisson bracket algebra may be used to determine that of F and G. This then suggests the form for the transformation to noninteracting fields. The argument is completed by verifying that the transformation is indeed canonical. We shall be using the Yang-Feldman formalism for an interacting field theory.[6]

In our second transformation, presented in Sec. III, we employ the inverse scattering method, already developed for this problem by Andreev.[5] We find that, owing to conformal invariance, the usual scattering data provide only half the action-angle variables for the system. The second half is identified and used to diagonalize the light-cone Hamiltonian. We construct all conserved quantities and recover Andreev's charges as a subset. The conformal constants of motion are related to those arising from the inverse approach. The Marchenko equations are derived, allowing for a reconstruction of the solution to the Liouville theory from the complete set of action-angle variables.

In Sec. IV, we use the above information to conclude that in the semiclassical approximation the energy spectrum is that of a free theory, i.e., it is continuous. This is consistent with conformal invariance. Moreover, on the basis of a formula for the exact effective potential, we argue that the energy spectrum of the complete quantum field theory, though bounded from below, does not possess a translationally invariant lowest-energy (ground) state.

II. CONFORMAL STRUCTURE

Under a conformal transformation the space-time coordinates x^μ change infinitesimally into

$$\begin{aligned} x^\mu &\to x^\mu + \delta x^\mu \,, \\ \delta x^\mu &= -f^\mu(x) \,, \end{aligned} \tag{2.1}$$

where f^μ is a two-dimensional conformal Killing vector, i.e., it satisfies

$$\partial_\mu f_\nu + \partial_\nu f_\mu - g_{\mu\nu}\partial_\alpha f^\alpha = 0 \,. \tag{2.2}$$

Any function f^μ solving this equation has a $+$ component depending only on x^+, and a $-$ component depending on x^-:

$$f^+ = f^+(x^+), \quad f^- = f^-(x^-) \,. \tag{2.3}$$

The infinitesimal transformations obey a composition law,

$$[\delta_f, \delta_g] = \delta_h \tag{2.4}$$

with f, g, and h conformal Killing vectors, the latter given by the Lie bracket of the former two:

$$h^\mu = f^\alpha \partial_\alpha g^\mu - g^\alpha \partial_\alpha f^\mu \,. \tag{2.5}$$

As we shall presently demonstrate, the Liouville equation shares with the free massless theory an invariance against these conformal transformations. Consequently, the boundary-value problem associated with the differential equations of the two theories written in terms of light-cone variables ($x^\pm, \Box = 2\partial_+\partial_-$) requires, for a complete specification of a unique solution, fixing values of the field for all x^- at fixed $x^+ = x_0^+$, supplemented by a specification of the field for all x^+ at fixed $x^- = x_0^-$. This determines the two arbitrary functions that are present in the conformal transformation. As a further consequence, the canonical light-cone formalism requires equal-x^+ Poisson brackets, supplemented by equal-x^- brackets.

To appreciate the conformal properties of the Liouville theory, it is useful to contrast them with those of the noninteracting theory. Also, we shall need the free field formalism, since the Liouville model will be expressed in those terms. We first review these well-known results.[7]

A. Free theory

The action for the free Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \tag{2.6}$$

is invariant against (infinitesimal) conformal transformations, provided the field change is the Lie derivative of the field,

$$\delta_f \phi = f^\alpha \partial_\alpha \phi \,. \tag{2.7}$$

The conserved currents, expressed in terms of a symmetric and traceless energy-momentum tensor,

$$\theta_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2} g_{\mu\nu}\partial_\alpha\phi\,\partial^\alpha\phi \tag{2.8}$$

take the familiar Bessel-Hagen form,[8]

$$j_f^\mu = \theta^{\mu\nu} f_\nu \,. \tag{2.9}$$

Since $\theta_{\mu\nu}$ is traceless and conserved when ϕ solves the wave equation,

$$\phi(x) = \phi^+(x^+) + \phi^-(x^-) \,, \tag{2.10}$$

it follows that θ_{--} depends only on x^-, and θ_{++} only on x^+. Consequently, two sets of constants of motion may be defined

$$Q_f^- = \int dx^- f^-(x^-)\theta_{--}(x^-) \,, \tag{2.11a}$$

$$Q_f^+ = \int dx^+ f^+(x^+)\theta_{++}(x^+) \,, \tag{2.11b}$$

$$Q_f = Q_f^+ + Q_f^- \,.$$

When a canonical formalism for the theory is constructed in light-cone coordinates, which are natural for the problem at hand, it is necessary to postulate both equal-x^+ and equal-x^- Poisson brackets:

$$\{\phi(x),\phi(y)\}\,|_{x^+=y^+}=\tfrac{1}{4}\epsilon(x^- - y^-)\ , \tag{2.12a}$$

$$\{\phi(x),\phi(y)\}\,|_{x^-=y^-}=\tfrac{1}{4}\epsilon(x^+ - y^+)\ . \tag{2.12b}$$

Equivalently, since ϕ is a wave field, we have

$$\{\phi^-(x^-),\phi^-(y^-)\}=\tfrac{1}{4}\epsilon(x^- - y^-)\ , \tag{2.13a}$$

$$\{\phi^+(x^+),\phi^+(y^+)\}=\tfrac{1}{4}\epsilon(x^+ - y^+)\ , \tag{2.13b}$$

$$\{\phi^+(x),\phi^-(y)\}=0\ . \tag{2.13c}$$

The Poisson brackets of two quantities A and B, which are functionals of $\phi^{\pm}$, are defined by

$$\{A,B\}=\tfrac{1}{4}\int dz\,dz'\epsilon(z-z')\times\left[\frac{\delta A}{\delta\phi^+(z)}\frac{\delta B}{\delta\phi^+(z')}+\frac{\delta A}{\delta\phi^-(z)}\frac{\delta B}{\delta\phi^-(z')}\right]. \tag{2.14}$$

All the usual properties hold: the brackets are linear and antisymmetric in (A,B), and satisfy the Jacobi identity.

The charges generate the transformation (2.7):

$$\{Q_f,\phi\}=f^\alpha\partial_\alpha\phi\ . \tag{2.15}$$

Moreover, the Poisson brackets of the charges (2.11) reproduce the composition law (2.4), provided surface integrals vanish (we shall always omit them):

$$\{Q_f^{\pm},Q_g^{\pm}\}=-Q_h^{\pm}\ , \tag{2.16a}$$

$$\{Q_f^+,Q_g^-\}=0\ . \tag{2.16b}$$

The massless, free theory also admits another (trivial) symmetry: the field may be shifted by an arbitrary wave field:

$$\delta\phi=\Omega\ , \tag{2.17}$$

$$\Box\Omega=0,\quad \Omega=\Omega^+(x^+)+\Omega^-(x^-)\ .$$

The conserved current is

$$j_\Omega^\mu=\partial^\mu\phi\,\Omega-\phi\,\partial^\mu\Omega\ . \tag{2.18}$$

Here again two independent charges may be defined:

$$q_\Omega^-=2\int dx^-\,\Omega^-\partial_-\phi^-\ , \tag{2.19a}$$

$$q_\Omega^+=2\int dx^+\,\Omega^+\partial_+\phi^+\ , \tag{2.19b}$$

$$q_\Omega=q_\Omega^+ + q_\Omega^-\ .$$

One may combine the field translation symmetry with the conformal symmetry. An especially interesting choice for Ω, in view of our later discussion of the Liouville theory, is proportional to the divergence of the conformal Killing vector, which satisfies the wave equation. Thus, we consider the following conserved quantities:

$$\tilde{Q}_f=Q_f+\frac{1}{\gamma}q_{\partial f}\ , \tag{2.20}$$

where γ is an arbitrary constant. They generate an inhomogeneous symmetry transformation on ϕ:

$$\{\tilde{Q}_f,\phi\}=f^\alpha\partial_\alpha\phi+\frac{1}{\gamma}\partial_\alpha f^\alpha\ . \tag{2.21}$$

Although the Poisson bracket algebra no longer closes,

$$\begin{aligned}\{\tilde{Q}_f,\tilde{Q}_g\}&=-\tilde{Q}_h+\frac{1}{\gamma^2}\Delta(f,g)\ ,\\ \Delta(f,g)&=\int dx^-(f^-\partial_-{}^3g^- - g^-\partial_-{}^3f^-)\\ &\quad+\int dx^+(f^+\partial_+{}^3g^+ - g^+\partial_+{}^3f^+)\ ,\end{aligned} \tag{2.22}$$

we may nevertheless adopt the transformation law (2.21) as the realization of conformal transformations on the field ϕ, because the additional term Δ in (2.22) is independent of dynamical variables—it is merely a center for the infinite-dimensional Lie algebra of the two-dimensional conformal group.[9]

Finally, let us observe that a current for the combined transformation takes the Bessel-Hagen expression

$$J_f^\mu=\Theta^{\mu\nu}f_\nu \tag{2.23}$$

with $\Theta^{\mu\nu}$ an improved, traceless energy-momentum tensor,[10] which differs from the canonical one (2.8) by a superpotential:

$$\begin{aligned}\Theta_{\mu\nu}&=\theta_{\mu\nu}+\frac{2}{\gamma}(g_{\mu\nu}\Box-\partial_\mu\partial_\nu)\phi\\ &=\partial_\mu\phi\,\partial_\nu\phi-\tfrac{1}{2}g_{\mu\nu}\partial_\alpha\phi\,\partial^\alpha\phi-\frac{2}{\gamma}\partial_\mu\partial_\nu\phi\ .\end{aligned} \tag{2.24}$$

(J_f^μ differs from $j_f^\mu+j_{\partial f}^\mu$ by a superpotential.) It follows that

$$\tilde{Q}_f^{\pm} = \int dx^{\pm} f^{\pm}(x^{\pm}) \left[\partial_\pm \phi^{\pm} \partial_\pm \phi^{\pm} - \frac{2}{\gamma} \partial_\pm{}^2 \phi^{\pm} \right] . \tag{2.25}$$

B. Liouville theory

The Liouville theory is also conformally invariant, as is seen from the fact that the explicit solution (1.4) (with $\beta = 1$) is form invariant against the transformation

$$\begin{aligned} &x^+ \to y^+(x^+) , \\ &x^- \to y^-(x^-) , \\ &\Phi(x^+, x^-) \to \Phi(y^+, y^-) - \ln(\partial_+ y^+)(\partial_- y^-) . \end{aligned} \tag{2.26}$$

The Lagrangian (1.1) changes by a total derivative under the following infinitesimal transformation, which we define to be the conformal transformation of the Liouville field Φ:

$$\delta_f \Phi = f^\alpha \partial_\alpha \Phi + \partial_\alpha f^\alpha . \tag{2.27}$$

Thus, we see that while the free theory is separately invariant against the usual conformal transformation, as well as against the field translation, in the Liouville model the two must be combined, with γ set equal to 1 (this we do henceforth), to achieve the symmetry transformation. Owing to conformal invariance we expect the conserved currents to be given in terms of a traceless energy-momentum tensor. This is indeed possible, but the canonical Noether tensor must be improved.[11]

$$\Theta_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - g_{\mu\nu}(\tfrac{1}{2}\partial_\alpha \Phi \partial^\alpha \Phi - m^2 e^{\Phi}) + 2(g_{\mu\nu}\Box - \partial_\mu \partial_\nu)\Phi , \tag{2.28}$$

$$\Theta^\mu_\mu = 2m^2 e^{\Phi} + 2\Box\Phi = 0 . \tag{2.29}$$

The current is

$$J_f^\mu = \Theta^{\mu\nu} f_\nu . \tag{2.30}$$

Again, two sets of charges may be defined, since Θ_{--} depends only on x^-, and Θ_{++} only on x^+ (Ref. 12):

$$\begin{aligned} Q_f^{\pm} &= \int dx^{\pm} f^{\pm}(x^{\pm}) \Theta_{\pm\pm}(x^{\pm}) \\ &= \int dx^{\pm} f^{\pm} (\partial_\pm \Phi \partial_\pm \Phi - 2\partial_\pm{}^2 \Phi) . \end{aligned} \tag{2.31}$$

Canonical Poisson brackets must be postulated for equal x^+ and equal x^-, owing to the conformal invariance of the theory, just as in the free field case,

$$\{\Phi(x), \Phi(y)\}\,|_{x^+ = y^+} = \tfrac{1}{4}\epsilon(x^- - y^-) , \tag{2.32a}$$

$$\{\Phi(x), \Phi(y)\}\,|_{x^- = y^-} = \tfrac{1}{4}\epsilon(x^+ - y^+) . \tag{2.32b}$$

With (2.32), one verifies that Q_f generates the transformation law (2.27),

$$\{Q_f, \Phi\} = f^\alpha \partial_\alpha \Phi + \partial_\alpha f^\alpha \tag{2.33}$$

as well as the fact that the algebra (2.22) is satisfied:

$$\begin{aligned} &\{Q_f, Q_g\} = -Q_h + \Delta(f,g) , \\ &Q_f = Q_f^+ + Q_f^- . \end{aligned} \tag{2.34}$$

We now consider $\Theta_{\pm\pm}(x^\pm)$ and substitute for Φ Liouville's solution (1.4). One finds

$$\Theta_{++} = 3\left[\frac{F''}{F'}\right]^2 - 2\frac{F'''}{F'} , \tag{2.35a}$$

$$\Theta_{--} = 3\left[\frac{G''}{G'}\right]^2 - 2\frac{G'''}{G'} . \tag{2.35b}$$

This shows yet again that Θ_{++} depends only on x^+, and Θ_{--} only on x^-. Finally, we define

$$\ln F'(x^+) = \phi^+(x^+) , \tag{2.36a}$$

$$\ln G'(x^-) = \phi^-(x^-) \tag{2.36b}$$

and recognize that in terms of $\phi^\pm$, $\Theta_{\mu\nu}$ takes the free field form (2.24)

$$\begin{aligned} &\Theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + 2(g_{\mu\nu}\Box - \partial_\mu \partial_\nu)\phi , \\ &\phi(x) = \phi^+(x^+) + \phi^-(x^-) . \end{aligned} \tag{2.37}$$

[The free field ϕ, defined here, does not in general coincide with ϕ_0, the one occurring in the Bäcklund transformation (1.3); rather there is a nonlocal relation between the two:

$$\beta\phi_0^+ = \beta\phi^+ - 2\ln\left[\partial_+{}^{-1} e^{\beta\phi^+} - \frac{2a}{m}\right] + \ln\frac{ab}{\alpha} ,$$

$$\beta\phi_0^- = -\beta\phi^- + 2\ln\left[\partial_-{}^{-1} e^{\beta\phi^-} + \frac{2}{am}\right] + \ln\frac{a}{\alpha b} ,$$

where a and b are arbitrary constants, and $\partial_\pm{}^{-1}$ are Green's functions.]

