Scattering in constraint relativistic quantum dynamics

L. P. Horwitz*
Eidgenössische Technische Hochschule, Hönggerberg, CH 8093 Zürich, Switzerland

F. Rohrlich
Syracuse University, Syracuse, New York 13210
(Received 12 April 1982)

A relativistic scattering theory is developed for a covariant constraint dynamics with direct interparticle interactions. Both time-dependent and time-independent formulations are presented, the latter being a generalization of the Lippmann-Schwinger equation. For the two-body problem, we study the simple case of maximal symmetry which, equivalently, admits both single- and two-time formulations. The two-time formalism illustrates the main features of the general case of $N \geq 3$ particles. Perturbation expansions are given for the wave function and for the S matrix. Their structure is similar to those in quantum field theory corresponding to skeleton diagrams.

I. INTRODUCTION

In a recent paper[1] (to be quoted as I) we formulated a relativistic quantum dynamics including the elements of a scattering theory which is in the framework of a constraint Hamiltonian theory.[2] As a relativistic direct-interaction theory it stands between nonrelativistic quantum mechanics and relativistic field theory. As a theory with a finite number of degrees of freedom it does not suffer from some of the basic mathematical difficulties that the infinite number of degrees of freedom imposes on quantum field theory; as a relativistic theory it permits use in phenomenological studies far beyond nonrelativistic quantum mechanics, both in particle physics and in nuclear physics.

As shown in I the system of equations for N particles

$$i\frac{\partial}{\partial\tau_a} \mid \Psi_S \rangle = K_a \mid \Psi_S \rangle \tag{1.1}$$

is a proper quantum version of classical constraint Hamiltonian dynamics.[2] Here $\mid \Psi_S \rangle$ is a state vector in the Schrödinger picture, K_a $(a=1,\ldots,N)$ are the constrainor operators,

$$K_a = {p_a}^2 + {m_a}^2 + \Phi_a \ , \tag{1.2}$$

and τ_a are a set of evolution parameters. Equation (1.2) has the meaning of a mass shell in the presence of interaction.

The integrability condition for the system (1.1) is

$$[K_a, K_b] \mid \Psi_S \rangle = 0 \quad \forall a,b \tag{1.3}$$

and it is assumed that $K_a \mid \Psi_S \rangle$ is a state which is also in the manifold satisfying (1.1) and (1.3). The classical analog of (1.3) is the first-class property of the K_a and the classical equations are indeed obtained in the $\hbar \to 0$ limit as shown in I.

The system (1.1) characterizes a "many-time" formulation with the set $[\tau] = \tau_1, \ldots, \tau_N$ as the set of evolution parameters for the N particles. The development of a scattering theory requires an asymptotic relation which is most conveniently stated in the Dirac picture (interaction picture),

$$\lim_{[\tau] \to -\infty} \mid [\tau] \rangle = \mid \psi_{\text{in}} \rangle \ . \tag{1.4}$$

It was shown in I that this $[\tau]$ limit is independent of the order in which the τ_a limits are taken among the various τ_a (corresponding results are valid for $\mid \psi_{\text{out}} \rangle$ when $[\tau] \to +\infty$). It was also shown how conditions for the existence of the many-time wave operators can be obtained, and that, if they exist, they are independent of the order of the limits. The S operator and S matrix were defined, and it was demonstrated that in case a subset of the N particles is moved far away from the others in a space-like direction, there is a cluster decomposition and the S matrix factors.

The two-body problem can, under certain conditions, be reduced to a single-time problem.[3] No such simplification seems to exist for the general case of $N \geq 3$ particles without making much stronger assumptions. Except for the fact that the potential is required to satisfy the first-class constraint condition, the single-time form of the two-body problem coincides with a problem that has been studied previously in the framework of a canonical formalism.[4] In that work, it was pointed out that a particular general functional form for the potential could be chosen which would exactly preserve the individual particle asymptotic masses. This form is precisely the one required to satisfy the first-class constraint conditions imposed in the constraint Hamiltonian formalism. With this condition, the two formalisms, with their apparently very different perturbation expansions, are equivalent.

In Sec. II we shall discuss the evolution of interaction-picture states in the many-time formalism, and in Sec. III we shall formulate the two-body problem in both the two-time and the equivalent single-time forms.

In Sec. IV, we shall give time-dependent perturbation expansions in both the single- and two-time formulation of the two-body problem. The latter exhibits a structure similar to that of the perturbation expansion of quantum field theory. In Sec. V a τ-independent analog of the Lippmann-Schwinger equation and "closed form" solutions will be given in terms of the T matrix. In Sec. VI we treat[5] the cases $N \geq 3$, and in Sec. VII we shall make some concluding remarks.

