Constraints in relativistic Hamiltonian mechanics

Joseph Samuel*
Department of Physics, Indian Institute of Science, Bangalore, 560 012 India
(Received 1 April 1982)

We propose a new scheme for the use of constraints in setting up classical, Hamiltonian, relativistic, interacting particle theories. We show that it possesses both Poincaré invariance and invariance of world lines. We discuss the transition to the physical phase space and the nonrelativistic limit.

INTRODUCTION

We present a new way of using constraints to set up a theory of relativistic, interacting particles. The method we propose is in some respects simpler than the ones currently in use in the literature while it achieves the same objectives. What one expects of a nontrivial relativistic theory are (1) a realization of the Poincaré algebra by functions on the physical phase space of the system, (2) invariant world lines, and (3) particle interaction. While these objectives are incompatible within the instant form of dynamics,[1] they can be achieved with a more general framework. This new framework uses the theory of constraints developed by Dirac[2] and is explained in Refs. 3 and 4.

In Sec. I we briefly review the material in Refs. 3 and 4 prior to setting forward our own ideas in Sec. II. Then we discuss an example of an N-particle system. Section III discusses the transition to the true physical phase space of the system. In Sec. IV we take the nonrelativistic limit of the N-particle system. Section V presents some formal aspects of the scheme. Section VI is a concluding discussion.

I. REVIEW OF PREVIOUS WORK

Komar[3] starts with an $8N$-dimensional phase space Γ spanned by

$$(x_{a\mu},p_{a\mu}),\ \ a=1,\ldots,N,\ \ \mu=0,\ldots,3$$

with basic Poisson brackets

$$\{x_{a\mu},p_{b\nu}\}=\delta_{ab}\eta_{\mu\nu}\,,\qquad \{x_{a\mu},x_{b\nu}\}=\{p_{a\mu},p_{b\nu}\}=0\,. \tag{1.1}$$

[We use the metric $\eta_{\mu\nu}$=diagonal $(1,-1,-1,-1)$.] The Poincaré algebra is realized on this space via the Poisson bracket by the ten generators

$$P_\mu=\sum_a p_{a\mu}\,,\qquad M_{\mu\nu}=\sum_a (x_{a\mu}\,p_{a\nu}-x_{a\nu}\,p_{a\mu})\,. \tag{1.2}$$

In order to define the system, Komar imposes N constraints

$$K_a\approx 0\,,$$

which are first class,

$$\{K_a,K_b\}\approx 0\,, \tag{1.3}$$

and Poincaré invariant,

$$\{G,K_a\}\approx 0\,, \tag{1.4}$$

where G stands for any of the 10 generators (1.2). These K_a by their vanishing define a $7N$-dimensional region in Γ. Further, as a consequence of Eqs. (1.3), they foliate this space into a $6N$-parameter family of N-dimensional sheets in a manner explained in Refs. 3 and 4. These sheets, while they are a $6N$-parameter family, are not yet in correspondence with the states of motion of an N-particle system.[5] In order to define states of motion and determine the particle world lines, one has to choose a "connection between syntactic and semantic observables." Once this is done, the phase-space trajectory and particle world lines are well defined.

Sudarshan, Mukunda, and Goldberg[4] point out that it is inappropriate to identify Komar's sheets with the phase-space trajectory of the system because a sheet does not determine a set of N world lines. In order to arrive at a phase-space trajectory, they impose $N-1$ "time" constraints χ_α, $\alpha=1,\ldots,N-1$, which serve to pick a one-dimensional line on each sheet. After introducing one more "labeling" constraint χ_N, which explicitly depends on τ (the evolution parameter), they have a set, K_a and χ_a, of $2N$ second-class constraints. One can define a Dirac bracket with respect to this set. It then transpires, as a result of Eqs. (1.3) and (1.4), that the 10 functions (1.2) satisfy the Poincaré algebra with respect to the Dirac bracket for any choice of the χ's. These authors also discuss a world-line condition (originally developed in Ref. 6) for use in the present context. Their discussion shows that one way to satisfy the world-line condition is to have the $N-1$ τ-independent time constraints χ_a Poincaré invariant.