Since the Poisson bracket algebra of constants of motion is known, as is that of the (improved) energy-momentum tensors' components, one may infer an algebra for the arbitrary functions $\ln F'$ and $\ln G'$. Comparing (2.37) and (2.24), and recalling

(2.12) and (2.13), suggests that we postulate canonical Poisson brackets:

$$\{\phi^-(x^-),\phi^-(y^-)\}=\tfrac{1}{4}\epsilon(x^- - y^-)\ , \tag{2.38a}$$

$$\{\phi^+(x^+),\phi^+(y^+)\}=\tfrac{1}{4}\epsilon(x^+ - y^+)\ , \tag{2.38b}$$

$$\{\phi^+(x^+),\phi^-(y^-)\}=0\ . \tag{2.38c}$$

It is clear that the general solution (1.4) may be written in the form

$$\Phi=\phi-2\ln\left[1+\frac{m^2}{4}(\partial_+{}^{-1}e^{\phi^+})(\partial_-{}^{-1}e^{\phi^-})\right]$$

$$=\phi-2\ln\left[1+\frac{m^2}{2}\Box^{-1}e^{\phi}\right]. \tag{2.39}$$

The Green's functions will be specified presently. One may view (2.39) as a transformation from the old set of variables Φ to a new set ϕ. We have already seen that this transformation makes the (improved) Φ-field's energy-momentum tensor take on the free (improved) form. Consequently, the Liouville Hamiltonian is mapped into the free Hamiltonian. It remains to verify that the transformation is canonical, i.e., that the canonical algebra of the $\phi^{\pm}$'s given in (2.38) ensures that the Poisson brackets of the Φ's are canonical, as in (2.32). To compute the Φ equal-x^+ and equal-x^- algebra from that of the ϕ's, we can use (2.14), but the Green's functions occurring in (2.39) must be specified.

A straightforward calculation indicates that the Green's functions should be of the Yang-Feldman variety,[6] so that we have the choice between

$$\partial^{-1}f=\int_{-\infty}^{x}dy\,f(y)$$

and

$$\partial^{-1}f=-\int_{x}^{\infty}dy\,f(y)\ . \tag{2.40}$$

In both cases the transformation is canonical. It may therefore be written as

$$\Phi(x)=\phi(x)-2\ln\left[1+\frac{m^2}{2}\int d^2y\,G(x-y)e^{\phi(y)}\right],$$
$$\Box\phi=0 \tag{2.41}$$

with $G(x-y)=\tfrac{1}{2}g(x^+-y^+)g(x^--y^-)$ and either $g(z)=\theta(z)$ or $g(z)=-\theta(-z)$. For both Green's functions, one may also verify that the equal-time algebra

$$\{\Phi(x),\Phi(y)\}\,|_{x^0=y^0}=\{\dot\Phi(x),\dot\Phi(y)\}\,|_{x^0=y^0}=0\ ,$$
$$\{\dot\Phi(x),\Phi(y)\}\,|_{x^0=y^0}=\delta(x^1-y^1) \tag{2.42}$$

is satisfied when Φ is given by (2.41) and ϕ obeys (2.38).

Equation (2.41) puts into evidence the very transparent significance of our transformation. Observe that the field equation (1.2) has a formal Yang-Feldman solution:

$$\Phi(x)=\phi(x)-m^2\int d^2y\,G(x-y)e^{\Phi(y)}\ . \tag{2.43}$$

Iterating this in powers of m^2 produces

$$\begin{aligned}\Phi(x)=\ &\phi(x)-m^2\int d^2y\,G(x-y)e^{\phi(y)}\\ &+m^4\int d^2y\,G(x-y)e^{\phi(y)}\\ &\times\int d^2z\,G(y-z)e^{\phi(z)}+\cdots\end{aligned} \tag{2.44}$$

which is the same power series as the one obtained by expanding the logarithm in (2.41).

III. INVERSE SCATTERING ANALYSIS

In this section, we develop the inverse scattering method for the Liouville system and we diagonalize the Hamiltonian. We work again in light-cone coordinates and Poisson brackets are defined as the above light-cone brackets.[13] In subsection A, we exhibit the peculiarities of the inverse scattering procedure for the Liouville equation. We discuss which boundary conditions must be specified so that the direct and inverse scattering transforms are well defined, and we state the form of the boundary conditions that we shall use. In subsection B, we show that, within the framework of the standard inverse scattering analysis, local Lax pairs cannot be used to derive solutions for Liouville's equation. Therefore, in subsection C, a nonlocal Lax pair is constructed from the local pair; it allows a complete reconstruction of solutions to the Liouville equation, as was shown in part by Andreev.[5] Subsection D is devoted to a derivation of general scattering theory results needed for the subsequent discussion. In subsection E, the Poisson brackets of the scattering data are computed and the action-angle variables identified. We diagonalize the Hamiltonian in subection F and express the local conserved charges in terms of the Liouville field Φ in subsection G. These charges are found to be related to the ones constructed in Sec. II, as well as to the conserved quantities found by Andreev.[5] Finally, in subsection H, the Marchenko equations are derived allowing for the reconstruction of the time-evolved fields.

A. Boundary conditions

The necessity of specifying both $\Phi(x_0^+,x^-)$, and $\Phi(x^+,x_0^-)$ as boundary conditions requires special care when applying the inverse scattering procedure to the Liouville system. That procedure makes a fundamental distinction between time and space coordinates. We shall choose x^+ as our "time" coordinate, so that Poisson bracketing with the light-cone Hamiltonian

$$P^- = \int dx^+ \Theta^{--} = \int dx^+ [\partial_+ \Phi(x^+,x_0^-)]^2 \tag{3.1}$$

reproduces the correct x^+ evolution, as seen from (2.33).

The first step of the inverse method is to map the initial data $\Phi(x_0^+,x^-)$ onto the scattering data of a linear problem. One also determines the Poisson brackets amongst the scattering data. These results, obtained at fixed time x_0^+, are entirely independent of the dynamical variables $\Phi(x^+,x_0^-)$ occurring in P^-. Consequently, time evolution cannot be computed using the Poisson brackets of the scattering data alone, since the Hamiltonian is not expressed in terms of these. We need also the Poisson brackets of the scattering data with the supplementary set of dynamical variables contained in $\Phi(x^+,x_0^-)$.

This is to be as contrasted with the case, e.g., of the sine-Gordon theory,[14–16] where the Hamiltonian is expressed solely in terms of the scattering data, so that time evolution, determined by Poisson bracketing with the Hamiltonian, can be evaluated from the Poisson brackets of the scattering data. For the Liouville system, the time evolution of the scattering data will depend functionally on $\Phi(x^+,x_0^-)$. Hence, direct and inverse scattering procedures must be executed with $\Phi(x^+,x_0^-)$ specified. We shall show that in this case, the Marchenko equations are solved uniquely in terms of the time-evolved scattering data and the given function $\Phi(x^+,x_0^-)$.

The important conclusion to be drawn from this discussion is that all the action-angle variables in the Liouville theory are not just, as is usually the case, the scattering data [determined by $\Phi(x_0^+,x^-)$], but also a supplementary set of action-angle variables exists [determined by $\Phi(x^+,x_0^-)$].

It will become clear in subsection B that, in order to apply the standard techniques of direct and inverse scattering, we must impose the following boundary conditions:

$$\lim_{x^-\to\infty} [\Phi(x^+,x^-) + 2\sqrt{V_\infty}\, x^- - \phi_{+\infty}(x^+)] = 0 , \tag{3.2a}$$

$$\lim_{x^-\to-\infty} [\Phi(x^+,x^-) - 2\sqrt{V_\infty}\, x^- - \phi_{-\infty}(x^+)] = 0 , \tag{3.2b}$$

where $\phi_{\pm\infty}(x^+)$ are limiting, asymptotic functions. Consideration of the Liouville equation shows that V_∞ is independent of x^+ but $\phi_{\pm\infty}(x^+)$ may vary with x^+. Of course both functions $\phi_{\pm\infty}(x^+)$ cannot be chosen arbitrarily; the differential equation determines one from the other. We shall restrict our investigation to functions $\phi_{+\infty}$ obeying the following limit:

$$\lim_{|x^+|\to\infty} \phi_{+\infty}(x^+) = -\infty . \tag{3.2c}$$

We may view $\phi_{+\infty}(x^+)$ [or $\phi_{-\infty}(x^+)$] as additional data specifying Φ at fixed x_0^-, viz., at positive [or negative] infinity. This supplements the initial data which specify Φ at fixed x_0^+ and which are converted into scattering data by the linear direct scattering problem.

The boundary conditions (3.2) have the further property that the potential energy density $m^2 e^\Phi$ vanishes in the limit $|x^-| \to \infty$.

B. Local Lax pairs

Let us first concentrate on a local Lax pair defined by the following 2×2 matrix operators:

$$L = -i\sigma_2 \partial_- - \tfrac{1}{2}\Pi\sigma_1 , \tag{3.3a}$$

$$B = (\sigma_1 + i\sigma_2)\frac{m^2}{8\lambda} e^\Phi . \tag{3.3b}$$

Here $\Pi = \partial_- \Phi$, the momentum canonically conjugate to Φ, and the σ_i are Pauli matrices. If and only if Φ satisfies the Liouville equation will the system of equations

$$L\psi = \lambda\psi , \tag{3.4a}$$

$$\partial_+ \psi = B\psi \tag{3.4b}$$

be compatible. This fact may alternatively be expressed as the vanishing of the curvature functional

$R[\Phi]$ of a non-Abelian connection $\Gamma_\mu[\Phi]$, determined by the gauge-covariant derivatives

$$D_- = \partial_- - \tfrac{1}{2}\Pi\sigma_3 - i\sigma_2\lambda = \partial_- - \Gamma_-[\Phi] \,, \tag{3.5a}$$

$$D_+ = \partial_+ - (\sigma_1 + i\sigma_2)\frac{m^2}{8\lambda}e^{\Phi} = \partial_+ - \Gamma_+[\Phi] \,, \tag{3.5b}$$

$$R[\Phi] = [D_+, D_-] = -\tfrac{1}{4}\sigma_3(\Box\Phi + m^2 e^{\Phi}) \,. \tag{3.5c}$$

The Lax pair (3.3) and (3.4) is by no means unique; any gauge transformation of the covariant derivatives in (3.5) will modify the connection and change the curvature, but if Φ satisfies the Liouville equation in one gauge, then it continues to do so in all gauges. Different Lax pairs need not even be gauge equivalent. For example, the following derivatives, gauge inequivalent to (3.5a) and (3.5b), produce a curvature which coincides with (3.5c):

$$\begin{aligned} \widetilde{D}_\mu &= \partial_\mu + \tfrac{1}{4}\epsilon_{\mu\nu}\partial^\nu\Phi\sigma^3 \\ &\quad + \left[\frac{\epsilon_{\mu\nu}k^\nu}{2\sqrt{2}}\sigma_1 + \frac{ik_\mu}{2\sqrt{2}}\right]e^{\Phi/2} \,, \\ k^\mu k_\mu &= m^2 \,. \end{aligned} \tag{3.6}$$

Because of its greater simplicity, we shall work only with the first Lax pair (3.3) and (3.4).

Let us now justify that the boundary conditions (3.2) are appropriate to the scattering problem $L\psi = \lambda\psi$. This is most easily done by using a second-order formulation, appropriate to the case $\lambda \neq 0$. The spinor

$$\psi = \begin{bmatrix} u \\ v \end{bmatrix} \quad \begin{cases} u = -(\partial_- + \tfrac{1}{2}\Pi)w, \\ v = \lambda w \end{cases} \tag{3.7a}$$

solves $L\psi = \lambda\psi$ provided w satisfies the second-order equation

$$(-\partial_-{}^2 + V)w = \lambda^2 w \,, \tag{3.7b}$$

where $V = \tfrac{1}{4}\Pi^2 - \tfrac{1}{2}\partial_-\Pi$. [Henceforth, upper (lower) components of spinors will be denoted by u (v).] In order that (3.7b) define a "Schrödinger" equation with scattering, we must demand that V converge towards a finite asymptote as $x^- \to \pm\infty$. Moreover, the values at $x^- = \pm\infty$ must be equal so that the straightforward inverse scattering method be applicable. Hence we shall assume that

$$\Pi^2 \underset{x^- \to \pm\infty}{\longrightarrow} 4V_\infty \geq 0 \,.$$

The above condition still leaves open the question whether Π tends towards the same limit as $x^- \to \pm\infty$ or whether the limits are opposite in sign. The differential equation requires the opposite limits: since the derivative term $\partial_+\partial_-\Phi$ vanishes asymptotically, so also must e^{Φ}.

This establishes the boundary conditions (3.2a) and (3.2b), which also ensure that a normalizable zero mode always exists for the equation $L\psi = \lambda\psi$:

$$\psi \propto \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\Phi/2} \tag{3.8}$$

Condition (3.2c) is set for later convenience; see below.

The time dependence of the scattering data is determined by the evolution equation—the second member of the pair in (3.4). However, since the matrix

$$\tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is nilpotent,

$$(\sigma_1 + i\sigma_2)\partial_+\psi = 0 \Longrightarrow \partial_+ w = 0 \,, \tag{3.9}$$

the scattering data contained in w is time independent. This result may be further understood by recognizing that according to the Liouville equation $\partial_+ V = 0$; hence the combination of fields contained in V does not evolve in time.[17] Only the $\lambda = 0$ mode can escape the above result, since the evolution equation is not defined at $\lambda = 0$. We conclude that the Lax pair (3.3) and (3.4) is inadequate for constructing solutions to the Liouville equation. We now turn to a Lax pair that can reconstruct solutions.