II. TWO EQUIVALENT EVOLUTIONS

One can give a formal solution for the system (1.1),

$$|\Psi_S([\tau])\rangle = U_S([\tau],[\sigma])\,|\Psi_S([\sigma])\rangle \ . \tag{2.1}$$

This follows by observing that the Heisenberg-picture states $|\Psi_H\rangle$ which are $[\tau]$ independent, are mapped into the Schrödinger-picture states $|\Psi_S([\tau])\rangle$ by

$$|\Psi_S([\tau])\rangle = U([\tau])\,|\Psi_H\rangle \ , \tag{2.2}$$

where

$$U([\tau]) = \prod_{a=1}^{N} e^{-iK_a\tau_a} \tag{2.3}$$

the K_a being the $[\tau]$-independent Schrödinger-picture operators (1.2). Therefore,

$$\begin{aligned}|\Psi_S([\tau])\rangle &= U([\tau])\,|\Psi_H\rangle \\ &= U([\tau])U^{-1}([\sigma])\,|\Psi_S([\sigma])\rangle \ .\end{aligned} \tag{2.4}$$

The evolution operator in (2.1) is thus

$$U_S([\tau],[\sigma]) = U([\tau])U^{-1}([\sigma]) \ . \tag{2.5}$$

The same evolution (2.1) can also be expressed in the Dirac-picture states $|[\tau]\rangle$. These states are mapped into the $|\Psi_S\rangle$ by

$$|\Psi_S([\tau])\rangle = U_0([\tau])\,|[\tau]\rangle \ , \tag{2.6}$$

where

$$U_0([\tau]) = \prod_{a} e^{-iK_a^0\tau_a} \ , \tag{2.7}$$

$$K_a^0 = p_a{}^2 + m_a{}^2 \ . \tag{2.8}$$

The evolution (2.4) in the Dirac picture is therefore

$$\begin{aligned}|[\tau]\rangle = U_0{}^{-1}([\tau])U([\tau])U^{-1}([\sigma]) \\ \times U_0([\sigma])\,|[\sigma]\rangle \ .\end{aligned} \tag{2.9}$$

It is convenient to introduce the operator $\Omega([\tau])$ by

$$\Omega([\tau]) = U^{-1}([\tau])U_0([\tau]) \ . \tag{2.10}$$

The evolution (2.9) can then be written in the form

$$\begin{aligned}|[\tau]\rangle &= \Omega^{-1}([\tau])\Omega([\sigma])\,|[\sigma]\rangle \\ &\equiv U([\tau],[\sigma])\,|[\sigma]\rangle \ .\end{aligned} \tag{2.11}$$

On the other hand, it was proven in I that the evolution of the states $|[\tau]\rangle$ can be expressed by

$$|[\tau]\rangle = U_{[\tau]}(\tau_1\sigma_1)U_{\sigma_1\tau_2\cdots\tau_N}(\tau_2\sigma_2)\cdots U_{\sigma_1\cdots\sigma_{N-1}\tau_N}(\tau_N\sigma_N)\,|[\sigma]\rangle \ , \tag{2.12}$$

where

$$U_{\sigma_1\cdots\sigma_{j-1},\tau_j,\cdots\tau_N}(\tau_j\sigma_j) = \left[\exp\left[-i\int_{\sigma_j}^{\tau_j}\phi_j(\sigma_1\cdots\sigma_{j-1}\tau_j'\tau_{j+1}\cdots\tau_N)d\tau_j'\right]\right]_+ \tag{2.13}$$

and ϕ_a $(a=1,\ldots,N)$ are the interaction functions in the Dirac picture. The subscript $+$ indicates positive time ordering. It was further proven in I that the evolution (2.12) is independent of the order in which the σ_j are taken to the τ_j.

It follows that the two evolutions (2.11) and (2.12) must be equal.

In the special case of $N=2$ this equality of evolutions reads

$$|\tau_1\tau_2\rangle = U_0^\dagger(\tau_1\tau_2)e^{-iK_1(\tau_1-\sigma_1)}e^{-iK_2(\tau_2-\sigma_2)}U_0(\sigma_1\sigma_2)\,|\,\sigma_1\sigma_2\rangle$$
$$= \left[\exp\left[-i\int_{\sigma_1}^{\tau_1}\phi_1(\tau_1'\tau_2)d\tau_1'\right]\right]_+ \left[\exp\left[-i\int_{\sigma_2}^{\tau_2}\phi_2(\sigma_1\tau_2')d\tau_2'\right]\right]_+ |\,\sigma_1\sigma_2\rangle \ . \tag{2.14}$$

III. FORMULATION OF THE TWO-BODY PROBLEM

In this section, we shall treat a particularly simple class of two-body problems, namely those for which $\Phi_1=\Phi_2\equiv\Phi$. In this case, there is a general form of the potential which satisfies the first-class constraint conditions. The results can be cast into a simple form, and an interesting connection can be made with the single-time formalism.[4] In Sec. VI, the general case is treated, since the two-body subproblem of an $N\geq 3$ body problem does not necessarily admit a simplification of this type.