II. THE PRESENT FORMALISM

We will show that one can fulfill the requirements of Poincaré invariance and the world-line condition by a different set of conditions on the constraints. We introduce $2N-1$ constraints φ_i, $i=1,\ldots,2N-1$, all on an equal footing. These replace the K_a's and the χ_a's of the previous section. We require them to be Poincaré invariant,

$$\{G,\varphi_i\}\approx 0 \ . \tag{2.1}$$

As before we need yet another labeling constraint $\varphi_{2N}(\tau)$ explicitly depending on τ. This last need not be Poincaré invariant. The set φ_i, $i=1,\ldots,2N$, is assumed to form a second-class set; i.e., the $2N\times 2N$ matrix

$$\{\varphi_i,\varphi_j\}$$

is invertible. Let us denote its inverse by C_{ij}.

The equations of motion are generated by a Hamiltonian $H=v_i\varphi_i$ via the Poisson brackets,

$$\begin{aligned}\frac{dx_{a\mu}}{d\tau}&\approx\{x_{a\mu},H\}\approx\{x_{a\mu},\varphi_i\}v_i \ ,\\ \frac{dp_{a\mu}}{d\tau}&\approx\{p_{a\mu},H\}\approx\{p_{a\mu},\varphi_i\}v_i \ .\end{aligned}\tag{2.2}$$

The requirement that H preserves the constraints in τ uniquely fixes v_i,

$$\begin{aligned}\frac{d\varphi_i}{d\tau}&\approx\frac{\partial\varphi_i}{\partial\tau}+\{\varphi_i,\varphi_j\}v_j\approx 0 \ ,\\ v_i&\approx -C_{ij}\frac{\partial\varphi_j}{\partial\tau}\approx -C_{i2N}\frac{\partial\varphi_{2N}}{\partial\tau} \ .\end{aligned}\tag{2.3}$$

Once the physical system is fixed by choice of the constraints, its phase-space trajectory and particle world lines are determined from Eqs. (2.2) and (2.3).

We need to check now that the 10 generators (1.2) satisfy the Poincaré algebra with respect to the Dirac bracket defined by the $2N$ second-class constraints:

$$\{f,g\}^*\approx\{f,g\}-\{f,\varphi_i\}C_{ij}\{\varphi_j,g\} \ . \tag{2.4}$$

One easily sees that if G and G' are any two of the Poincaré generators,

$$\{G,\varphi_i\}C_{ij}\{\varphi_j,G'\}\approx 0 \ . \tag{2.5}$$

Since, by Eq. (2.1), $\{G,\varphi_i\}$ and $\{\varphi_j,G'\}$ are nonzero only if $i=j=2N$, and then C_{ij} vanishes by its antisymmetry. Thus we have

$$\{G,G'\}^*\approx\{G,G'\} \ . \tag{2.6}$$

The generators (1.2) do satisfy the Poincaré algebra with respect to the Poisson bracket. From Eq. (2.6) we see that they also do so with respect to the Dirac bracket. Thus, the requirement of Poincaré invariance is fulfilled.

We come to the requirement of world-line invariance. Notice that the Poisson bracket of $x_{a\mu}$ with the generators (1.2) coincides with its geometrical change under Poincaré transformations. The world-line condition (WLC) (Refs. 1, 4, and 6) demands that the canonical transformation property of $x_{a\mu}$ via the *Dirac* bracket should coincide with its geometrical one apart from a translation of each particle (by possibly different amounts) along its world line. It reads

$$\{x_{a\mu},G\}^*\approx\{x_{a\mu},G\}+\{x_{a\mu},H\}\delta_a\tau \ , \quad a=1,\ldots,N, \ (\text{no} \textstyle\sum \text{ on } a) \ . \tag{2.7}$$