C. Nonlocal Lax pair

We must find a new operator $\widetilde{B}$ instead of B, to determine the time evolution of the zero-eigenvalue data. This has already been done by Andreev,[5] who replaced B by a nonlocal $\widetilde{B}$, which is free of singularities in λ:

$$\widetilde{B}\psi(x^-;\lambda) = \frac{m^2}{8}e^{\Phi(x^+,x^-)/2}\int_{-\infty}^{x^-} dy^- e^{\Phi(x^+,y^-)/2}(1+\sigma_3)\psi(y^-;\lambda) \,. \tag{3.10}$$

(We shall not exhibit explicitly the x^+ dependence of ψ.) The compatibility condition on the system

$$L\psi=\lambda\psi \;, \tag{3.11a}$$

$$\partial_+\psi=\widetilde{B}\psi \tag{3.11b}$$

is still the Liouville equation, provided that $\psi(-\infty;\lambda)\exp[\frac{1}{2}\Phi(x^+,-\infty)]=0$. Since the boundary conditions (3.2) imply that $\Phi(x^+,\pm\infty)=-\infty$, this condition is automatically satisfied.

System (3.10) and (3.11) does not define a connection, since the operator $\widetilde{B}$ is nonlocal. In particular, one cannot find a local gauge transformation taking (3.10) and (3.11) into (3.3) and (3.4) or *vice versa,* not even on the solution manifold. Nevertheless, (3.3) and (3.4) are a consequence of (3.10) and (3.11) when $\lambda\neq0$. To see this, suppose $L\psi=\lambda\psi$ ($\lambda\neq0$) and $\partial_+\psi=\widetilde{B}\psi$. Then

$$\partial_+\psi(x^-;\lambda)=\frac{m^2}{8\lambda}\int_{-\infty}^{x^-}dy^-e^{[\Phi(x^+,y^-)+\Phi(x^+,x^-)]/2}(1+\sigma_3)[-i\sigma_2\partial_y-\tfrac{1}{2}\Pi(x^+,y^-)]\psi(y^-;\lambda) \tag{3.12a}$$

simplifies with the help of boundary condition (3.2) to give

$$\partial_+\psi=\frac{m^2}{8\lambda}(\sigma_1+i\sigma_3)e^{\Phi}\psi \tag{3.12b}$$

or $\partial_+\psi=B\psi$. So the Lax pair (3.3) and (3.4) is equivalent to the pair (3.10) and (3.11) when $\lambda\neq0$, but only the nonlocal one is defined when $\lambda=0$.

D. Scattering analysis

Now that we have constructed the appropriate Lax pair, and settled upon definite boundary conditions, the direct and inverse scattering analyses may be executed.

For every real value of λ such that $\lambda^2\geq V_\infty$, the system $L\psi=\lambda\psi$ has two independent eigenvectors. We define the spinors ψ_+ and $\bar{\psi}_+$ by their behavior at $x^-\to+\infty$, while the spinors ψ_- and $\bar{\psi}_-$ have a prescribed behavior at $x\to-\infty$:

$$\psi_+(x^-;\lambda)\to\begin{bmatrix}1\\ ie^{i\delta}\end{bmatrix}e^{ikx^-},\quad \bar{\psi}_+(x^-;\lambda)\to\begin{bmatrix}1\\ -ie^{-i\delta}\end{bmatrix}e^{-ikx^-},\quad x^-\to+\infty\;, \tag{3.13a}$$

$$\psi_-(x^-;\lambda)\to\begin{bmatrix}1\\ -ie^{i\delta}\end{bmatrix}e^{-ikx^-},\quad \bar{\psi}_-(x^-;\lambda)\to\begin{bmatrix}1\\ ie^{-i\delta}\end{bmatrix}e^{ikx^-},\quad x^-\to-\infty\;, \tag{3.13b}$$

$$k=(\lambda^2-V_\infty)^{1/2},\quad e^{i\delta(\lambda)}=\frac{\lambda}{k+i\sqrt{V_\infty}}\;. \tag{3.13c}$$

It is seen that for $\lambda^2\geq V_\infty$ barred quantities coincide with complex conjugated ones. The function $k(\lambda)$ is defined on a double-sheeted Riemann surface with branch cuts $(-\infty,-\sqrt{V_\infty})$, $(\sqrt{V_\infty},\infty)$. On the upper sheet we have $\operatorname{Im}k>0$.

The spinors $\psi_\pm$ and $\bar{\psi}_\pm$ are the Jost functions for the equation $L\psi=\lambda\psi$. Since $(\psi_-,\bar{\psi}_-)$ are linearly dependent on $(\psi_+,\bar{\psi}_+)$, the former may be expressed in terms of the latter:

$$\psi_-(x^-;\lambda)=a(\lambda)\bar{\psi}_+(x^-;\lambda)+b(\lambda)\psi_+(x^-;\lambda)\;,$$
$$\bar{\psi}_-(x^-;\lambda)=a^*(\lambda)\psi_+(x^-;\lambda)+b^*(x)\bar{\psi}_+(x^-;\lambda)\;. \tag{3.14}$$

The coefficients $a(\lambda)$ and $b(\lambda)$ also depend on x^+, but this dependence is not exhibited. Using the Wronskian

$$W(\psi_1,\psi_2)=u_1v_2-u_2v_1\;, \tag{3.15}$$

which is x^- independent for solutions belonging to the same eigenvalue, we derive the unitarity relation

$$|a(\lambda)|^2-|b(\lambda)|^2=1\;, \tag{3.16}$$

and express a and b in terms of the Jost functions:

$$a(\lambda)=\frac{W(\psi_-,\psi_+)}{W(\bar{\psi}_+,\psi_+)},\quad b(\lambda)=\frac{W(\psi_-,\bar{\psi}_+)}{W(\psi_+,\bar{\psi}_+)}\;. \tag{3.17}$$

For $\lambda^2 \leq V_\infty$, where k becomes purely imaginary, as well as in the full upper complex k plane, the Jost functions ψ_- and ψ_+ are defined by analytic continuation. Analyticity of $u_-(x^-;\lambda)$ and $u_+(x^-;\lambda)$ in the upper-half k plane is seen to hold directly from the second-order formulation (3.7), provided that

$$\int_{-\infty}^{\infty} dx^-[V(x^+,x^-)-V_0]=\int_{-\infty}^{\infty} dx^- \left| \tfrac{1}{4}\Pi^2(x^+,x^-)-\tfrac{1}{2}\partial_-\Pi(x^+,x^-)-V_\infty \right| < \infty \ . \tag{3.18}$$

Furthermore, from Eq. (3.7) it is clear that the functions $(\lambda/k)v_-(x^-;\lambda)$ and $(\lambda/k)v_+(x^-;\lambda)$ are also analytic in the upper half plane. The scattering data $a(\lambda)$ is now seen to be analytic in the upper-half k plane. We shall prove in subsection G that the analyticity of u_-, u_+, $(\lambda/k)v_-$, and $(\lambda/k)v_+$ suffices to derive the Marchenko equations. Equation (3.18) provides a sufficient, but by no means necessary, condition for analyticity. Indeed, by working with the first-order equation a weaker sufficient condition can be established:

$$\int_{-\infty}^{\infty} dx^- \left| \Pi(x^+,x^-)-2\sqrt{V_\infty}\,\epsilon(x^-) \right| < \infty \ . \tag{3.19}$$

The analytically continued Jost functions possess asymptotes identical to (3.13), where δ and k are still given in terms of λ by (3.13c), but where k may now vary throughout the complete upper-half complex k plane.

Of course, $a(\lambda)$ may have zeros in the upper-half k plane. We show in Appendix A that these zeros are simple. However, for real λ, with $\lambda^2 \geq V_\infty$, the unitarity relation $|a(\lambda)|^2-|b(\lambda)|^2=1$ implies that $a(\lambda)\neq 0$. From the second-order formulation for the equation $L\psi=\lambda\psi$, it is clear that the scattering problem can only have continuum and/or bound states, but no resonances. Hence all the zeros of $a(\lambda)$ must lie in the interval $-\sqrt{V_\infty}<\lambda<\sqrt{V_\infty}$, and each zero corresponds to a bound state. So, no breathers[18] are present amongst the solutions of the Liouville equation; only scattering ($|\lambda|>\sqrt{V_\infty}$) and soliton ($|\lambda|<\sqrt{V_\infty}$) solutions, and superpositions thereof can occur. This is to be contrasted with the nonlinear Schrödinger and sine-Gordon equations.[14–16]

The variables $a(\lambda)$, $b(\lambda)$, the positions ζ_n of the zeros in the upper-half k plane of $a(\lambda)$, and the normalizations of the bound states

$$c_n=\frac{b(\lambda_n)}{ia'(\lambda_n)} \tag{3.20}$$

are collectively called the scattering data. Here the dash stands for differentiation with respect to k, and $\lambda_n=\lambda(\zeta_n)$. We shall show in subsection H that the above defined list of scattering data, together with the specification of $\phi_{+\infty}(x^+)$ (see subsection A), suffices to reconstruct uniquely the solution of the Liouville equation.

The time evolution of the scattering data has been derived by Andreev[5] only in the case where $\phi_{+\infty}(x^+)=-Bx^+$ and $\phi_{-\infty}(x^+)=Bx^+$. We need the time-evolved data for arbitrary $\phi_{+\infty}$.

It has already been remarked, on the basis of the Lax pair (3.3) and (3.4), that the function w in (3.7) is time independent. Hence the scattering data $a(\lambda)$ and $b(\lambda)$ also do not depend on time for $\lambda\neq 0$. This remains true for the Lax pair (3.10) and (3.11) in view of the equivalence of both pairs for $\lambda\neq 0$, as can also be seen explicitly from (3.10) when the orthogonality of the Jost functions $\psi(x^-;0)$ ($\propto e^{\Phi/2}$) and $\psi(x^-;\lambda)$, $\lambda\neq 0$, is taken into account. So we are left with a possible time dependence only in the data of the bound state at $\lambda=0$. The object of interest for the inverse scattering method is

$$c_0(x^+)=\frac{b(0;x^+)}{ia'(0;x^+)} \ .$$

Here we have explicitly indicated the time dependence of a', b, and c_0, which we shall now determine.

Because a is analytic in the upper-half k plane, we can represent it by Cauchy's theorem:

$$a(\lambda(k);x^+)=\int_C \frac{dk'}{2\pi i}\frac{a(\lambda(k');x^+)}{k'-k} \ . \tag{3.21}$$

The curve C lies in the upper-half complex k plane, and encircles the point $\lambda=0$. The right-hand side is time independent since $\lambda(k')$ is nonzero. Consequently, the left-hand side also is time independent; so we conclude that $a(\lambda;x^+)$ for all λ, including zero, does not depend on x^+, and henceforth we omit the x^+ argument of a.

In order to determine the time dependence of $b(0;x^+)$, we relate the Jost functions at $\lambda=0$ to the Liouville field Φ according to (3.8) (unlike a, b is not an analytic function, hence it can retain a time dependence at $\lambda=0$):

$$\psi_-(x^-;0)=N(x^+)\begin{bmatrix}1\\0\end{bmatrix}e^{\Phi(x^+,x^-)/2} . \quad (3.22)$$

Here $N(x^+)$ is an x^+-dependent normalization. At $\lambda=0$, ψ_- and ψ_+ are proportional:

$$\psi_-(x^-;0)=b(0;x^+)\psi_+(x^-;0) . \quad (3.23a)$$

ψ_+ is defined so that [see (3.13)]

$$\psi_+(x^-;0) \underset{x^-\to\infty}{\to} \begin{bmatrix}1\\0\end{bmatrix}e^{-(V_\infty)^{1/2}x^-} . \quad (3.23b)$$

Hence we have

$$b(0;x^+)=N(x^+)e^{\phi_{+\infty}(x^+)/2} . \quad (3.24)$$

The time-evolution equation then implies

$$\partial_+ b(0;x^+)=\frac{1}{4b(0;x^+)}e^{\phi_{+\infty}(x^+)}\int dx^- u_-^2(x^-;0). \quad (3.25a)$$

We use Eq. (3.23a) to express u_- in terms of u_+, and then Eq. (A8) to evaluate the integral over u_-^2 in terms of the scattering data (v_+ vanishes at $\lambda=0$):

$$\partial_+ b(0;x^+)=\frac{i}{2}a'(0)e^{\phi_{+\infty}(x^+)} . \quad (3.25b)$$

Since $a'(0)$ does not depend on x^+, we find

$$c_0=\frac{1}{2}\partial_+^{-1}e^{\phi_{+\infty}} , \quad (3.26)$$

which generalizes Andreev's result.[5] We have now derived the time dependence of all the scattering data as a functional of $\phi_{+\infty}$.

The time dependence of the scattering data can be checked by calculating the Poisson bracket with the Hamiltonian

$$P^-=\int dx^+(\partial_+\phi_{+\infty}(x^+))^2 . \quad (3.27)$$

[This is the x^--independent formula (3.1), with the integrand evaluated at $x^-=\infty$.] The brackets between $\phi_{+\infty}$'s with different arguments are deduced from those of Φ by letting $x^-\to+\infty$:

$$\{\phi_{+\infty}(x^+),\phi_{+\infty}(y^+)\}=\tfrac{1}{4}\epsilon(x^+-y^+) . \quad (3.28)$$

Using this bracket and Hamiltonian P^-, we derive the time dependence of c_0:

$$\{P^-,c_0\}=\tfrac{1}{2}\partial_+^{-1}(e^{\phi_{+\infty}}\{P^-,\phi_{+\infty}\}) . \quad (3.29a)$$

Imposing the boundary condition, stated in (3.2c), that $\phi_{+\infty}(x^+)\to-\infty$ as $|x^+|\to\infty$, we have

$$\{P^-,c_0(x^+)\}=\partial_+c_0(x^+) . \quad (3.29b)$$

The next ingredient needed for the inverse scattering technique is the complex-conjugated scattering data. From $L\psi=\lambda\psi$, we see that $L\sigma_3\psi=-\lambda\sigma_3\psi$. Using the asymptotic behavior of the functions ψ, and the fact that $(k(\lambda))^*=-k(-\lambda^*)$,[19] we find

$$\sigma_3(\psi_\pm(x^-;\lambda))^*=\psi_\pm(x^-;-\lambda^*) , \quad (3.30a)$$

$$\sigma_3(\bar\psi_\pm(x^-;\lambda))^*=\bar\psi_\pm(x^-;-\lambda^*) \quad (3.30b)$$

so that, we get from (3.17)

$$\begin{aligned}&(a(\lambda))^*=a(-\lambda^*) ,\\ &(b(\lambda))^*=b(-\lambda^*) .\end{aligned} \quad (3.31a)$$

For real λ and $\lambda^2\geq V_\infty$, multiplication of a and b by the same x^--independent phase factor merely amounts to a redefinition of the asymptotic behavior of the Jost functions. It will turn out to be convenient in what follows to choose $\mathrm{Im}b(\sqrt{V_\infty})=0$ so that

$$\mathrm{Im}b(-\sqrt{V_\infty})=0 . \quad (3.31b)$$

Finally, we need the dispersion relation between $\arg(\lambda)$ and $\ln|a(\lambda)|$:

$$\begin{aligned}\arg a(\xi)=&-\frac{1}{\pi}⨍ dl\frac{\ln|a(\lambda(l))|}{l-k(\xi)}\\ &+i\sum_{n=1}^{N}\ln\frac{k(\xi)-\zeta_n}{k(\xi)-\zeta_n^*} ,\end{aligned} \quad (3.32)$$

where $a(\lambda(\zeta_n))=0$ and $\mathrm{Im}\zeta_n>0$.