For $\Phi_1=\Phi_2$, the first-class constraint condition (1.3) becomes

$$[K_1^0-K_2^0,\Phi]\,|\,\Psi_S\rangle=0 \ . \tag{3.1}$$

With the help of the commutation relations [signature $(-,+,+,+)$]

$$[q_a^\mu,p_b^\nu]=ig^{\mu\nu}\delta_{ab} \tag{3.2}$$

and for Φ a function of $q=q_1-q_2$ by translation invariance, p_1 and p_2, the commutator in (3.1) is ($P^\mu=p_1^\mu+p_2^\mu$, the conserved center-of-mass momentum, commutes with q)

$$[K_1^0-K_2^0,\Phi]=2P_\mu[p_1^\mu,\Phi]=-2P_\mu[p_2^\mu,\Phi]$$
$$=-2iP^\mu\frac{\partial\Phi}{\partial q^\mu} \ . \tag{3.3}$$

The expression (3.3) vanishes identically (on its domain of definition) when Φ is a function of q only in the combination

$$q_\perp^\mu\equiv q^\mu-q^\nu P_\nu P^\mu/P^2 \ . \tag{3.4}$$

With this choice, (3.1) becomes an operator identity, i.e., valid for any $|\,\Psi\rangle$ (in the domain of definition of the commutator). We shall assume this strong integrability condition to be valid in the following.

Since

$$K_1^0\tau_1+K_2^0\tau_2=\tfrac{1}{2}(K_1^0+K_2^0)(\tau_1+\tau_2)$$
$$+\tfrac{1}{2}(K_1^0-K_2^0)(\tau_1-\tau_2) \tag{3.5}$$

and $K_1^0-K_2^0$ commutes with Φ, the interaction operator in the Dirac picture is

$$\phi(\tau_1\tau_2)=e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}\Phi e^{-iK_1^0\tau_1}e^{-iK_2^0\tau_2}$$
$$=e^{i(K_1^0+K_2^0)(\tau_1+\tau_2)/2}\Phi e^{-i(K_1^0+K_2^0)(\tau_1+\tau_2)/2}$$
$$\equiv\phi(\tau_1+\tau_2) \ . \tag{3.6}$$

Moreover, since by Eq. (2.9),

$$|\,\tau_1\tau_2\rangle=e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}e^{-iK_1\tau_1}e^{-iK_2\tau_2}\,|\,\Psi_H\rangle \tag{3.7}$$

and $K_1-K_2=K_1^0-K_2^0$, the interaction-picture states evolve according to

$$|\,\tau_1\tau_2\rangle=e^{i(K_1^0+K_2^0)(\tau_1+\tau_2)/2}e^{-i(K_1+K_2)(\tau_1+\tau_2)/2}\,|\,\Psi_H\rangle$$
$$\equiv|\,\tau_1+\tau_2\rangle \ . \tag{3.8}$$

In fact, it follows directly from the fundamental equations (1.1) for the two-body case,

$$i\frac{\partial}{\partial\tau_1}\,|\,\tau_1\tau_2\rangle=i\frac{\partial}{\partial\tau_2}\,|\,\tau_1\tau_2\rangle$$
$$=\phi\,|\,\tau_1\tau_2\rangle \ , \tag{3.9}$$

that $(\partial/\partial\tau_1-\partial/\partial\tau_2)\,|\,\tau_1\tau_2\rangle=0$ i.e., that $|\,\tau_1\tau_2\rangle$ must be a function of $\tau_1+\tau_2$ alone.

The evolution Eq. (2.11) reads, for the two-body case,

$$|\,\tau_1+\tau_2\rangle=e^{i(K_1^0+K_2^0)(\tau_1+\tau_2)/2}e^{-i(K_1+K_2)(\tau_1+\tau_2)/2}e^{i(K_1+K_2)(\sigma_1+\sigma_2)/2}e^{-i(K_1^0+K_2^0)(\sigma_1+\sigma_2)/2}\,|\,\sigma_1+\sigma_2\rangle \ . \tag{3.10}$$

For the alternative form Eq. (2.14), we note that a change of variables results in

$$\left[\exp\left[-i\int_{\sigma_2}^{\tau_2}\phi(\sigma_1+\tau_2')d\tau_2'\right]\right]_+=\left[\exp\left[-i\int_{\sigma_1+\sigma_2}^{\sigma_1+\tau_2}\phi(\tau')d\tau'\right]\right]_+ , \tag{3.11}$$

where the sense of τ ordering is the same in $\tau=\tau_1+\tau_2$ as in τ_2 (for τ_1 fixed), and

$$\left[\exp\left[-i\int_{\sigma_1}^{\tau_1}\phi(\tau_1'+\tau_2)d\tau_1'\right]\right]_+=\left[\exp\left[-i\int_{\sigma_1+\tau_2}^{\tau_1+\tau_2}\phi(\tau')d\tau'\right]\right]_+ .$$

Hence, in the usual terminology of interaction-picture evolution operators,

$$U_{\tau_1\tau_2}=U(\tau_1+\tau_2,\sigma_1+\tau_2)\ ,\quad U_{\sigma_1\tau_2}(\tau_2\sigma_2)=U(\sigma_1+\tau_2,\sigma_1+\sigma_2) \tag{3.12}$$

and Eq. (2.14) then reads

$$|\tau_1+\tau_2\rangle=U(\tau_1+\tau_2,\sigma_1+\tau_2)U(\sigma_1+\tau_2,\sigma_1+\sigma_2)\,|\,\sigma_1+\sigma_2\rangle\ . \tag{3.13}$$