Here we have replaced the geometric transformation of $x_{a\mu}$ by $\{x_{a\mu},G\}$ in accordance with the above remark. If there exist numbers $\delta_a\tau$ for each of the 10 generators (1.2) such that Eq. (2.7) is satisfied, then the theory has objective world lines. In the present formalism we see that the WLC is in fact satisfied,

$$\begin{aligned}\{x_{a\mu},G\}^*&\approx\{x_{a\mu},G\}-\{x_{a\mu},\varphi_i\}C_{ij}\{\varphi_j,G\}\\ &\approx\{x_{a\mu},G\}-\{x_{a\mu},\phi_i\}C_{i2N}\{\varphi_{2N},G\}\\ &\approx\{x_{a\mu},G\}\\ &\quad-\{x_{a\mu},H\}\frac{\{\varphi_{2N},G\}}{-\partial\varphi_{2N}/\partial\tau} \ .\end{aligned}\tag{2.8}$$

From here one sees that the uniform choice

$$\delta_a\tau=\frac{\{\varphi_{2N},G\}}{\partial\varphi_{2N}/\partial\tau}, \ \text{ all } a \tag{2.9}$$

satisfies Eq. (2.7). The WLC, as developed in Refs. 1, 4, and 6, permits a translation of each particle independently along its world line. Thus, points on two world lines which are "simultaneous" with respect to τ in one frame need not be so in another. In the present formalism, since $\delta_a \tau$ does not depend on a, the notion of simultaneity is not changed by a change of frame. Thus we have here an invariant notion of simultaneity as well as invariant world lines.

Notice that we do not assume that the K_a's are first class or even that their brackets vanish modulo all the constraints. While we do give up these Komar-Todorov equations (1.3), we need to assume that Eqs. (2.1) are fulfilled. Thus we have a different set of conditions on the constraints from those used in the Komar-Todorov formalism.

We next discuss an example of an N-particle system using this framework. This system can be thought of as having "instantaneous interaction" in its rest frame. The constraints that define our system are

$$K_a = p_a{}^2 - m_a{}^2 - V_a \ , \qquad \chi_a = \hat{P}\cdot x_a - \tau \ , \tag{2.10}$$

where $\hat{P}_\mu = P_\mu / P$, and $P = (P_\mu P^\mu)^{1/2}$. V_a are N Poincaré-invariant functions constructed from the x variables alone. By taking the differences of the χ_a's one sees that all but one of them are Poincaré invariant and τ independent. So, the constraints (2.10) describe a system which does satisfy Poincaré invariance and world-line invariance.

We remark that the K_a's are not required to form a first-class set and so the system falls outside the scope of the Komar-Todorov formalism. In fact,

$$\{K_a, K_b\} = F_{ab} = 2p_{a\mu}\frac{\partial V_b}{\partial x_{a\mu}} - 2p_{b\mu}\frac{\partial V_a}{\partial x_{b\mu}} \neq 0 \ . \tag{2.11}$$

One works out the remaining Poisson brackets as

$$\{\chi_a, \chi_b\} = 0, \qquad \{K_a, \chi_b\} = -2p_a\cdot\hat{P}\delta_{ab} \ , \tag{2.12}$$

so that

$$[\{\varphi_i, \varphi_j\}] = \begin{bmatrix} F_{ab} & -2p_a\cdot\hat{P}\delta_{ab} \\ 2p_a\cdot\hat{P}\delta_{ab} & 0 \end{bmatrix}$$

and

$$C_{ij} = \begin{bmatrix} 0 & \dfrac{\delta_{ab}}{2p_a\cdot\hat{P}} \\ \dfrac{-\delta_{ab}}{2p_a\cdot\hat{P}} & \dfrac{F_{ab}}{4p_a\cdot\hat{P}\; p_b\cdot\hat{P}} \end{bmatrix} \ . \tag{2.13}$$

The Hamiltonian of the system is given by

$$H = -\varphi_i C_{ij}\frac{\partial \varphi_j}{\partial \tau} = \sum_a (u_a K_a + \omega_a \chi_a) \ ,$$

where

$$u_a = \frac{1}{2p_a\cdot\hat{P}} \text{ and } \omega_a = \sum_b \frac{F_{ab}}{4p_a\cdot\hat{P}\; p_b\cdot\hat{P}} \ .$$

The equations of motion are given by Eqs. (2.2).