E. Poisson brackets of the scattering data

Using light-cone Poisson brackets for the field Φ at equal x^+, we shall show that the quantities $\rho(\lambda)$ and $\omega(\lambda)$ are canonically conjugate,

$$\{\rho(\lambda),\omega(\lambda')\}=\delta(\lambda-\lambda'),\quad \lambda,\lambda'>\sqrt{V_\infty} , \quad (3.33)$$

where $\rho(\lambda)=(8/\pi\lambda)\ln|a(\lambda)|$ and $\omega(\lambda)=\arg b(\lambda)$. In formula (3.33), we have restricted λ and λ' to be larger than $\sqrt{V_\infty}$, because the canonical variables can be chosen to have positive argument. This presents no loss of generality since ρ and ω are odd functions of their argument in view of (3.31). Similar conjugate pairs result from the variables associated with the zeros of $a(\lambda)$. Of course, the bracket of a canonical variable with itself vanishes. The method used here was devised by Zakharov and Manakov[15] and was applied by them to the nonlinear Schrödinger and Kortweg-de Vries equations.

We shall use a form for the light-cone Poisson brackets at equal x^+ which differs from (2.32):

$$\{A,B\}=-\tfrac{1}{4}\int dx^-\left[\frac{\delta A}{\delta\Pi(x^-)}\partial_-\frac{\delta B}{\delta\Pi(x^-)}-\frac{\delta B}{\delta\Pi(x^-)}\partial_-\frac{\delta A}{\delta\Pi(x^-)}\right]. \tag{3.34}$$

(In what follows the time dependence of Φ and Π will be suppressed.) Expression (3.34) is equivalent to the definition (2.32), but is more convenient here, because only Π occurs in the equation $L\psi=\lambda\psi$. The field Φ is a dependent variable at fixed x^+ and can be eliminated.[20] Since the data are given by Jost functions, we shall need their functional derivatives with respect to Π, which we now determine.

The equation $L\psi=\lambda\psi$ may alternatively be written as

$$\psi_+(x^-;\lambda)=\psi_0(x^-;\lambda)+\int_{x^-}^{\infty}dy^-K(x^-,y^-;\lambda)\tfrac{1}{2}\Pi(y^-)\sigma_3\psi_+(y^-;\lambda)\ , \tag{3.35a}$$

where

$$(\partial_--i\sigma_2\lambda)\begin{cases}\psi_0(x^-;\lambda)=0\ ,\\ K(x^-,y;^-\lambda)=\delta(x^--y^-)\end{cases} \tag{3.35b}$$

with

$$K(x^-,y^-;\lambda)=0\ ,\quad x^->y^-\ . \tag{3.35c}$$

This may also be expressed as a Volterra series,

$$\psi_+(x^-;\lambda)=\psi_0(x^-;\lambda)+\sum_{n=1}^{\infty}\psi_n(x^-;\lambda)\ , \tag{3.36a}$$

where

$$\psi_n(x^-;\lambda)=\int_{x^-}^{\infty}dy_1^-\int_{y_1^-}^{\infty}dy_2^-\cdots\int_{y_{n-1}^-}^{\infty}dy_n^-K(x^-,y_1^-;\lambda)\tfrac{1}{2}\Pi(y_1^-)\sigma_3\cdots K(y_{n-1}^-,y_n^-;\lambda)\times\tfrac{1}{2}\Pi(y_n^-)\sigma^3\psi_0(y_n^-;\lambda)\ ,\quad n\geq 1\ . \tag{3.36b}$$

From Eq. (3.36) it is easy to see that the Jost functions $\psi_+(x^-;\lambda)$ and $\bar\psi_+(x^-;\lambda)$, which were defined by their asymptotic behavior at $x^-=+\infty$, are independent of Π when $x^->y^-$.

$$\frac{\delta\psi_+(x^-;\lambda)}{\delta\Pi(x^--\epsilon)}=\frac{\delta\bar\psi_+(x^-;\lambda)}{\delta\Pi(x^--\epsilon)}=0\ ,\quad \epsilon>0\ . \tag{3.37a}$$

Similarly,

$$\frac{\delta\psi_-(x^-;\lambda)}{\delta\Pi(x^-+\epsilon)}=\frac{\delta\bar\psi_-(x^-;\lambda)}{\delta\Pi(x^-+\epsilon)}=0\ ,\quad \epsilon>0\ . \tag{3.37b}$$

Using Eqs. (3.35) and (3.37) we get

$$\frac{\delta\psi_+(x^-;\lambda)}{\delta\Pi(x^-+\epsilon)}=K(x^-,x^-+\epsilon;\lambda)\tfrac{1}{2}\sigma_3\psi_+(x^-+\epsilon;\lambda)+\int_{x^-}^{x^-+\epsilon}dy^-K(x^-,y^-;\lambda)\times\tfrac{1}{2}\Pi(y^-)\sigma_3\frac{\delta\psi_+(y^-;\lambda)}{\delta\Pi(x^-+\epsilon)}$$

or, in the limit $\epsilon\to 0$,

$$\frac{\delta\psi_+(x^-;\lambda)}{\delta\Pi(x^-+0)}=-\tfrac{1}{2}\sigma_3\psi_+(x^-;\lambda)\ . \tag{3.38a}$$

Similarly,

$$\frac{\delta\bar\psi_+(x^-;\lambda)}{\delta\Pi(x^-+0)}=-\tfrac{1}{2}\sigma_3\bar\psi_+(x^-;\lambda) \tag{3.38b}$$

and

$$\frac{\delta\psi_-(x^-;\lambda)}{\delta\Pi(x-0)}=\tfrac{1}{2}\sigma_3\psi_-(x^-;\lambda)\,. \tag{3.38c}$$

These suffice to compute the functional derivatives of the scattering data $a(\lambda),b(\lambda)$ with respect to Π:

$$\frac{\delta a(\lambda)}{\delta\Pi(x^-)}=\frac{1}{4i\cos\delta(\lambda)}[u_-(\lambda)v_+(\lambda)+v_-(\lambda)u_+(\lambda)]\,, \tag{3.39a}$$

$$\frac{\delta b(\lambda)}{\delta\Pi(x^-)}=\frac{-1}{4i\cos\delta(\lambda)}[u_-(\lambda)\bar{v}_+(\lambda)+v_-(\lambda)\bar{u}_+(\lambda)]\,. \tag{3.39b}$$

(We have suppressed the x^- dependence in the Jost functions.) Using the equation $L\psi=\lambda\psi$, we derive

$$\partial_-\frac{\delta a(\lambda)}{\delta\Pi(x^-)}=\frac{\lambda}{2i\cos\delta(\lambda)}[v_-(\lambda)v_+(\lambda)-u_-(\lambda)u_+(\lambda)]\,, \tag{3.40a}$$

$$\partial_-\frac{\delta b(\lambda)}{\delta\Pi(x^-)}=\frac{-\lambda}{2i\cos\delta(\lambda)}[v_-(\lambda)\bar{v}_+(\lambda)-u_-(\lambda)\bar{u}_+(\lambda)]\,. \tag{3.40b}$$

With these formulas, we can express all Poisson brackets between scattering data in terms of Jost functions alone.

Let us start by computing $\{a(\lambda),b(\lambda)\}$. One may establish the following equality:

$$8\left[\frac{\lambda}{\xi}\frac{\delta a(\lambda)}{\delta\Pi(x^-)}\partial_-\frac{\delta b(\xi)}{\delta\Pi(x^-)}+\frac{\xi}{\lambda}\frac{\delta b(\xi)}{\delta\Pi(x^-)}\partial_-\frac{\delta a(\lambda)}{\delta\Pi(x^-)}\right]$$
$$=\frac{1}{\cos\delta(\lambda)\cos\delta(\xi)}\partial_-[u_-(\lambda)u_+(\lambda)v_-(\xi)\bar{v}_+(\xi)+u_-(\xi)\bar{u}_+(\xi)v_-(\lambda)v_+(\lambda)]\,. \tag{3.41}$$

This is derived by inserting in the left-hand side expressions (3.39) and (3.40), and then using the equation $L\psi=\lambda\psi$. After adding multiples of $[\delta a(\lambda)/\delta\Pi][\delta b(\xi)/\delta\Pi]$, we can reexpress, in terms of a's and b's alone, the appropriate combination occurring for $\{a(\lambda),b(\xi)\}$, when this Poisson bracket is evaluated from its definition (3.34). Collecting all terms we obtain

$$\{a(\lambda),b(\xi)\}=\int dx^-\partial_-\left[\frac{1}{16}\frac{\lambda\xi}{\xi^2-\lambda^2}\left\{\frac{1}{\cos\delta(\lambda)\cos\delta(\xi)}[u_-(\lambda)u_+(\lambda)v_-(\xi)\bar{v}_+(\xi)+u_-(\xi)\bar{u}_+(\xi)v_-(\lambda)v_+(\lambda)]\right\}+\frac{1}{4}\frac{\xi^2+\lambda^2}{\xi^2-\lambda^2}\left\{\frac{\delta a(\lambda)}{\delta\Pi(x^-)}\frac{\delta b(\xi)}{\delta\Pi(x^-)}\right\}\right]\,, \tag{3.42a}$$

and defining

$$\{a(\lambda),b(\xi)\}_\Lambda=\left[\frac{1}{16}\frac{\lambda\xi}{\xi^2-\lambda^2}\left\{\frac{1}{\cos\delta(\lambda)\cos\delta(\xi)}[u_-(\lambda)u_+(\lambda)v_-(\xi)\bar{v}_+(\xi)+u_-(\xi)\bar{u}_+(\xi)v_-(\lambda)v_+(\lambda)]\right\}+\frac{1}{4}\frac{\xi^2+\lambda^2}{\xi^2-\lambda^2}\left\{\frac{\delta a(\lambda)}{\delta\Pi(x^-)}\frac{\delta b(\xi)}{\delta\Pi(x^-)}\right\}\right]_{x^-=\Lambda} \tag{3.42b}$$

we have

$$\{a(\lambda),b(\xi)\}=\lim_{\Lambda\to\infty}[\{a(\lambda),b(\xi)\}_\Lambda-\{a(\lambda),b(\xi)\}_{-\Lambda}]\,. \tag{3.42c}$$

To evaluate the limits, we use the identity

$$\lim_{\Lambda\to+\infty}\frac{e^{i\Lambda x}}{x}=i\pi\delta(x) \tag{3.43}$$

as well as the relations (3.14) between the Jost functions:

$$\begin{aligned}\{a(\lambda),b(\xi)\}_\infty=&\left[\frac{1}{8}\frac{\lambda^2\xi^2}{\xi^2-\lambda^2}-\frac{1}{16}\frac{\lambda^2+\xi^2}{\xi^2-\lambda^2}V_\infty\right]\frac{1}{k(\lambda)k(\xi)}a(\lambda)b(\xi)\\&+\frac{\pi i V_\infty}{16k(\lambda)}a(\lambda)a(\xi)\delta(k(\xi))+\frac{\pi i V_\infty}{16k(\xi)}b(\lambda)b(\xi)\delta(k(\lambda))\\&+\frac{\pi i\lambda^2}{8k(\lambda)}a(\xi)b(\lambda)\delta(k(\lambda)-k(\xi))\,,\end{aligned} \tag{3.44a}$$

$$\begin{aligned}\{a(\lambda),b(\xi)\}_{-\infty}=&\left[-\frac{1}{8}\frac{\lambda^2\xi^2}{\xi^2-\lambda^2}+\frac{1}{16}\frac{\lambda^2+\xi^2}{\xi^2-\lambda^2}V_\infty\right]\frac{1}{k(\lambda)k(\xi)}a(\lambda)b(\xi)\\&-\frac{\pi i V_\infty}{16k(\lambda)}a(\lambda)a^*(\xi)\delta(k(\xi))+\frac{\pi i V_\infty}{16k(\xi)}b(\xi)b^*(\lambda)\delta(k(\lambda))\\&+\frac{\pi i\lambda^2}{8k(\lambda)}b^*(\lambda)a^*(\xi)\delta(k(\lambda)+k(\xi))\,.\end{aligned} \tag{3.44b}$$

The relation (3.31) between complex conjugates reduces the above to

$$\begin{aligned}\{a(\lambda),b(\xi)\}=&\frac{1}{8}\frac{1}{\xi^2-\lambda^2}\left[\frac{\lambda^2k(\xi)}{k(\lambda)}+\frac{\xi^2k(\lambda)}{k(\xi)}\right]a(\lambda)b(\xi)\\&+\frac{\pi i V_\infty}{16k(\lambda)}a(\lambda)[a(\xi)+a(-\xi)]\delta(k(\xi))\\&+\frac{\pi i\lambda^2}{8k(\lambda)}[a(\xi)b(\lambda)\delta(k(\lambda)-k(\xi))-b(-\lambda)a(-\xi)\delta(k(\lambda)+k(\xi))]\,.\end{aligned} \tag{3.45}$$

This expression implies

$$\{\ln|a(\lambda)|,b(\xi)\}=\frac{\pi}{8}i\lambda b(\xi)[\delta(\lambda-\xi)-\delta(\lambda+\xi)] \tag{3.46a}$$

or, for λ and ξ greater than $\sqrt{V_\infty}$,

$$\{\ln|a(\lambda)|,\arg b(\xi)\}=\frac{\pi}{8}\lambda\delta(\lambda-\xi)\,. \tag{3.46b}$$

Equation (3.46b) leads to the result quoted at the beginning of this subsection, Eq. (3.33).