We see from Eqs. (3.6)–(3.13) that the two-τ constraint Hamiltonian formalism reduces to a single-τ theory,[3] for which $\tau=\tau_1+\tau_2$, and the fundamental equation in the Dirac picture is

$$i\frac{\partial}{\partial\tau}\,|\,\tau\rangle=\phi(\tau)\,|\,\tau\rangle\ . \tag{3.14}$$

The evolution operator for this equivalent single-τ theory is

$$K=\tfrac{1}{2}(K_1+K_2)=K^0+\Phi\ , \tag{3.15}$$

where

$$K^0\equiv\tfrac{1}{2}(K_1^0+K_2^0)\ . \tag{3.16}$$

We emphasize, however, that this particular choice of τ, for which the two-time equations (3.9) reduce in a simple way to an equivalent single-time theory, does not necessarily parametrize the actual physical motion of the system. As pointed out in I (and in, for example, Ref. 5 for the classical case), it is possible to choose a "gauge" which specifies τ_1 and τ_2 in terms of a single parameter τ (not necessarily coinciding with the above choice), thus determining the actual evolution of the system. As argued in I, however, the wave operators and S matrix are independent of this choice, and are, therefore, given as well by computations based on Eq. (3.14).

Equations of the type (3.14) have been studied previously[4] in the framework of a canonical single-τ formalism. Starting with Eq. (3.14), with Φ an arbitrary (sufficiently well-behaved) Lorentz-invariant function of $q_1^\mu-q_2^\mu$, there is no *a priori* guarantee that the individual particle masses are conserved in a collision. It was pointed out by Horwitz and Lavie,[4] however, that when Φ is a function of $q_1^\mu-q_2^\mu$ only through dependence on $q_\perp^\mu$, the individual particle masses will be exactly conserved. This follows from the fact that in the form (3.15) the evolution operator can be written in terms of total and relative momenta, P^μ and

$$p^\mu\equiv\tfrac{1}{2}(p_1^\mu-p_2^\mu)\ , \tag{3.17}$$

and relative coordinates alone. Equation (3.14), in the Schrödinger picture associated with the single-τ theory, is

$$i\frac{\partial}{\partial\tau}\,|\,\Psi_S(\tau)\rangle=[\tfrac{1}{4}P^2+p^2+\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\Phi]\,|\,\Psi_S(\tau)\rangle \tag{3.18}$$

and in the (space-time) coordinate representation, for which $P\to-i\partial/\partial Q$ and $p\to-i\partial/\partial q$, where $Q=\frac{1}{2}(q_1+q_2)$, we obtain

$$i\frac{\partial}{\partial\tau}\psi_\tau(q,Q)=[\tfrac{1}{4}P^2+p^2+\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\Phi]\psi_\tau(q,Q)\ . \tag{3.19}$$

Taking the Fourier transform with respect to Q,P takes on a numerical value (in the function Φ as well), and Eq. (3.19) becomes

$$i\frac{\partial}{\partial\tau}\tilde{\psi}_\tau(q,P)=[\tfrac{1}{4}P^2+p^2+\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\Phi]\tilde{\psi}_2(q,P)\ . \tag{3.20}$$

The operator p^2 commutes with the S matrix of this reduced motion problem. We now note that the differences between initial and final masses are given by[4]

$$\begin{aligned}p_1{}^2-p_1'{}^2&=(\tfrac{1}{2}P+p)^2-(\tfrac{1}{2}P'+p')^2\\&=p^2-p'^2+\Delta\cdot P\ ,\\p_2{}^2-p_2'{}^2&=(\tfrac{1}{2}P-p)^2-(\tfrac{1}{2}P'-p')^2\\&=p^2-p'^2-\Delta\cdot P\ ,\end{aligned}$$

where $\Delta=p_1-p_1'=p-p'$ is the momentum transfer, and we have used the fact that P^μ is absolutely conserved. Since $p^2=p'^2$ *asymptotically*, the individual particle masses will be conserved if $\Delta\cdot P=0$. This can be guaranteed by the choice $\Phi=\Phi(q_\perp,p,P)$, since the matrix element $\langle p\,|\,\Phi\,|\,p'\rangle$ (Fourier transformation) of a potential of this form contains the factor $\delta(\Delta\cdot P)$.

The condition that the individual particle masses are precisely conserved (asymptotically) in a single-τ canonical relativistic scattering theory makes this theory equivalent to a two-time relativistic constraint Hamiltonian scattering theory.

We wish now to discuss briefly the structure of the hierarchy of wave operators (introduced in I for the N-body problem) for the special case of $N=2$ when Eq. (3.1) is valid as an operator identity. In I, it was shown that the N-body wave operator, defined by

$$\Omega_+ = \lim_{[\tau]\to-\infty} \prod_a e^{iK_a\tau_a} \prod_b e^{-iK_b^0\tau_b} \tag{3.21}$$

is independent of the order of taking the limits in the τ_a. For the two-body problem and the case $\Phi_1=\Phi_2\equiv\Phi$ one obtains

$$\Omega_+ = \lim_{\tau_1,\tau_2\to-\infty} e^{i(K_1+K_2)(\tau_1+\tau_2)/2} e^{-i(K_1^0+K_2^0)(\tau_1+\tau_2)/2} , \tag{3.22}$$