III. TRANSITION TO THE PHYSICAL PHASE SPACE

For convenience in setting up the theory we have started with an $8N$-dimensional phase space and imposed constraints to reduce the degrees of freedom. Once the theory is set up we wish to find the physical phase space of the system with $6N$ dimensions. On this space we expect the equations of motion to assume Hamiltonian form with respect to the Dirac bracket. In order to do this we need to find $6N$ independent functions u_α, $\alpha = 1, \ldots, 6N$, on the original phase space with the property that their Dirac brackets are τ independent,

$$\frac{\partial'}{\partial \tau}(\{u_\alpha, u_\beta\}^*) \approx 0 \ . \tag{3.1}$$

(The τ derivative here denotes that the u_α are held constant during the differentiation.) Then, as proved in Ref. 7, there exists a function $\mathscr{H}$ of these variables so that the τ derivatives of these variables as computed from Eqs. (2.2) are given in Hamiltonian form with respect to the Dirac bracket,

$$\frac{du_\alpha}{d\tau} = \{u_\alpha, \mathscr{H}\}^* \ . \tag{3.2}$$

Thus the u_α's span the true physical phase space. $\{u_\alpha, u_\beta\}^*$ gives its symplectic structure and $\mathscr{H}$ is the Hamiltonian function. While the functions u_α and $\mathscr{H}$ do exist, finding them can in practice be quite cumbersome. For the system discussed above, we will show that the physical phase space is spanned by the $6N$ variables

$$y_{a\mu} = O_{\mu\nu}x_a^\nu, \quad g_{a\mu} = O_{\mu\nu}p_a^\nu, \quad \hat{P}_\mu \ , \tag{3.3}$$

where $O_{\mu\nu} = \eta_{\mu\nu} - \hat{P}_\mu \hat{P}_\nu$. To see this, notice that

$$\{f, -P\} \approx 0 \ , \tag{3.4}$$

where f is any of the $6N$ variables (3.3). Also

$$\{K_a,-P\}=0=\frac{\partial K_a}{\partial\tau}\,,$$
$$\{\chi_a,-P\}=-1=\frac{\partial\chi_a}{\partial\tau}\,. \qquad (3.5)$$

Now,

$$\frac{df}{d\tau}\approx\{f,H\}$$
$$\approx-\{f,\varphi_i\}C_{ij}\frac{\partial\varphi_j}{\partial\tau}$$
$$\approx\{f,-P\}-\{f,\varphi_i\}C_{ij}\{\varphi_j,-P\}$$
$$\approx\{f,-P\}^*\,. \qquad (3.6)$$

Thus $-P$ generates the τ evolution of the $6N$ variables (3.3) via the Dirac bracket. The variables (3.3) are the u_α's and $\mathscr{H}=-P$. $\mathscr{H}$ can be expressed in terms of the u_α's by use of the constraints (2.10). One can easily reverse the arguments of Ref. 7 and show from Eq. (3.6) that the variables (3.3) have τ-independent Dirac brackets.

IV. NONRELATIVISTIC LIMIT OF THE N-PARTICLE SYSTEM

We want to discuss here how one takes the nonrelativistic limit of the system set out in Sec. II. That it should reduce in this limit to the familiar Hamiltonian mechanics is a useful check on the relativistic theory. Our method is to take the limit $c\to\infty$ in the $8N$-dimensional phase space. We discuss how the Poincaré group action and the constraints behave in this limit. Then we show that the resulting formalism coincides with nonrelativistic mechanics. One by-product of this procedure is that one sees the familiar nonrelativistic mechanics cast in the language of the constraint formalism. This leads to a better understanding of the constraint formalism of relativistic mechanics.