Next, we find the brackets of $\ln|a(\lambda)|$ and $\arg b(\xi)$ with themselves. An identity similar to (3.41) is used to compute the bracket $\{a(\lambda),a(\xi)\}$, and it is found to vanish:

$$\{a(\lambda),a(\xi)\}=0\,. \tag{3.47}$$

This then implies the vanishing of the bracket between the canonical variables $\ln|a(\lambda)|$—a fact which also follows directly from (3.16), (3.31a), and (3.46a). Similarly, the bracket $\{b(\lambda),b(\xi)\}$ can be computed from yet another generalization of (3.41); but there is a simpler route. Let us compute $\{A(\lambda),B(\xi)\}$ where

$$\begin{aligned}A(\lambda)&=a(\lambda)+b(\lambda)\,,\\B(\lambda)&=a(\lambda)-b(\lambda)\,.\end{aligned} \tag{3.48a}$$

The calculation of $\{A(\lambda),B(\xi)\}$ is then identical to that of $\{a(\lambda),b(\xi)\}$, provided the substitutions

$$\begin{aligned}\psi_+\to\psi'_+&=\psi_+-\bar\psi_+\,,\\\bar\psi_+\to\bar\psi'_+&=-\psi_+-\bar\psi_+\end{aligned} \tag{3.48b}$$

are made. The result of this calculation is that

$$\{\arg b(\lambda), \arg b(\xi)\} = 0 \ . \tag{3.49}$$

Equations (3.46b), (3.47), and (3.49) hold for $\lambda^2 \geq V_\infty$, which defines the independent canonical variables. We must still compute the Poisson brackets of the canonical variables associated with the zeros of $a(\lambda)$.

We have proven (in Appendix A) that the zeros $\lambda_n = \lambda(\zeta_n)$ of $a(\lambda)$ are simple when λ is real. At the zeros, ψ_+ and ψ_- are linearly dependent:

$$\psi_-(\lambda_n) = b(\lambda_n)\psi_+(\lambda_n) \tag{3.50}$$

and

$$\frac{\delta b(\lambda_n)}{\delta\Pi(x^-)} = \frac{-1}{4i\cos\delta(\lambda_n)}[u_-(\lambda_n)\bar{v}_+(\lambda_n) + v_-(\lambda_n)\bar{u}_+(\lambda_n)]\,|_{x^-} \ . \tag{3.51}$$

Conjugate to ζ_n is c_n, defined in (3.20). We now determine how ζ_n varies under a variation $\delta\Pi$ of Π. To do so, we use the equation $L\psi = \lambda\psi$:

$$\begin{aligned} \partial_- u'_+ - \tfrac{1}{2}[\Pi(x^-) + (\delta\Pi)\delta(x^- - z)]u'_+ &= (\lambda_n + d\lambda_n)v'_+ \ , \\ -\partial_- v'_+ - \tfrac{1}{2}[\Pi(x^-) + (\delta\Pi)\delta(x^- - z)]v'_+ &= (\lambda_n + d\lambda_n)u'_+ \ . \end{aligned} \tag{3.52}$$

Since the zeros of $a(\lambda)$ are simple, the functions $\psi_-(\lambda_n + d\lambda_n)$ and $\psi_+(\lambda_n + d\lambda_n)$ are linearly independent and the solution of (3.52) must be of the form

$$\psi'(\lambda_n + d\lambda_n, x^-) = \begin{cases} C_1\psi_+(\lambda_n + d\lambda_n, x^-), & x^- > z \ , \\ C_2\psi_-(\lambda_n + d\lambda_n, x^-), & x^- < z \ . \end{cases} \tag{3.53}$$

Equation (3.52) is compatible with (3.53) only if C_1 and C_2 match at $x^- = z$:

$$\begin{aligned} C_1(u_+ - \tfrac{1}{2}\delta\Pi u_+) - C_2 u_- &= 0 \ , \\ C_1(v_+ + \tfrac{1}{2}\delta\Pi v_+) - C_2 v_- &= 0 \ , \end{aligned} \tag{3.54}$$

which has a solution provided

$$\begin{vmatrix} u_+ - \frac{1}{2}\delta\Pi u_+ & -u_- \\ v_+ + \frac{1}{2}\delta\Pi v_+ & -v_- \end{vmatrix} = 0, \quad x^- = z \tag{3.55a}$$

or

$$W(\psi_+, \psi_-) - \tfrac{1}{2}\delta\Pi(u_+v_- + v_+u_-)\,|_{x^- = z} = 0 \ . \tag{3.55b}$$

Hence we have

$$a'(\lambda_n)d\zeta_n W(\psi_+, \bar{\psi}_+) - \tfrac{1}{2}\delta\Pi(u_+v_- + v_+u_-)\,|_{x^- = z} = 0$$

or

$$\frac{\delta\zeta_n}{\delta\Pi(x^-)} = \frac{1}{4ia'(\lambda_n)\cos\delta(\lambda_n)} \times [u_-(x^-;\lambda_n)v_+(x^-;\lambda_n) + v_-(x^-;\lambda_n)u_+(x^-;\lambda_n)] \ . \tag{3.56}$$

This is seen to be the continuation of (3.39a), with the help of the formal identity

$$\left.\frac{\delta a(\lambda)}{\delta\Pi(x^-)}\right|_{\lambda=\lambda_n} = a'(\lambda_n)\frac{\delta\zeta_n}{\delta\Pi(x^-)} \ . \tag{3.57}$$

It is then straightforward to derive the brackets

$$\begin{aligned} &\{\zeta_n, a(\lambda)\} = \{\zeta_n, b(\lambda)\} = \{c_n, a(\lambda)\} = 0 \ , \\ &\{\zeta_n, \zeta_m\} = \{c_n, c_m\} = \{c_n, b(\lambda)\} = 0 \ , \\ &\{\zeta_n, c_m\} = 0, \quad n \neq m \ . \end{aligned} \tag{3.58}$$

Finally, adapting the calculation of the bracket $\{a(\lambda), b(\xi)\}$ to the case of $\lambda^2 < V_\infty$ we find

$$\{\lambda_n, c_n\} = -\tfrac{1}{8}\lambda_n c_n \ . \tag{3.59}$$

Consequently, the canonical variables are

$$\begin{aligned} &\rho(\lambda) = \frac{8}{\pi\lambda}\ln|a(\lambda)|, \quad \omega(\lambda) = \arg b(\lambda) \ , \quad \lambda \geq \sqrt{V_\infty} \ , \\ &P_n = \ln\lambda_n, \quad R_n = -8\ln c_n, \quad n \neq 0 \ . \end{aligned} \tag{3.60}$$

The only nonvanishing brackets are $\{\rho(\lambda), \omega(\lambda')\} = \delta(\lambda - \lambda')$ and $\{P_n, R_m\} = \delta_{nm}$ when $n \neq 0$. In addition, we have $\{\zeta_0, \ln c_n\} = 0$ for all n, so that ζ_0 commutes with all canonical variables and consequently has no time dependence. It can therefore be regarded as a c number. The same is not true however for c_0. Although c_0 commutes with the canonical variables $a(\lambda)$, $b(\lambda)$, c_n, and ζ_n, it does have a nontrivial time evolution, because it does not commute with the additional canonical variables which we now construct.

At this point, we can fully appreciate the discussion in subsection A of the boundary conditions on the Lax pair. The scattering data $a(\lambda)$, $b(\lambda)$, ζ_n, and c_n $(n>0)$ do not suffice in characterizing the most general solution to the system. The function $\phi_{+\infty}(x^+)$ must also be specified. Phrased differently: the scattering data are only *half* of the dynamical variables and the other half is precisely spanned by $\phi_{+\infty}(x^+)$.

Although we have $\{\phi_{+\infty}(x^+),\phi_{+\infty}(y^+)\}=\frac{1}{4}\epsilon(x^+-y^+)$, this is not yet in the form appropriate to action-angle variables. Separation is however easily achieved by Fourier transforming $\phi_{+\infty}(x^+)$:

$$\phi_{+\infty}(x^+)=\int \frac{dk}{\sqrt{2\pi}}e^{ikx^+}\frac{d_k}{\sqrt{2|k|}} . \tag{3.61}$$

Since $\phi_{+\infty}$ is required to be real, $d_k^*=d_{-k}$; it is then straightforward to check that

$$\{d_k,d_{k'}\}=\{d_k^*,d_{k'}^*\}=0 , \tag{3.62a}$$

$$\{d_k,d_{k'}^*\}=-\frac{ik}{2}\left[\frac{1}{k+i\epsilon}+\frac{1}{k-i\epsilon}\right]\delta(k-k') ,\quad k,k'\geq 0 . \tag{3.62b}$$

Here again, d_0 commutes with everything in the Poisson bracket algebra and has no time dependence, so that it can be regarded as a c number and not as a dynamical variable. Thus (3.62) becomes

$$\{d_k,d_{k'}\}=\{d_k^*,d_{k'}^*\}=0 , \tag{3.63a}$$

$$\{d_k,d_{k'}^*\}=-i\delta(k-k'),\quad k,k'>0 . \tag{3.63b}$$

It is now readily seen that $\mathscr{P}_k$ and $\mathscr{R}_k$, defined below, are action-angle variables:

$$\mathscr{P}_k=d_k^*d_k,\quad \mathscr{R}_k=\arg d_k,\quad k>0 \tag{3.64}$$

satisfying the Poisson brackets

$$\{\mathscr{P}_k,\mathscr{P}_{k'}\}=\{\mathscr{R}_k,\mathscr{R}_{k'}\}=0 , \tag{3.65a}$$

$$\{\mathscr{P}_k,\mathscr{R}_{k'}\}=\delta(k-k') . \tag{3.65b}$$

Using the expression for $\phi_{+\infty}(x^+)$ in terms of the action-angle variables $\mathscr{P}_k$ and $\mathscr{R}_k$, we find the dependence of c_0 on $\mathscr{P}_k$ and $\mathscr{R}_k$:

$$c_0(x^+)=\int dy^+\partial_+^{-1}(x^+-y^+)\exp\left[\int_0^\infty \frac{dk}{\sqrt{\pi k}}\sqrt{\mathscr{P}_k}\cos(\mathscr{R}_k+ky^+)\right] . \tag{3.66}$$

Equations (3.60) and (3.64) comprise all the action-angle variables and we can diagonalize the Hamiltonian.

F. Diagonalization of the Hamiltonian

We have already done all the work here: the Fourier transformation (3.61), that leads to the canonical pairs d_k,d_k^*, also diagonalizes the Hamiltonian (3.1),

$$P^-=\int_0^\infty dk\, k d_k^* d_k=\int_0^\infty dk\, k\mathscr{P}_k . \tag{3.67}$$

The Hamiltonian is completely independent of the scattering data $a(\lambda)$, $b(\lambda)$, ζ_n, and c_n $(n>0)$. This situation should again be contrasted with the case of the sine-Gordon theory where the Hamiltonian is entirely expressed in terms of the scattering data.[14–16] Clearly then, for the Liouville theory, the above scattering data are conserved quantities. We shall now derive their explicit formulas in terms of the field Φ.

G. Conserved charges

As usual, the local conserved charges are found by expanding $\ln a(\lambda)$ in a power series in $1/k$.[14,16] This is done using the relation

$$a(\lambda)=\frac{1}{2i\cos\delta(\lambda)}[u_-(\lambda)v_+(\lambda)-u_+(\lambda)v_-(\lambda)] \tag{3.68}$$

and the asymptotic form of u_+ and v_+,

$$[iu_-(x^-,\lambda)e^{i\delta(\lambda)}-v_-(x^-,\lambda)]e^{ikx^-}\underset{x^-\to\infty}{\longrightarrow} 2i\cos\delta(\lambda)a(\lambda) . \tag{3.69}$$

The combination $\sigma=ie^{i\delta}u_- -v_-$ satisfies the equation

$$\partial_-\left[\frac{\partial_-\sigma+ik\sigma}{\frac{1}{2}\Pi-\lambda\sin\delta}\right]=\frac{1}{2}\Pi\sigma+\lambda\sigma\sin\delta+2ik\frac{\partial_-\sigma+ik\sigma}{\Pi-2\lambda\sin\delta} . \tag{3.70}$$

Defining τ such that $\sigma=2ie^{\tau}e^{-ikx^-}\cos\delta$, we have $\tau\to\ln a(\lambda)$ as $x^-\to+\infty$. The differential equations satisfied by τ reads

$$(\partial_-\tau)^2-2ik(\partial_-\tau)+(\Pi-2\sqrt{V_\infty})\partial_-\left[\frac{\partial_-\tau}{\Pi-2\sqrt{V_\infty}}\right]=\tfrac{1}{4}\Pi^2-V_\infty\ . \quad (3.71)$$

Upon expanding

$$\tau(x^-,k)=\sum_{n=1}^{\infty}\frac{\tau_n(x^-)}{(2ik)^n}\ , \quad (3.72a)$$

we find the recursion relation for the τ_n's,

$$(\Pi+2\sqrt{V_\infty})\partial_-\left[\frac{\partial_-\tau_n}{\Pi+2\sqrt{V_\infty}}\right]+\sum_{m=1}^{n-1}(\partial_-\tau_m)(\partial_-\tau_{n-m})-\partial_-\tau_{n+1}=0\ ,\quad n>1\ , \quad (3.72b)$$

$$\partial_-f_1=-\tfrac{1}{4}(\Pi^2-4V_\infty)\ .$$

The charges $r_n=\int_{-\infty}^{\infty}dx^-(\partial_-\tau_n)$ are conserved and

$$\ln a(\lambda)=\sum_{n=1}^{\infty}\frac{r_n}{(2ik)^n}\ . \quad (3.73)$$

The first few charges read

$$\begin{aligned}
r_1&=-\tfrac{1}{4}\int_{-\infty}^{\infty}dx^-(\Pi^2-4V_\infty)\ ,\\
r_2&=2V_\infty\ ,\\
r_3&=\tfrac{1}{16}\int_{-\infty}^{a}dx^-[(\Pi^2-4V_\infty)^2+4(\partial_-\Pi)^2]\ ,\\
r_4&=4V_\infty{}^2\ .
\end{aligned} \quad (3.74)$$

The charge r_1 is clearly proportional to the finite part of Q_1^- defined in (2.31), and the higher charges r_3, r_5, etc., are related to higher powers of Θ^{++}.[21] Note that, although $\partial_-\tau_{2n}$ are total derivatives of combinations of the field Φ, the even charges do not vanish but are c numbers. Furthermore, a change of V_∞ rearranges the conserved charges in linear combinations with coefficients proportional to the length of the system. Also, note that all charges are finite.[22]

Nonlocal charges are usually found by expanding $\ln a(\lambda)$ in a Taylor series at $\lambda=0$. Here however, due to lack in analyticity of $\ln a(\lambda)$ at $\lambda=0$, we cannot expand that function in a Laurent series, but we can expand $\ln a(\lambda(k))$ around $k=0$. For the series to exist and converge a condition stronger than (3.19) is required:

$$\int_{-\infty}^{\infty}dy\,|y|\,|\Pi(y)+2\sqrt{V_\infty}\epsilon(y)|<\infty\ . \quad (3.75)$$

The expansion is defined by

$$\tau(x^-;k)=\sum_{n=0}^{\infty}(2ik)^n\tilde{\tau}_n(x^-)\ , \quad (3.76a)$$

and the $\tilde{\tau}_n$'s satisfy a recursion relation which follows from (3.71):

$$[\Pi+2(V'_\infty)^{1/2}]\partial_-\left[\frac{\partial_-\tilde{\tau}_n}{\Pi+2\sqrt{V_\infty}}\right]+\sum_{m=0}^{n}(\partial_-\tilde{\tau}_n)(\partial_-\tilde{\tau}_{n-m})-\partial_-\tilde{\tau}_{n-1}=0\ ,\quad n>1\ , \quad (3.76b)$$

$$(\Pi+2\sqrt{V_\infty})\partial_-\left[\frac{\partial_-\tilde{\tau}_0}{\Pi+2\sqrt{V_\infty}}\right]+(\partial_-\tilde{\tau}_0)^2=\tfrac{1}{4}(\Pi^2-4V_\infty)\ . \quad (3.76c)$$

This expansion gives rise to a set of nonlocal conserved quantities. The Ricatti equation (3.76c) can be reduced to a linear equation, which however does not seem to have a general, closed-form, solution for arbitrary Π.