coinciding precisely with the single-time wave operator for the problem defined by Eqs. (3.15) and (3.16). The condition for the existence of this wave operator is[4]

$$\int_{-\infty}^0 ||\Phi e^{-i(K_1^0+K_2^0)\tau/2}\psi||d\tau < \infty \tag{3.23}$$

for all ψ in some dense set. In I, however, a different approach was followed. As a first step in determining the existence of the wave operator, it was required that

$$\int_{-\infty}^0 ||\Phi e^{-iK_1^0\tau_1}\psi||d\tau_1 < \infty \tag{3.24}$$

and as a second step (the last one needed for the two-body problem),

$$\int_{-\infty}^0 ||(K_2\Omega_+^{(1)} - \Omega_+^{(1)}K_2^0)e^{-iK_2^0\tau_2}\psi||d\tau_2 < \infty , \tag{3.25}$$

where

$$\Omega_+^{(1)} = \lim_{\tau_1\to-\infty} e^{iK_1\tau_1} e^{-iK_1^0\tau_1} . \tag{3.26}$$

Replacing $e^{-iK_1^0\tau_1}$ by

$$e^{-i(K_1^0+K_2^0)\tau_1/2} e^{-i(K_1^0-K_2^0)\tau_1/2}$$

in (3.24), one sees that this condition is the same as that of (3.23). Furthermore,

$$K_2\Omega_+^{(1)} - \Omega_+^{(1)}K_2^0 = (K_2-K_1)\Omega_+^{(1)} - \Omega_+^{(1)}(K_2^0-K_1^0) + K_1\Omega_+^{(1)} - \Omega_+^{(1)}K_1^0 ,$$

vanishes in (3.25), since $\Omega_+^{(1)}$ intertwines K_1, K_1^0, and $K_2^0-K_1^0$ commutes with $\Omega_+^{(1)}$, a functional of Φ. Condition (3.25) is therefore identically, and trivially, satisfied. The second step in the formation of Ω_+ is unnecessary, in fact, since

$$\Omega_+^{(1)} = \lim_{\tau_1\to-\infty} e^{i(K_1+K_2)\tau_1/2} e^{i(K_1-K_2)\tau_1/2} e^{-i(K_1^0-K_2^0)\tau_1/2} e^{-i(K_1^0+K_2^0)\tau_1/2} = \Omega_+ \tag{3.27}$$

and $e^{iK_2\tau_2}\Omega_+ e^{-iK_2^0\tau_2} = \Omega_+$.

In the next section we shall review briefly the structure of the perturbation expansion for the calculation of Ω_+ and the S matrix in the single-time formalism based on Eq. (3.14).[4] However, as a prototype illustrating many of the properties of the general N-body case, we shall also work out a perturbation expansion in the two-time description of the two-body problem. To do this, consider again the Eqs. (3.9). Differentiating the first with respect to τ_2, one finds the second-order equation

$$-\frac{\partial^2}{\partial\tau_1\partial\tau_2} |\tau_1\tau_2\rangle = V(\tau_2\tau_1)|\tau_1\tau_2\rangle , \tag{3.28}$$

where

$$V(\tau_2\tau_1)|\tau_1\tau_2\rangle = i\frac{\partial}{\partial\tau_2}[\phi(\tau_1\tau_2)|\tau_1\tau_2\rangle] = \{[\phi(\tau_1\tau_2),K_2^0]+\phi^2\}|\tau_1\tau_2\rangle$$
$$= ([\phi,K_1^0]+\phi^2)|\tau_1\tau_2\rangle = V(\tau_1\tau_2)|\tau_1\tau_2\rangle . \tag{3.29}$$

The last equality follows from the strong integrability condition. In fact, $V(\tau_1,\tau_2)=V(\tau_1+\tau_2)$. Formally integrating Eq. (3.28) twice, we obtain the integral equation

$$|\tau_1\tau_2\rangle = |\sigma_1\tau_2\rangle + |\tau_1\sigma_2\rangle - |\sigma_1\sigma_2\rangle - \int_{\sigma_1}^{\tau_1} d\tau_1' \int_{\sigma_2}^{\tau_2} d\tau_2' V(\tau_1'\tau_2')|\tau_1'\tau_2'\rangle . \tag{3.30}$$

This equation can also be obtained from the integral form of Eq. (3.14),

$$|\tau\rangle = |\sigma\rangle - i\int_\sigma^\tau d\tau'\phi(\tau')|\tau'\rangle \tag{3.31}$$

if we take first $\tau=\tau_1+\tau_2$, $\sigma=\sigma_1+\sigma_2$, then $\tau=\tau_1+\sigma_2$, $\sigma=\sigma_1+\sigma_2$ and subtract the second from the first:

$$\begin{aligned}|\tau_1+\tau_2\rangle - |\tau_1+\sigma_2\rangle &= |\sigma_1+\tau_2\rangle - |\sigma_1+\sigma_2\rangle - i\left[\int_{\sigma_1+\tau_2}^{\tau_1+\tau_2} d\tau'\phi(\tau')|\tau'\rangle - \int_{\sigma_1+\sigma_2}^{\tau_1+\sigma_2} d\tau_1'\phi(\tau')|\tau'\rangle\right]\\ &= |\sigma_1+\tau_2\rangle - |\sigma_1+\sigma_2\rangle - \int_{\sigma_1}^{\tau_1} d\tau_1'\int_{\sigma_2}^{\tau_2} d\tau_2' i\frac{\partial}{\partial\tau_2'}[\phi(\tau_1'+\tau_2')|\tau_1'+\tau_2'\rangle] .\end{aligned} \tag{3.32}$$

As we shall see, the forms of the perturbation expansion for Eqs. (3.30) and (3.31) are quite different, the former having a structure close to that of quantum field theory.