We start with the $8N$-dimensional phase space Γ with coordinates

$$x_{a\mu}=(ct_a,x_{ai}),\;\; p_{a\mu}=\left[\frac{E_a}{c},p_{ai}\right]. \qquad (4.1)$$

(c is restored for the duration of this section.) In the limit $c\to\infty$, we must ensure that

$$t_a,\; x_{ai},\; p_{ai},\text{ and } e_a=E_a-m_ac^2 \qquad (4.2)$$

are kept finite. We use these as coordinates on Γ since they remain finite in the limit. Their nonvanishing brackets are

$$\{x_{ai},p_{bj}\}=-\delta_{ab}\delta_{ij}\,,$$
$$\{t_a,e_b\}=\delta_{ab}\,. \qquad (4.3)$$

The generators P_0, P_i, $K_i=M_{0i}$, and $J_i=\frac{1}{2}\epsilon_{ijk}M_{jk}$ satisfy the Poincaré algebra on Γ via the Poisson bracket. In the nonrelativistic limit this algebra reduces to that of the Galilei group in a manner explained in Ref. 8. We mention that some of the generators need to be redefined in order that they remain finite in the limit, namely,

$$K_i'=\frac{K_i}{c}\,,$$
$$h=c(P_0-Mc)\,,$$

where $M=\sum_a m_a$. The generators can now be expressed in terms of the $8N$ variables (4.2), and after the limit $c\to\infty$ is taken, they assume the form

$$P_i=\sum_a p_{ai}\,,$$
$$h=\sum_a e_a\,,$$
$$K_i'=\sum_a(t_ap_{ai}-m_ax_{ai})\,, \qquad (4.4)$$
$$J_i=\sum_a\epsilon_{ijk}x_{aj}p_{ak}\,.$$

These provide a realization [via the Poisson brackets (4.3)] of the Galilei group with a neutral element M.

We now discuss the constraints. We start with the χ_a's. After rewriting τ as $c\tau$, these are

$$\chi_a=\hat{P}\cdot x_a-c\tau\,.$$

In the limit of $c\to\infty$, we have to leading order in c^2, after division by c (we use the same symbol χ_a to denote the new constraint so obtained),

$$\chi_a=t_a-\tau\,. \qquad (4.5)$$

So, modulo the χ_a's we have $t_a=\tau$ and we recover the "instant form" of dynamics. Coming now to the K_a's, we have (with some convenient redefinitions)

$$K_a=p_a{}^2-m_a{}^2c^2-2m_aV_a\,. \qquad (4.6)$$

Expressed in terms of the variables (4.2),

$$K_a=m_a{}^2c^2-2m_ae_a-p_{ai}p_{ai}-m_a{}^2c^2$$
$$-2m_aV_a+\frac{e_a}{c^2}\,, \qquad (4.7)$$

so that its nonrelativistic limit is (after division by

$2m_a$)

$$K_a = e_a - \frac{p_{ai}p_{ai}}{2m_a} - V_a \ . \tag{4.8}$$

Now V_a must be constructed entirely in terms of position differences in view of Poincaré invariance of the relativistic theory. It is easily seen that modulo the χ_a equations[9] the time differences vanish the V_a is expressed in terms of spatial position differences only.

Equations (4.3)–(4.5) and (4.8) describe a nonrelativistic constraint formalism. We work out the relevant Poisson brackets:

$$\begin{aligned} \{\chi_a,\chi_b\} &= 0 \ , \\ \{K_a,\chi_b\} &= -\delta_{ab} \ , \\ \{K_a,K_b\} &= \frac{p_{ai}}{m_a}\frac{\partial V_b}{\partial x_a{}^i} - \frac{p_{bi}}{m_b}\frac{\partial V_a}{\partial x_b{}^i} = F_{ab} \ . \end{aligned} \tag{4.9}$$

So,

$$(\{\varphi_i,\varphi_j\}) = \begin{bmatrix} F_{ab} & -\delta_{ab} \\ \delta_{ab} & 0 \end{bmatrix}$$

and

$$C_{ij} = \begin{bmatrix} 0 & \delta_{ab} \\ -\delta_{ab} & F_{ab} \end{bmatrix} , \tag{4.10}$$

$$H = -\varphi_i C_{ij}\frac{\partial \varphi_j}{\partial \tau} = \sum_a (u_a K_a + \omega_a \chi_a) \ , \tag{4.11}$$

where $u_a = 1$ and $\omega_a = \sum_b F_{ab}$. One easily checks that Eqs. (2.2) give the nonrelativistic equations of motion for x_{ai} and p_{ai} that describe a collection of particles interacting via a potential

$$V = \sum_a V_a \ ,$$

depending only on Galilei-invariant functions of the spatial position variables.