In addition to $a(\lambda)$,$b(\lambda)$ also is conserved. In general $b(\lambda)$ is not analytic everywhere, but suppose it is analytic at k_0. Then defining

$$\frac{1}{2i\cos\delta}(v_-+ie^{-i\delta}u_-)e^{-ikx^-}=s(x^-,k) \quad (3.77)$$

we have

$$s(x^-,k)\underset{x^-\to\infty}{\longrightarrow}b(k)\ ,$$

where k is in the region of analyticity. The quantity s is determined by an equation analogous to (3.70):

$$\partial_-{}^2s-\frac{\partial_-\Pi}{\Pi+2\sqrt{V_\infty}}\partial_-s=\tfrac{1}{4}(\Pi^2-4V_\infty)s\ . \quad (3.78)$$

Upon expanding $s(x^-,k)$ about the point k_0,

$$s(x^-;k)=\sum_{n=0}^{\infty}s_n(x^-)[2i(k-k_0)]^n\ , \quad (3.79)$$

we see now that every s_n must obey (3.78) separately. Since the linear equation (3.78) has only two independent solutions, of which one is e^{τ_0}, we conclude that the whole set of conserved quantities $b(\lambda)$ in fact collapses to just one additional, independent conserved charge. This additional conserved charge is a nonlocal functional of the field Π, but we have not been able to express it in closed form.

Next, we derive the most general conserved quantity built out of the action-angle variables $\mathscr{P}_k, \mathscr{R}_k$. For $S(\mathscr{P}_k, \mathscr{R}_k)$ to be conserved, we must have

$$\{P^-, S\} = \int_0^\infty dk\, k \frac{\delta S}{\delta \mathscr{R}_k} = 0 . \qquad (3.80)$$

In order that also all Poisson brackets with $\{P^-, S\}$ vanish, it is necessary and sufficient that the integrand vanish. Hence $S(\mathscr{P}_k, \mathscr{R}_k) = S_0(\mathscr{P}_k)$ is the most general conserved quantity built on $\mathscr{P}_k$ and $\mathscr{R}_k$.

In conclusion, the most general conserved quantity is of the type

$$\mathscr{S} = \mathscr{S}[a(\lambda), b(\lambda), \zeta_n, c_n, \mathscr{P}_k] \qquad (3.81)$$

for $n \neq 0$, $\lambda \geq \sqrt{V_\infty}$, and $k > 0$.

H. The Marchenko equations and inverse scattering

Knowing the scattering data at a time x^+, we can construct the scattering potential at that time. Since $u_+(x^-;\lambda)$ is analytic in the upper-half k plane (see subsection C) we can define kernels K_i, $i = 1,2$, such that

$$u_+(x^-;\lambda)e^{-ikx^-} - 1 = \int_{x^-}^\infty ds\, K_1(x^-;s)e^{ik(s-x^-)} , \qquad (3.82a)$$

and

$$\frac{\lambda}{k} v_+(x^-,\lambda)e^{-ikx^-} + i = i \int_{x^-}^\infty ds\, K_2(x^-;s)e^{-ik(s-x^-)} . \qquad (3.82b)$$

Upon defining

$$F(x) = \frac{1}{2\pi} \int dk \frac{b(k)}{a(k)} e^{ikx} + \sum_{n=1}^N c_n e^{i\zeta_n x} + c_0[d_k, d_k^*] e^{-(V_\infty)^{1/2} x} \qquad (3.83)$$

the Marchenko equations can be deduced using the fact that u_+ and $(\lambda/k)v_+$ are analytic in the upper half plane[16]:

$$K_1^*(x;y) = -F(x+y) - \int_x^\infty ds\, K_1(x;s)F(y+s), \quad x < y , \qquad (3.84a)$$

$$K_2^*(x;y) = -F(x+y) + \int_x^\infty ds\, K_2(x;s)F(y+s), \quad x < y , \qquad (3.84b)$$

$$K_1(x;y) = K_2(x;y) = 0 , \quad x > y . \qquad (3.84c)$$

From the leading term of an expansion for ψ_+ in $1/k$, it can be deduced[23] that

$$\Pi(x^-) = -\sqrt{V_\infty} - 2[K_1(x^-;x^-) + K_2(x^-;x^-)] . \qquad (3.85)$$

Using (3.31), (A8), and the fact that ζ_n is purely imaginary, we see that F is real, so that K_1 and K_2 are real; the function Π is then real. The equation satisfied by K_i is

$$K_i(x;y) = -F(x+y) - \epsilon_i \int_x^\infty ds\, K_i(x;s)F(y+s) , \quad \epsilon_1 = 1, \ \epsilon_2 = -1 \qquad (3.86)$$

for $x < y$. Formally, in operator notation, this reads $K = -F - \epsilon KF$ and is solved by the series

$$K = -F + \epsilon F^2 - \epsilon^2 F^3 + \cdots \qquad (3.87)$$

provided the magnitudes of F's eigenvalues are less than 1. F is diagonal in the momentum representation and $|[b(k)/a(k)]| < 1$ for real k. When $a(k)$ has zeros in the upper half plane, the series (3.87) is not absolutely convergent in general, but equation (3.86) can be solved uniquely by continuing the solution in the variables c_n. Hence (3.86) is uniquely solvable for Π in terms of d_k as well as the scattering data $b(k)/a(k)$ (for real k), c_n and ζ_n (which is purely imaginary). The most general solution to the Liouville equation is then

$$\Phi(x^+;x^-) = \int dy^- \partial_-^{-1}(x^- - y^-)\Pi(x^+,y^-) + \Phi_0(x^+) , \qquad (3.88)$$

and $\Phi_0(x^+)$ is determined as a functional of the d_k's with the help of (3.2). The solution we obtain here is of the general type (1.4).

IV. QUANTUM THEORY

There are no apparent obstacles to a canonical equal-time or light-cone quantization of the Liouville theory where Poisson brackets are replaced by commutators.[24] Of course, a solution requires developing calculational techniques, approximate ones, if need be. We shall now describe the results of different approaches to this problem. While various difficulties are encountered, it is possible to make a plausible conjecture about the complete theory.

A. Semiclassical quantization

The canonical transformations, constructed in Secs. II and III, allow for a semiclassical analysis. The canonical variables obtained from both transformations appear unconstrained, as in a canonical massless free field. [There does exist a constraint on ϕ, as defined in (2.36) and (2.37): the form invariance (1.6) should be respected; however, this does not seem to limit the canonical nature of the action-angle variables.] Consequently, semiclassical quantization yields a continuous spectrum, as in a free field theory, consistent with the conformal invariance of the theory.

B. Quantum inverse scattering method

For the sine-Gordon theory, Sklyanin, Takhtadzhyan, and Faddeev[23] have proposed a quantum inverse method, enabling them to construct the exact S matrix (except in the soliton sector) as well as the quantal conservation laws. Also they diagonalize the Hamiltonian. In attempting to apply this procedure to the Liouville theory, the following obstacles are encountered.

For Lax pair (3.10) and (3.11), defined in light-cone coordinates, the fields Φ and Π are promoted to quantum operators obeying light-cone commutation relations. Following Ref. 23, we first concentrate on the quantum version of the L operator, and construct the similarity matrix $R(\lambda,\mu)$. By using light-cone commutators, we unfortunately lose the crucial property of ultralocality and the quantum inverse method no longer applies.

If we insist upon formulating Lax pair (3.10) and (3.11) in t,x coordinates, with Φ and $\dot{\Phi}$ obeying equal-time commutators, we find that both equations of the corresponding Lax pair are nonlocal. Again, the method is not defined in this case.

In conclusion, we have not succeeded in applying the quantum inverse method to the Liouville theory.

C. Perturbation theory

Perturbation theory in β for vacuum matrix elements is clearly vitiated by the term linear in $(1/\beta)\Phi$ appearing when $(m^2/\beta^2)e^{\beta\Phi}$ is expanded in powers of β. This tadpole cannot be removed by a shift in the field, since shifting Φ only redefines m^2.

Perturbation theory in m^2 is ultraviolet finite after mass renormalization (for $\beta^2<4\pi$),[25] but is severely infrared divergent. These divergences are to be expected, since m^2 is dimensionful. Moreover, a power series in m^2 cannot be uniformly defined, since by shifting Φ by $\ln m^2$, m^2 may be removed from the theory.

We interpret these failures of perturbative computations of vacuum amplitudes as evidence that the theory possesses no ground state. This conjecture is further substantiated by the following considerations.

D. Effective potential and nonexistence of a ground state

In this subsection, we show that no translationally invariant ground state exists. This can already be surmised from the quantum equation of motion

$$\Box\Phi+\frac{m^2}{\beta}e^{\beta\Phi}=0\ . \tag{4.1}$$

If the theory possesses a translationally invariant, normalizable ground state $|0\rangle$, then $\langle 0|\Box\phi|0\rangle=0$, so that

$$\frac{m^2}{\beta}\langle 0|e^{\beta\Phi}|0\rangle=0 \tag{4.2}$$

which violates the formal positivity of the exponential. This suggests that no translationally invariant ground state exists.

We now confirm this by using the effective potential, which can be evaluated in the loop expansion. Here we need not select a ground state initially; no infrared divergences appear and a well-defined expression is obtained. In tree approximation, the effective potential is just the classical potential $(m^2/\beta^2)e^{\beta\Phi}$, which has no minimum (except at $\Phi=-\infty$). Quantum corrections do not modify this conclusion.

The exact effective potential has been computed by Goldstone,[26] using a simple argument that relies on the fact that a shift in the field Φ is equivalent to a redefinition of the only dimensionful parameter in the theory, m^2, which can in turn be compensated for by normal reordering. Goldstone's result for the complete effective potential is

$$V(\beta,\Phi)=\frac{m_r^{\,2}}{\tilde{\beta}^2}e^{\tilde{\beta}\Phi},\quad \tilde{\beta}^{-1}=\beta^{-1}+\frac{\hbar\beta}{8\pi}\;. \tag{4.3}$$

Here m_r is the renormalized mass.

We have checked this formula in the loop expansion and find, to the two-loop, $O(\hbar^2)$ approximation,

$$V(\beta,\Phi)=\frac{m^2}{\beta^2}e^{\beta\Phi}\left[1+\frac{\hbar\beta^2}{8\pi}\left[\ln\frac{\Lambda^2}{m^2}-\beta\Phi+1\right]+\frac{1}{2}\left[\frac{\hbar\beta^2}{8\pi}\right]^2\left[\ln\frac{\Lambda^2}{m^2}-\beta\Phi\right]^2+C\right]+O(\hbar^3),\quad C=3.05208\cdots\;, \tag{4.4}$$

where Λ is an ultraviolet cutoff.[27] After renormalizing the bare mass m, we see that to $O(\hbar^3)$ (4.4) agrees with (4.3).[28]

Consequently, the effective potential is monotonic, and no translationally invariant ground state exists. The spectrum has a greatest lower bound—zero—which however is not attainted by any eigenstate. In Appendix B, we show that the Liouville theory, in one fewer dimension has a continuous spectrum but no ground state. This quantum mechanical model can also be considered as the high-temperature limit of the field theory.

Upon combining the semiclassical results with the effective potential analysis, we are led to conclude that the energy spectrum, consistent with conformal invariance, is continuous, bounded from below by an unattained, vanishing, greatest lower bound. Evidently this peculiar behavior does not violate any known quantum-mechanical principles; it reflects the physical fact that the lowest-energy classical configuration requires a [negatively] infinite value for the dynamical variable. It may happen that the effective potential vanishes for some definite values of β; presumably for small generic β this does not occur. Also there remains the possibility, about which we have nothing to say, that a translationally noninvariant ground state exists.

V. ADDED NOTE

There has now appeared an investigation by Curtright and Thorn[29] of the quantized Liouville model on a finite spatial interval, with periodic boundary conditions. While we do not agree with their assertions concerning properties of the ground state[30]—no such state exists—their analysis of the conformal algebra (2.34) may be used to the same end for the infinite-space theory. They show that the quantum algebra closes, viz., various commutator anomalies merely redefine parameters. We present here a derivation of the corresponding result for the full Minkowski-space theory. Our method initially does not employ normal ordering as Curtright and Thorn do, and we make explicit an assumption which underlies their approach.