To relate $|\tau_1\tau_2\rangle$ to the in state, we take the limit $\sigma_1,\sigma_2\to-\infty$ in Eq. (3.30). For every finite τ_1,τ_2, we have the relations

$$\begin{aligned}|-\infty\,\tau_2\rangle = |\tau_1-\infty\rangle &= |-\infty,-\infty\rangle\\ &= |\psi_{\rm in}\rangle ,\end{aligned} \tag{3.33}$$

and we therefore obtain the integral equation

$$\begin{aligned}|\tau_1\tau_2\rangle &= |\psi_{\rm in}\rangle\\ &\quad - \int_{-\infty}^{\tau_1} d\tau_1'\int_{-\infty}^{\tau_2} d\tau_2' V(\tau_1'\tau_2')|\tau_1'\tau_2'\rangle .\end{aligned} \tag{3.34}$$

IV. TIME-DEPENDENT PERTURBATION SOLUTIONS FOR THE TWO-BODY PROBLEM

The S matrix for two-body scattering can be obtained from the integral form (3.31) of the single-τ equation (3.14). We shall first briefly review the structure of the perturbation expansion obtained from this equation,[4] and then study the structure of the corresponding expansion in the two-time form. The perturbative equivalence of these two expansions is demonstrated in the Appendix.

Since $|\sigma\rangle\to|\psi_{\rm in}\rangle$ for $\sigma\to-\infty$, Eq. (3.31) becomes

$$|\tau\rangle = |\psi_{\rm in}\rangle - i\int_{-\infty}^{\tau}\phi(\tau')|\tau'\rangle d\tau' . \tag{4.1}$$

Iteration yields the series

$$|\tau\rangle = |\psi_{\rm in}\rangle - i\int_{-\infty}^{\tau}\phi(\tau')|\psi_{\rm in}\rangle d\tau' + (-i)^2\int_{-\infty}^{\tau}d\tau'\int_{-\infty}^{\tau'}d\tau''\phi(\tau')\phi(\tau'')|\psi_{\rm in}\rangle + \cdots . \tag{4.2}$$

The last integration in each term of the expansion contains the absolutely convergent integral assuring the existence of the wave operator. We may therefore insert a factor $\exp(\epsilon\tau^{(n)})$, and take $\epsilon\to0_+$ after the integrations (this limit is implicit in all of the formulas which follow). The nth term in the expansion of $\psi(p_1,p_2;\tau)$, the momentum-space representation of $|\tau\rangle$, is

$$\begin{aligned}\psi^{(n)}(p_1p_2;\tau) = (-i)^n\int_{-\infty}^{\tau}d\tau^{(1)}\int_{-\infty}^{\tau^{(1)}}d\tau^{(2)}\cdots\int_{-\infty}^{\tau^{(n-1)}}&d\tau^{(n)}(dp_1^{(1)})(dp_2^{(1)})\cdots(dp_1^{(n-1)})(dp_2^{(n-1)})\\ &\times(dp_1')(dp_2')\langle p_1p_2|\phi(\tau^{(1)})|p_1^{(1)}p_2^{(1)}\rangle\\ &\times\langle p_1^{(1)}p_2^{(1)}|\phi(\tau^{(2)})|p_1^{(2)}p_2^{(2)}\rangle\cdots\\ &\times\langle p_1^{(n-1)}p_2^{(n-1)}|\phi(\tau^{(n)})|p_1'p_2'\rangle e^{\epsilon\tau^{(n)}}\psi_{\rm in}(p_1'p_2') .\end{aligned} \tag{4.3}$$

The last integral, over $\tau^{(n)}$, is, using Eq. (3.6),

$$\begin{aligned}&\int_{-\infty}^{\tau^{(n-1)}}d\tau^{(n)}\exp\left[\frac{i}{2}[(p_1^{(n-1)})^2+(p_2^{(n-1)})^2-p_1'^2-p_2'^2-i\epsilon]\tau^{(n)}\right]\langle p_1^{(n-1)}p_2^{(n-1)}|\Phi|p_1'p_2'\rangle\psi_{\rm in}(p_1'p_2')\\ &\quad=\frac{2}{i}\frac{\exp\{(i/2)[(p_1^{(n-1)})^2+(p_2^{(n-1)^2})-p_1'^2-p_2'^2-i\epsilon]\tau^{(n-1)}\}}{(p_1^{(n-1)})^2+(p_2^{(n-1)})^2-p_1'^2-p_2'^2-i\epsilon}\langle p_1^{(n-1)}p_2^{(n-1)}|\Phi|p_1'p_2'\rangle\psi_{\rm in}(p_1'p_2') .\end{aligned} \tag{4.4}$$