Notice also that the physical phase space in this case is the nonrelativistic phase space we are used to dealing with. This space is spanned by the $6N$ variables

$$x_{ai},\ p_{ai} \ . \tag{4.12}$$

The function h in Eqs. (4.4) satisfies the relations

$$\{f,-h\} \approx 0$$

[where f is any of the $6N$ variables (4.12)] and

$$\{K_a,-h\} = 0 = \frac{\partial K_a}{\partial \tau} \ ,$$

$$\{\chi_a,-h\} = -1 = \frac{\partial \chi_a}{\partial \tau} \ .$$

So,

$$\begin{aligned} \frac{df}{d\tau} = \{f,H\} &\approx -\{f,\varphi_i\}C_{ij}\frac{\partial\varphi_j}{\partial\tau} \\ &\approx \{f,-h\} - \{f,\varphi_i\}C_{ij}\{\varphi_j,-h\} \\ &\approx \{f,-h\}^* \ . \end{aligned} \tag{4.13}$$

So τ evolution on the physical phase space is generated via the Dirac bracket by $-h$ and the $\mathscr{H}$ of the previous section is $-h$. Expressing h in terms of the variables (4.12) by use of the constraints (4.8) gives

$$h = \sum_a \frac{p_{ai}p_{ai}}{2m_a} + \sum_a V_a \ . \tag{4.14}$$

From the fact that $\{f,\chi_a\} = 0$ and the form of the C_{ij} matrix (4.10), one sees that

$$\{x_{ai},p_{bj}\}^* = \{x_{ai},p_{bj}\} = -\delta_{ab}\delta_{ij} \ . \tag{4.15}$$

Equations (4.13)–(4.15) completely reproduce the nonrelativistic theory for a position-dependent potential V.

V. FORMAL ASPECTS

In this section we discuss some formal aspects of the scheme presented. The $2N-1$ constraints φ_i, $i = 1, \ldots, 2N-1$ define a $(6N+1)$-dimensional region of Γ. We call this hypersurface M. We need to assume as a condition on the φ_i's that the $2N-1 \times 2N-1$ matrix

$$\{\varphi_i,\varphi_j\} \tag{5.1}$$

has maximal rank all over M. This means that this matrix has only one independent null eigenvector,[10]

$$\{\varphi_i,\varphi_j\}v_j \approx 0 \ . \tag{5.2}$$

$H = v_j\varphi_j$ is the unique first-class combination of the φ_i's. Its flow is tangential to M.[11] One can define equivalence classes on M by declaring two points to be equivalent if they can be joined by a curve whose tangent is along the flow of H. Thus M splits up into a $6N$-parameter family of one-dimensional curves or trajectories. These trajectories can be directly identified with the states of motion of the system since they are one-dimensional objects and form a $6N$-parameter family. The foliation of M into trajectories is similar to the foliation of the $7N$-dimensional surface (in the Komar approach) into sheets of dimension N. The difference is that here the sheets are one-dimensional objects and we need no analog of Eqs. (1.3) to ensure integrability. The space of equivalence classes is a $6N$-dimensional space and represents in this context the frozen phase space of Bergmann and Komar. This is also known as the space of initial conditions or the space of constants of motion.