An equal-time quantization is adopted, and the time variable is henceforth suppressed. The conformal algebra (2.34) will be valid if and only if nonvanishing components of the (conformally improved) energy-momentum tensor, Θ_{++} and Θ_{--}, satisfy the following commutation relations:

$$\frac{1}{i\hbar}[\Theta_{++}(x),\Theta_{++}(y)]=2(\Theta_{++}(x)+\Theta_{++}(y))\delta'(x-y)-c\delta'''(x-y)\;, \tag{N1a}$$

$$\frac{1}{i\hbar}[\Theta_{--}(x),\Theta_{--}(y)]=-2(\Theta_{--}(x)+\Theta_{--}(y))\delta'(x-y)+c\delta'''(x-y)\;, \tag{N1b}$$

$$\frac{1}{i\hbar}[\Theta_{++}(x),\Theta_{--}(y)]=0\;. \tag{N1c}$$

The triple-differentiated δ function multiplies a c number. We discuss only the first commutator; the remaining two behave in an entirely similar way.

The classical formula for Θ_{++} is given by (2.28):

$$\begin{aligned}\Theta_{++}&=\partial_+\Phi\partial_+\Phi-\frac{2}{\beta}\partial_+{}^2\Phi\\ &=\tfrac{1}{2}(\Pi+\Phi')^2-\frac{2}{\beta}(\Pi+\Phi')'+\frac{m^2}{\beta^2}e^{\beta\Phi}\;.\end{aligned} \tag{N2}$$

Here Π is the equal-time canonical momentum Φ. Quantal anomalies arise, as always, from singularities in products of operators at the same point. This problem afflicts the first and third terms in the last line of (N2). To proceed, we shall assume that the singularities are no worse than in free field theory, and that no singularities are associated with the coincidence of time variables.

In particular, upon defining the dimension-1 combination,

$$A=\Pi+\Phi' \tag{N3}$$

we extract the short-distance singularity in the product $A(x)A(y)$:

$$A(x)A(y)+A(y)A(x)=-\frac{2\hbar a}{\pi(x-y)^2}+{:}A(x)A(y){:}+{:}A(y)A(x){:}\ . \tag{N4}$$

The colons do not signify normal ordering, rather they merely indicate a product in which the coincident-point limit can be taken. The c number singularity has dimension 2, and is governed by the constant a.

Next we consider the field exponential. In the free theory it is multiplicatively renormalizable; hence we assert that here too the combination $m^2e^{\beta\Phi}$ is finite, since the (infinite) renormalization may be absorbed in the bare mass m, to give the renormalized mass M:

$$m^2e^{\beta\Phi}=M^2{:}e^{\beta\Phi}{:}\ . \tag{N5}$$

Finally, we shall encounter the product of the exponential with A. Again dimensional analysis suggests the appropriate expression:

$$m^2[A(x)e^{\beta\Phi(y)}+e^{\beta\Phi(y)}A(x)]=\frac{-\hbar\beta b}{\pi(x-y)}M^2{:}e^{\beta\Phi(y)}{:}+M^2[{:}A(x)e^{\beta\Phi(y)}{:}+{:}e^{\beta\Phi(y)}A(x){:}]\ . \tag{N6}$$

The dimension-1 singularity is governed by the constant b.

The quantum formula which replaces (N2) is

$${:}\Theta_{++}{:}=\tfrac{1}{2}{:}A^2{:}-\frac{2}{\gamma}A'+\frac{M^2}{\beta^2}{:}e^{\beta\Phi}{:}+\text{const.} \tag{N7}$$

We have inserted an as yet undetermined parameter γ, which in the free theory is arbitrary and in the classical Liouville theory is β; for the quantum Liouville theory, it will be fixed presently. Also, we have allowed a c number constant in (N7), which is necessary for the closure of the algebra.

To evaluate the commutators (N1), we need

$$C_1(x,y)\equiv\frac{1}{i\hbar}\left[\tfrac{1}{2}{:}A^2(x){:}-\frac{2}{\gamma}A'(x)+\text{const},\ \tfrac{1}{2}{:}A^2(y){:}-\frac{2}{\gamma}A'(y)+\text{const}\right], \tag{N8a}$$

$$C_2(x,y)\equiv\frac{1}{i\hbar}\left[\tfrac{1}{2}{:}A^2(x){:}-\frac{2}{\gamma}A'(x)+\text{const},\ \frac{M^2}{\beta^2}{:}e^{\beta\Phi(y)}{:}\right]-(x\leftrightarrow y)\ . \tag{N8b}$$

Since the commutator of two operators is insensitive to additions of c numbers to the operators, we may use (N4) and (N5) to rewrite (N8):

$$C_1(x,y)=\frac{1}{i\hbar}\left[\tfrac{1}{2}A^2(x)-\frac{2}{\gamma}A'(x),\ \tfrac{1}{2}A^2(y)-\frac{2}{\gamma}A'(y)\right], \tag{N9a}$$

$$C_2(x,y)=\frac{1}{i\hbar}\left[\tfrac{1}{2}A^2(x)-\frac{2}{\gamma}A'(x),\frac{m^2}{\beta^2}e^{\beta\Phi(y)}\right]-(x\leftrightarrow y)\ . \tag{N9b}$$

Use of the canonical commutator

$$\frac{1}{i\hbar}[\Phi(x),\Pi(y)]=\delta(x-y) \tag{N10}$$

yields

$$C_1(x,y)=\left[A(x)A(y)+A(y)A(x)-\frac{4}{\gamma}A'(x)-\frac{4}{\gamma}A'(y)\right]\delta'(x-y)-\frac{8}{\gamma^2}\delta'''(x-y)\ , \tag{N11a}$$

$$C_2(x,y)=-\frac{m^2}{2\beta}[A(x)e^{\beta\Phi(y)}+e^{\beta\Phi(y)}A(x)-A(y)e^{\beta\Phi(x)}-e^{\beta\Phi(x)}A(y)]\delta(x-y)+\frac{2m^2}{\beta\gamma}(e^{\beta\Phi(x)}+e^{\beta\Phi(y)})\delta'(x-y)\ . \tag{N11b}$$

Next one wants to set x equal to y in operators multiplying δ functions; however, before doing so,the singularity at $x=y$ must be extracted. We have, according to (N4), (N5), and (N6),

$$C_1(x,y)=\left[-\frac{2\hbar a}{\pi(x-y)^2}+:A(x)A(y):+:A(y)A(x):-\frac{4}{\gamma}A'(x)-\frac{4}{\gamma}A'(y)\right]\delta'(x-y)-\frac{8}{\gamma^2}\delta'''(x-y)$$

$$=2\left[\tfrac{1}{2}:A^2(x):-\frac{2}{\gamma}A'(x)+\tfrac{1}{2}:A^2(y):-\frac{2}{\gamma}A'(y)\right]\delta'(x-y)$$

$$-\frac{2\hbar a}{\pi(x-y)^2}\delta'(x-y)-\frac{8}{\gamma^2}\delta'''(x-y)\,, \tag{N12a}$$

$$C_2(x,y)=-\frac{M^2}{2\beta}\Bigg[:A(x)e^{\beta\Phi(y)}:+:e^{\beta\Phi(y)}A(x):-:A(y)e^{\beta\Phi(x)}:-:e^{\beta\Phi(x)}A(y):$$

$$-\frac{\hbar\beta b}{\pi(x-y)}:e^{\beta\Phi(y)}:-\frac{\hbar\beta b}{\pi(x-y)}:e^{\beta\Phi(x)}:\Bigg]\delta(x-y)$$

$$+\frac{2M^2}{\beta\gamma}(:e^{\beta\Phi(x)}:+:e^{\beta\Phi(y)}:)\delta'(x-y)$$

$$=M^2(:e^{\beta\Phi(x)}:+:e^{\beta\Phi(y)}:)\left[\frac{2}{\beta\gamma}\delta'(x-y)+\frac{\hbar b}{2\pi(x-y)}\delta(x-y)\right]. \tag{N12b}$$

Finally, the following properties of one-dimensional δ functions

$$\frac{\delta(x)}{x}=-\delta'(x)\,, \tag{N13a}$$

$$\frac{\delta'(x)}{x^2}=\frac{\delta'''(x)}{6}+\alpha\delta'(x) \tag{N13b}$$

(α is an arbitrary constant) give the result

$$C_1(x,y)=2\left[\tfrac{1}{2}:A^2(x):-\frac{2}{\gamma}A'(x)-\frac{\hbar\alpha a}{2\pi}+\tfrac{1}{2}:A^2(y):-\frac{2}{\gamma}A'(y)-\frac{\hbar\alpha a}{2\pi}\right]\delta'(x-y)$$

$$-\left[\frac{8}{\gamma^2}+\frac{\hbar a}{3\pi}\right]\delta'''(x-y)\,, \tag{N14a}$$

$$C_2(x,y)=\frac{2M^2}{\beta^2}\left[\frac{\beta}{\gamma}-\frac{\hbar\beta^2 b}{4\pi}\right](:e^{\beta\Phi(x)}:+:e^{\beta\Phi(y)}:)\delta'(x-y)\,. \tag{N14b}$$

The commutator anomaly in C_1 is the familiar Schwinger term of the free theory.[7] The anomaly in C_2, characteristic of the Liouville theory, was identified by Curtright and Thorn.[29] For the conformal algebra, to close, the bracketed term of (N14b) must equal 1; this will be true if γ is given by

$$\frac{1}{\gamma}=\frac{1}{\beta}+\frac{\hbar\beta b}{4\pi}\,. \tag{N15}$$

Since $M^2:e^{\beta\Phi}:$ commutes with itself, (N1a) is seen to follow with

$$c=8\left[\frac{1}{\gamma^2}+\frac{\hbar a}{24\pi}\right] \tag{N16}$$

provided γ is evaluated by (N15), and the undetermined constant in (N7) is identified with $-\hbar\alpha a/2\pi$, which is the undetermined constant occurring in (N14a).

It is important to appreciate that we have not really answered the question whether the conformal algebra is truly realized in the quantum theory. To effect the above derivation, we had to assume a free field form for singularities in operator products.

This assumption is difficult to assess for the Liouville theory. On the one hand, it is known that perturbation theory for the two-dimensional exponential interaction with an additional mass term is rendered ultraviolet finite by normal ordering.[25] This leaves operator product singularities in their free field form. On the other hand, in the Liouville theory (without an additional mass term) perturbation theory is ill defined owing to infrared divergences, and it is not clear whether the perturbative (massive) results are relevant. While in conformally noninvariant theories it is plausible that ultraviolet behavior should be independent of infrared behavior, in the conformally invariant Liouville theory it may well be impossible to separate ultraviolet and infrared behavior. In the absence of explicit calculation, the argument remains incomplete.

Let us now present a normal-ordering analysis, following Curtright and Thorn.[29] The result is consistent with the above, and provides an evaluation of the constants a and b in (N4) and (N6). But the fundamental question about short-distance singularities is not addressed, since normal ordering presupposes their free field form. Moreover, a new question arises: is it possible to define in the Liouville-theory Hilbert space a Fock space with respect to which normal ordering is performed?

The canonical commutation relation (N10) is realized by the following expansions for Φ and Π.

$$\Phi=\Phi^{(+)}+\Phi^{(-)}\,,$$

$$\Phi^{(+)}(x)=\int dk\left[\frac{\hbar}{4\pi\omega(k)}\right]^{1/2}e^{ikx}a(k)\,, \tag{N17a}$$

$$\Phi^{(-)}(x)=\int dk\left[\frac{\hbar}{4\pi\omega(k)}\right]^{1/2}e^{-ikx}a^{\dagger}(k)=\left[\Phi^{(+)}(x)\right]^{\dagger}, \tag{N17b}$$

$$\Pi=\Pi^{(+)}+\Pi^{(-)}\,,$$

$$\Pi^{(+)}(x)=-i\int dk\left[\frac{\hbar\omega(k)}{4\pi}\right]^{1/2}e^{ikx}a(k)\,, \tag{N18a}$$

$$\Pi^{(-)}(x)=i\int dk\left[\frac{\hbar\omega(k)}{4\pi}\right]^{1/2}e^{-ikx}a^{\dagger}(k)=\left[\Pi^{(+)}(x)\right]^{\dagger}, \tag{N18b}$$

$$\begin{aligned}&[a(k),a(k')]=0=[a^{\dagger}(k),a^{\dagger}(k')],\\&[a(k),a^{\dagger}(k')]=\delta(k-k')\,.\end{aligned} \tag{N19}$$

Here $\omega(k)$ is an arbitrary positive, even function of k. While $\omega(k)$ would be $(k^2+\mu^2)^{1/2}$ if normal ordering is performed with reference to a free field with mass μ, we shall not need this formula. Nevertheless we shall assume conventional large-k behavior for $\omega(k)$:

$$\omega(k)\underset{k\to\infty}{\sim}|k|\,. \tag{N20}$$

Normal ordering is defined with respect to ω:

$$A(x)A(y)+A(y)A(x)=\hbar\int\frac{dk}{2\pi}\left[\frac{k^2}{\omega(k)}+\omega(k)\right]\cos k(x-y)+{:}A(x)A(y){:}\,|_{\omega}+{:}A(y)A(x){:}\,|_{\omega}\,, \tag{N21}$$

$$\begin{aligned}m^2e^{\beta\Phi(x)}&=m^2\exp\left[\frac{\beta}{2}[\Phi^{(+)}(x),\Phi^{(-)}(x)]\right]e^{\beta\Phi^{(-)}(x)}e^{\beta\Phi^{(+)}(x)}\\&=m^2\exp\left[\hbar\beta^2\int\frac{dk}{8\pi\omega(k)}\right]{:}e^{\beta\Phi(x)}{:}\,|_{\omega}=M_{\omega}^2{:}e^{\beta\Phi(x)}{:}\,|_{\omega}\,,\end{aligned} \tag{N22}$$

$$\begin{aligned}m^2(A(x)e^{\beta\Phi(y)}+e^{\beta\Phi(y)}A(x))=&\,M_{\omega}^2({:}A(x)e^{\beta\Phi(y)}{:}\,|_{\omega}+{:}e^{\beta\Phi(y)}A(x){:}\,|_{\omega})\\&-\hbar\beta M_{\omega}^2{:}e^{\beta\Phi(y)}{:}\,|_{\omega}\int\frac{dk}{2\pi}\frac{k}{\omega(k)}\sin k(x-y)\,.\end{aligned} \tag{N23}$$