The exponential factor $\exp\{\frac{1}{2}i[(p_1^{(n-2)})^2+(p_2^{(n-2)})^2-(p_1^{(n-1)})^2-(p_2^{(n-1)})^2]\tau^{(n-1)}\}$ from the second to last term cancels $(p_1^{(n-1)})^2+(p_2^{(n-1)})^2$ in the exponent of (4.4) and replaces them by $(p_1^{(n-2)})^2+(p_2^{(n-2)})^2$. The process can then be continued, with the result

$$\psi^{(n)}(p_1p_2;\tau)=(-2)^n \int (dp_1^{(1)})(dp_2^{(1)}) \cdots (dp_1^{(n-1)})(dp_2^{(n-1)})(dp_1')(dp_2')e^{i(p_1{}^2+p_2{}^2-p_1'^2-p_2'^2)\tau/2}$$
$$\times \langle p_1p_2 | \Phi | p_1^{(1)}p_2^{(1)}\rangle \langle p_1^{(1)}p_2^{(1)} | \Phi | p_1^{(2)}p_2^{(2)}\rangle \cdots \langle p_1^{(n-1)}p_2^{(n-1)} | \Phi | p_1'p_2'\rangle$$
$$\times \frac{1}{p_1{}^2+p_2{}^2-p_1'^2-p_2'^2-i\epsilon}\,\frac{1}{(p_1^{(1)})^2+(p_2^{(1)})^2-p_1'^2p_2'^2-i\epsilon} \cdots$$
$$\times \frac{1}{(p_1^{(n-1)})^2+(p_2^{(n-1)})^2-p_1'^2-p_2'^2-i\epsilon}\psi_{\text{in}}(p_1'p_2') \ . \quad (4.5)$$

To obtain the S matrix, we must take the limit $\tau \rightarrow +\infty$. By a well-known relation,[6] one finds that the last integration yields a δ function:

$$\lim_{\tau\rightarrow\infty} \frac{\exp[\frac{1}{2}i(p_1{}^2+p_2{}^2-p_1'^2-p_2'^2)\tau]}{p_1{}^2+p_2{}^2-p_1'^2-p_2'^2-i\epsilon}=2\pi i\delta(p_1{}^2+p_2{}^2-p_1'^2-p_2'^2) \ . \quad (4.6)$$

The S matrix is, therefore, given by

$$\langle p_1p_2 | S | p_1'p_2'\rangle=\delta_4(p_1-p_1')\delta_4(p_2-p_2')+2\pi i\delta(p_1{}^2+p_2{}^2-p_1'^2-p_2'^2)$$
$$\times \Bigg[\cdots +(-2)^n \int (dp_1^{(1)})(dp_2^{(1)}) \cdots (dp_1^{(n-1)})(dp_2^{(n-1)})$$
$$\times \langle p_1p_2 | \Phi | p_1^{(1)}p_2^{(1)}\rangle \langle p_1^{(1)}p_2^{(1)} | \Phi | p_1^{(2)}p_2^{(2)}\rangle \cdots$$
$$\times \langle p_1^{(n-1)}p_2^{(n-1)} | \Phi | p_1'p_2'\rangle \frac{1}{(p_1^{(1)})^2+(p_2^{(1)})^2-p_1'^2-p_2'^2-i\epsilon} \cdots$$
$$\times \frac{1}{(p_1^{(n-1)})^2+(p_2^{(n-1)})^2-p_1'^2-p_2'^2-i\epsilon} + \cdots \Bigg] \ . \quad (4.7)$$

The structure of Eqs. (4.5) and (4.7) is essentially different from that found in the perturbation expansions of quantum field theory with gauge field interactions. The two-body intermediate-state propagators in Eqs. (4.7) are of the form

$$[(p_1^{(i)})^2+(p_2^{(i)})^2-p_1'^2-p_2'^2-i\epsilon]^{-1} \quad (4.8)$$

and not of the product form

$$[(p_1^{(i)})^2-p_1'^2-i\epsilon]^{-1}[(p_2^{(i)})^2-p_2'^2-i\epsilon]^{-1} \quad (4.9)$$

associated with the usual Feynman rules. The δ function accompanying the scattering part of the S matrix, moreover, conserves the sum of the squared masses of the particles, so that the conservation of individual particle masses is a dynamical question. As pointed out in Sec. III, the choice $\Phi=\Phi(q_\perp)$ satisfying the first-class constraint requirements, assures (if the wave operators exist in the usual sense), in fact, that individual particle masses are precisely conserved.