We mention that the equations of motion (2.2) can be compactly stated in the form of an action principle. If p_1 and p_2 are two points of M, a solution to Eqs. (2.2) that passes through them is an extremum of the action

$$I=\int_{p_1}^{p_2}\sum_a p_{a\mu}dx_a{}^{\mu} \tag{5.3}$$

with respect to variations that lie in M. This is reminiscent of Maupertuis's principle in mechanics.[12] Since M, by Eqs. (2.1) is Poincaré invariant and the action (5.3) is also Poincaré invariant, the Poincaré invariance of the resulting equations of motion is manifest.

As stated above, the formalism makes no reference to any evolution parameter. The phase-space trajectory and the particle world lines are well defined with only the $2N-1$ φ_i's given. If we wish to introduce an evolution parameter, we can do so in a number of ways and this constitutes the reparametrization invariance of the theory. Once a definite choice of evolution parameter is made via a definite choice of $\varphi_{2N}(\tau)$, H, previously undetermined up to a multiplicative factor, becomes completely determined. This reparametrization invariance is to be expected of a relativistic theory and is an indication of the fact that τ is without physical significance.

VI. DISCUSSION

We have discussed a method of setting up relativistic interacting theories which is somewhat simpler than existing methods. In doing this, we have combined Poincaré invariance, world-line invariance, and particle interaction. As an illustration, we discussed an explicit system of N relativistic particles in interaction. We assumed for simplicity that the interaction potentials V_a depend only on position variables. Such an assumption would be incompatible with the first-class conditions in the Komar-Todorov formalism. In our approach potentials depending only on position variables are quite acceptable. Of course, momentum-dependent potentials too can be considered, but we do not do so here.

We then showed how one transits to the $6N$-dimensional physical phase space of the system. This involves a suitable choice of variables. We have also exhibited the Hamiltonian $\mathscr{H}$ which generates τ evolution on this phase space via the Dirac bracket. Finally, we showed that the system has a sensible nonrelativistic limit.

The essential difference between this and similar efforts is that we do not assume that the mass-shell conditions are first class. The requirement that the K_a's are first class constitutes a set of quadratic differential conditions on the interaction potentials. Since we dispense with these conditions we are able to choose the interaction potentials with more liberty. The resulting simplicity is labor saving in practical applications. Also, in dealing with other conceptual issues in relativistic particle interactions like separability, the need for many-body forces, quantization, etc., the formalism described above may prove useful. Concerning separability we find that with the methods discussed above we are able to make some progress. The application of these ideas to the construction of separable interactions is discussed elsewhere.[13] Thus, the method we propose may contribute towards resolving important physical problems in relativistic particle mechanics. In view of this and of the relative simplicity of our assumptions we feel that we have presented a viable alternative to the Komar-Todorov formalism.

ACKNOWLEDGMENTS

I am indebted to Professor N. Mukunda for his guidance in this subject and a critical reading of the manuscript. I have learned much from discussion with Professor E.C.G. Sudarshan and Dr. Rajaram Nityananda.

*Present address: Raman Research Institute, Bangalore 560080, India.

[1]D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963).

[2]P. A. M. Dirac, Can. J. Math. 2, 129 (1950).

[3]A. Komar, Phys. Rev. D 18, 1881 (1978); 18, 1887 (1978); 18, 3617 (1978).

[4]E. C. G. Sudarshan, N. Mukunda, and J. Goldberg, Phys. Rev. D 23, 2218 (1981).

[5]Except in the case of free particles.

[6]A. Kihlberg, R. Marnelius, and N. Mukunda, Phys. Rev. D 23, 2201 (1981).

[7]N. Mukunda, Phys. Scr. 21, 801 (1980); A. Hanson and T. Regge, Ann. Phys. (N.Y.) 87, 498 (1974).

[8]E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics—A Modern Perspective* (Wiley, New York, 1974), Chap. 20.

[9]One needs to go up to the second term in the expansion of χ in powers of $1/c^2$.

[10]v_j is arbitrary up to a multiplicative factor, not necessarily constant on M.

[11]M would be known to mathematicians as a "contact" structure.

[12]L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, New York, 1976), p. 141.

[13]J. Samuel, following paper, Phys. Rev. D 26, 3482 (1982).