Next we isolate the singularity at $x=y$ in the integrals occurring in (N21) and (N23), by using (N20):

$$\int \frac{dk}{2\pi}\left[\frac{k^2}{\omega(k)}+\omega(k)\right]\cos k(x-y)=\int \frac{dk}{2\pi}\left[\frac{k^2}{\omega(k)}+\omega(k)-2|k|\right]\cos k(x-y)-\frac{2}{\pi}\frac{1}{(x-y)^2}\,, \tag{N24}$$

$$\int \frac{dk}{2\pi}\frac{k}{\omega(k)}\sin k(x-y)=\int \frac{dk}{2\pi}\left[\frac{k}{\omega(k)}-\epsilon(k)\right]\sin k(x-y)+\frac{1}{\pi(x-y)}\,. \tag{N25}$$

A comparison with (N4) and (N6) gives the (ω-independent) definitions

$$:A(x)A(y):+:A(y)A(x):=\,:A(x)A(y):|_\omega+:A(y)A(x):|_\omega$$
$$+\hbar\int \frac{dk}{2\pi\omega(k)}[\omega(k)-|k|]^2\cos k(x-y)\,, \tag{N26}$$

$$M^2\left[:A(x)e^{\beta\Phi(y)}:+:e^{\beta\Phi(y)}A(x):\right]=M_\omega{}^2\left[:A(x)e^{\beta\Phi(y)}:|_\omega+:e^{\beta\Phi(y)}A(x):|_\omega\right]$$
$$-\hbar\beta M_\omega{}^2:e^{\beta\Phi(y)}:|_\omega\int\frac{dk}{2\pi}\left[\frac{k}{\omega(k)}-\epsilon(k)\right]\sin k(x-y)\,. \tag{N27}$$

The singularity in the operator products is indeed as in free field theory, and both constants a and b are evaluated at unity. Moreover, (N22) shows that for any ω, $m^2e^{\beta\Phi}=M^2{:}e^{\beta\Phi}{:}$ is a positive operator:

$$m^2e^{\beta\Phi}=M^2:e^{\beta\Phi}:=M_\omega{}^2\left[e^{\beta\Phi^{(+)}}\right]^\dagger\left[e^{\beta\Phi^{(+)}}\right]. \tag{N28}$$

The commutators with fields may also be given:

$$\frac{i}{\hbar}[:\Theta_{++}(x):,\Phi(y)]=[\Pi(x)\pm\Phi'(x)]\delta(x-y)-\frac{2}{\gamma}\delta'(x-y)\,, \tag{N29a}$$

$$\frac{i}{\hbar}[:\Theta_{\pm\pm}(x):,\Pi(y)]=\mp[\Pi(x)\pm\Phi'(x)]\delta(x-y)-\frac{M^2}{\beta}:e^{\beta\Phi(x)}:\delta(x-y)+\frac{2}{\gamma}\delta''(x-y)\,. \tag{N29b}$$

This shows that the Hamiltonian and momentum operators are

$$H=\tfrac{1}{2}\int dx[:\Theta_{++}(x):+:\Theta_{--}(x):]\,, \tag{N30a}$$

$$P=-\tfrac{1}{2}\int dx[:\Theta_{++}(x):-:\Theta_{--}(x):] \tag{N30b}$$

while the Heisenberg equation of motion reads

$$\frac{i}{\hbar}[P^\mu,\frac{i}{\hbar}[P_\mu,\Phi]]=\Box\Phi$$
$$=-\frac{M^2}{\beta}:e^{\beta\Phi}:\,. \tag{N31}$$

Note that the conformal algebra is consistent with results established in the main body of this paper. According to (N31) no normalizable energy-momentum eigenstate exists, since $(M^2/\beta){:}e^{\beta\Phi}{:}$ is a positive operator. Since $\int dx\, e^{ipx}\Theta_{\pm\pm}(x)$ act as raising and lowering operators for H and P, the energy and momentum spectra are continuous.

The normal ordering procedure that we have adopted for evaluating equal-time commutators produces c-number quantum anomalies which are twice those given by Curtright and Thorn[29] (after differences between our notation and theirs is taken into account). This discrepancey comes about from different ways of regulating products of distributions, as in (N13). Indeed, in the free field theory, if one uses the Bjorken-Johnson-Low (BJL) definition of the equal-time commutator,[31] the magnitude of the quantum correction to the center in C_1 is $\frac{1}{2}$ of what we give here.[7] Similarly the $O(\hbar)$ correction to C_2 can be evaluated in the free field theory; again $\frac{1}{2}$ of the present number is found.

Of course, in the absence of a dynamical calculation, the BJL procedure cannot be used for the Liouville theory; fortunately, the closure of the conformal algebra does not depend on numerical values of the quantum corrections, provided they are c numbers.[32] Nevertheless, this highlights once again the convention dependence of regulator methods and the uncertainty about the applicability of canonical free field techniques to an interacting theory.

Finally let us observe that the conformal symmetry should allow mapping the theory from Minkowski space to other spaces, where alternate quantization schemes can be carried out. Especially interesting is $O(2)\times O(2)$ which is the maximal compact subgroup of O(2,2), the restricted conformal group of the model. [Similar techniques have been employed for the point magnetic monopole, which is O(2,1) invariant, and for the Yang-Mills theory, which is O(4,2) invariant on the classical level.[33]

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While this paper was readied for publication we received a report from J.-L. Gervais and A. Neveu [Nucl. Phys. B199, 59 (1982)] which addresses similar questions concerning the Liouville model on a finite spatial interval (in contradistinction to our infinite interval). While no disagreements are evident, these authors present results which we do not find to hold in the infinite-space theory: they use a second-order Lax pair, which would be unacceptable for us; see Ref. 17. This research is supported through funds provided by the U.S. Department of Energy (DOE) under Contract No. DE-AC02-76ER03069.

APPENDIX A

Here we show that the zeros of the scattering data $a(\lambda)$ in the upper-half k plane are simple. We consider

$$W\left[\frac{\partial\psi_+}{\partial\lambda},\psi_-\right]=\frac{\partial u_+}{\partial\lambda}v_- - \frac{\partial v_+}{\partial\lambda}u_- \,. \tag{A1}$$

We have omitted the λ dependence of the Jost functions. Using the equation $L\psi=\lambda\psi$, we find

$$\partial_- W\left[\frac{\partial\psi_+}{\partial\lambda},\psi_-\right]=-(u_+u_-+v_+v_-)\,, \tag{A2a}$$

$$\partial_- W\left[\psi_+,\frac{\partial\psi_-}{\partial\lambda}\right]=u_+u_-+v_+v_- \tag{A2b}$$

so that

$$W\left[\psi_+,\frac{\partial\psi_-}{\partial\lambda}\right]\Bigg|_{y^-}^{L}=-\int_{y^-}^{L}dx^-(u_+u_-+v_+v_-)\,, \tag{A3a}$$

$$W\left[\frac{\partial\psi_+}{\partial\lambda},\psi_-\right]\Bigg|_{-L}^{y^-}=\int_{-L}^{y^-}dx^-(u_+u_-+v_+v_-)\,. \tag{A3b}$$

The Wronskians vanish as $L\to\infty$, so it follows that

$$\frac{d}{d\lambda}W(\psi_+,\psi_-)=-\int_{-\infty}^{\infty}dx^-(u_+u_-+v_+v_-) \tag{A4}$$

Now

$$\frac{d}{d\lambda}a(\lambda)=\frac{1}{W(\bar\psi_+,\psi_-)}\int_{-\infty}^{\infty}dx^-(u_+u_-+v_+v_-) + \frac{W(\psi_+,\psi_-)}{W(\bar\psi_+,\psi_+)^2}\frac{d}{d\lambda}W(\bar\psi_+,\psi_+)\,, \tag{A5}$$

but the second term is proportional to $a(\lambda)$ and must vanish when $a(\lambda)=0$. Furthermore $\psi_-(x^-,\lambda)=b(\lambda)\psi_+(x^-,\lambda)$ and

$$\frac{d}{d\lambda}a(\lambda)=\frac{b(\lambda)}{W(\bar\psi_+,\psi_+)}\int_{-\infty}^{\infty}dx^-(u_+{}^2+v_+{}^2)\,. \tag{A6}$$

Using the explicit expression of the Wronskian

$$W(\bar\psi_+,\psi_+)=2ik/\lambda \tag{A7}$$

we finally obtain

$$a'(\lambda)=\frac{d}{dk}a(\lambda) = -\frac{i}{2}b(\lambda)\int_{-\infty}^{\infty}dx^-(u_+{}^2+v_+{}^2)\,. \tag{A8}$$

When $|\lambda|<\sqrt{V_\infty}$ and λ is real, k must be purely imaginary and the Jost functions ψ are real. Furthermore $b(\lambda)\neq 0$, since by virtue of $a(\lambda)=0$, we would find $\psi_-(\lambda)=0$, if also $b(\lambda)=0$. This is contradictory. Hence $a(\lambda)=0$ implies $a'(\lambda)\neq 0$. So all the zeros on the real axis are simple.

APPENDIX B: ONE-DIMENSIONAL LIOUVILLE THEORY

The quantum-mechanical Liouville theory is defined by the Hamiltonian

$$H=\tfrac{1}{2}p^2+\frac{m^2}{\beta^2}e^{\beta q}\,. \tag{B1}$$

The corresponding Schrödinger equation yields a continuous spectrum with (continuum energy-normalized) wave functions

$$\psi_E(q)=\left[\frac{4}{\pi\beta}\sinh\pi\mid\nu\mid\right]^{1/2}K_\nu\left[\frac{2\sqrt{2}m}{\hbar\beta^2}e^{\beta q/2}\right],$$

$$\nu=i\frac{2\sqrt{2E}}{\hbar\beta}, \quad \text{(B2)}$$

where K_ν is a modified Bessel function.[34] We see that no zero-energy state exists, which reflects the fact that the equation

$$\left[-\frac{\hbar^2}{2}\frac{d^2}{dq^2}+\frac{m^2}{\beta^2}e^{\beta q}\right]\psi_0(q)=0 \quad \text{(B3)}$$

has the solutions $I_0((2\sqrt{2}m/\hbar\beta^2)e^{\beta q/2})$ and $K_0((2\sqrt{2}m/\hbar\beta^2)e^{\beta q/2})$ both of which are unbounded for large $|q|$. Furthermore, it is clear from (B2) that the spectrum is continuous and that $E=0$ is a greatest lower bound.[35] Note that the wave function has no expansion in powers of m^2, since K_ν is not analytic at the origin.

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[9] It is interesting to note that in the quantized theory there is a further, positive contribution to the center of the Poisson bracket algebra owing to anomalous commutators between components of the energy-momentum tensor; the quantal term is of the same form as Δ. Thus, even the conventional charges Q_f, when quantized, do not reproduce the classical algebra (2.4) (Ref. 7).

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[11] The improved tensor for this problem, Eq. (2.28), as well as the Poincaré and dilatation charges are constructed in Ref. 4.

[12] The expression (2.31) is formally the same as Q_f in (2.25), except that the latter involves the wave fields which depend only on $x^\pm$, while the former contains the interacting field Φ which depends on both x^+ and x^-. However, it is easy to verify from the field equation (1.2) that $\partial_\pm\Phi\partial_\pm\Phi-2\partial_\pm{}^2\Phi$ is independent of $x^\mp$.

[13] Using light-cone Poisson brackets in the light-cone coordinate formalism, we have reproduced all the usual (Ref. 14) results for the sine-Gordon theory—like the masses of breathers and solitons.

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[17] Let us observe that the second-order eigenvalue equations (3.7b), together with the trivial evolution equation (3.9), cannot be used as a Lax pair. The compatibility condition $\partial_+V=0$ which follows, does not imply only the Liouville equations, but is also solved by any wave field $\Phi=\phi^+(x^+)+\phi^-(x^-)$. Such a field does not satisfy Liouville's equation (except when $\Phi=-\infty$).

[18] By definition, a breather solution has an internal structure and is characterized by parameters of internal motion.

[19] The sign is established by setting $\lambda=0$.

[20] It is easy to check that the Dirac bracket is just the Poisson bracket, so that elimination of Φ may be performed either before or after the Poisson bracket is taken.

[21] Since the nonvanishing components of a traceless energy-momentum tensor in two-dimensional space-time depend only on one variable (x^+ or x^-), an infinite set of constants of motion is gotten by integrating powers of the appropriate component over the single variable on which it depends.

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[27]The calculation is done by the method developed in R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[28]An explicit functional $\Psi[\Phi]$ which has the property that

$$\int \mathscr{D}\Phi\Psi^*[\Phi]\Phi\Psi[\Phi] \Big/ \int \mathscr{D}\Phi\Psi^*[\Phi]\Psi[\Phi]=\varphi$$

and

$$\int \mathscr{D}\Phi\Psi^*[\Phi]H\Psi[\Phi] \Big/ \int \mathscr{D}\Phi\Psi^*[\Phi]\Psi[\Phi] = V(\beta,\varphi)\int_{-\infty}^{\infty} dx$$

can be given. It is

$$\Psi[\Phi]=\exp\left[-\frac{1}{2\hbar}\int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dy[\Phi(x)-\varphi]\omega_\varphi(x-y) \times[\Phi(y)-\varphi]\right],$$

where

$$\omega_\varphi(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}[k^2+\mu^2(\varphi)]^{1/2},$$

$$\mu^2(\varphi)=m_r{}^2e^{\tilde{\beta}\varphi},$$

and the Schrödinger picture formula for the Liouville Hamiltonian is

$$H=\int_{-\infty}^{\infty} dx\left[-\tfrac{1}{2}\hbar^2\frac{\delta^2}{\delta\Phi\delta\Phi}+\tfrac{1}{2}(\Phi')^2+\frac{m^2}{\beta^2}e^{\beta\Phi}\right].$$

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[32]The Curtright-Thorn values for the anomalies have the appealing feature that γ equals $\tilde{\beta}$, i.e., the quantal renormalization of β in the improved energy-momentum tensor coincides with the renormalization of β in the effective potential.

[33]R. Jackiw, C. Nohl, and C. Rebbi, in *Particles and Fields,* edited by D. Boal and A. Kamal (Plenum, New York, 1978); R. Jackiw, Ann. Phys. (N.Y.) 129, 183 (1980).

[34]H. Bethe and R. Jackiw, *Intermediate Quantum Mechanics* (Benjamin/Cummings, Reading, Mass. 1968), p. 16.

[35]Another quantum-mechanical system whose spectrum has a greatest lower bound, but where there is no ground state, is the one-dimensional particle subject to an infinite potential barrier on half the plane. In this simple problem, the complete effective potential can be calculated. It is proportional to q^{-2}, and has no minimum at finite q.