We now turn to a discussion of the expansion in the two-time formalism, based on Eq. (3.34). Iteration yields the series

$$| \tau_1\tau_2\rangle = | \psi_{\text{in}}\rangle - \int_{-\infty}^{\tau_1} d\tau_1' \int_{-\infty}^{\tau_2} d\tau_2' V(\tau_1'\tau_2') | \psi_{\text{in}}\rangle$$
$$+(-1)^2 \int_{-\infty}^{\tau_1} d\tau_1' \int_{-\infty}^{\tau_2} d\tau_2' \int_{-\infty}^{\tau_1'} d\tau_1'' \int_{-\infty}^{\tau_2'} d\tau_2'' V(\tau_1'\tau_2')V(\tau_1''\tau_2'') | \psi_{\text{in}}\rangle + \cdots \ . \quad (4.10)$$

The last integral in each term is similar in form to the absolutely convergent integral, assuring the existence of the wave operator, discussed in Sec. IV of I.[7] As before, we may therefore insert factors $\exp[\epsilon_1\tau_1^{(n)}]\exp[\epsilon_2\tau_2^{(n)}]$ in the last integration of the nth term of the series, and take $\epsilon_1,\epsilon_2 \rightarrow 0_+$ after carrying out the integration. The nth term in the expansion of $\psi(p_1p_2;\tau_1\tau_2)$, the momentum representation of $| \tau_1\tau_2\rangle$, is

$$\psi^{(n)}(p_1p_2;\tau_1\tau_2)=(-1)^n\int_{-\infty}^{\tau_1}d\tau_1^{(1)}\int_{-\infty}^{\tau_2}d\tau_2^{(2)}\cdots$$
$$\times\int_{-\infty}^{\tau_1^{(n-1)}}d\tau_1^{(n)}\int_{-\infty}^{\tau_2^{(n-1)}}d\tau_2^{(n)}\int(dp_1^{(1)})(dp_2^{(1)})\cdots(dp_1^{(n-1)})(dp_2^{(n-1)})(dp_1')(dp_2')$$
$$\times\langle p_1p_2\,|\,V(\tau_1^{(1)}\tau_2^{(1)})\,|\,p_1^{(1)}p_2^{(1)}\rangle\langle p_1^{(1)}p_2^{(1)}\,|\,V(\tau_1^{(2)}\tau_2^{(2)})\,|\,p_1^{(2)}p_2^{(2)}\rangle\cdots$$
$$\times\langle p_1^{(n-1)}p_2^{(n-1)}\,|\,V(\tau_1^{(n)}\tau_2^{(n)})\,|\,p_1'p_2'\rangle e^{\epsilon_1\tau_1^{(n)}}e^{\epsilon_2\tau_2^{(n)}}\psi_{\text{in}}(p_1'p_2')\;. \qquad (4.11)$$

The integrations in $\tau_1^{(n)}$ and $\tau_2^{(n)}$ each produce a propagator factor; in place of the single propagator obtained in Eq. (4.4), the last integral in (4.11) contains the product of two propagators. The ϵ_1,ϵ_2 factors recur in the exponential in the upper limit for the $\tau_1^{(n-1)},\tau_2^{(n-1)}$ integrations, and replacement of $(p_1^{(n-1)})^2,(p_2^{(n-1)})^2$ by $(p_1^{(n-2)})^2,(p_2^{(n-2)})^2$ occurs as in the calculation leading to Eq. (4.5). Carrying out the sequence of integrations, we find

$$\psi^{(n)}(p_1p_2;\tau_1\tau_2)=\int(dp_1^{(1)})(dp_2^{(1)})\cdots(dp_1^{(n-1)})(dp_2^{(n-1)})(dp_1')(dp_2')e^{i(p_1{}^2-p_1'{}^2)\tau_1}$$
$$\times e^{i(p_2{}^2-p_2'{}^2)\tau_2}\frac{1}{p_1{}^2-p_1'{}^2-i\epsilon_1}\,\frac{1}{p_2{}^2-p_1'{}^2-i\epsilon_2}\cdots\frac{1}{(p_1^{(n-1)})^2-p_1'{}^2-i\epsilon_1}$$
$$\times\frac{1}{(p_2^{(n-1)})^2-p_2'{}^2-i\epsilon_2}\langle p_1p_2\,|\,V\,|\,p_1^{(1)}p_2^{(2)}\rangle\langle p_1^{(1)}p_2^{(1)}\,|\,V\,|\,p_1^{(2)}p_2^{(2)}\rangle\cdots$$
$$\times\langle p_1^{(n-1)}p_2^{(n-1)}\,|\,V\,|\,p_1'p_2'\rangle\psi_{\text{in}}(p_1'p_2')\;. \qquad (4.12)$$

In the limit $\tau_1,\tau_2\rightarrow\infty$, we may again use the relation (4.6) to obtain

$$\langle p_1p_2\,|\,S\,|\,p_1'p_2'\rangle=\delta^4(p_1-p_1')\delta^4(p_2-p_2')-(2\pi)^2\delta(p_1{}^2-p_1'{}^2)\delta(p_2{}^2-p_2'{}^2)$$
$$\times\Bigg[\cdots+\int(dp_1^{(1)})(dp_2^{(1)})\cdots(dp_1^{(n-1)})(dp_2^{(n-1)})$$
$$\times\frac{1}{(p_1^{(1)})^2-p_1'{}^2-i\epsilon_1}\,\frac{1}{(p_2^{(1)})^2-p_2'{}^2-i\epsilon_2}\cdots\frac{1}{(p_1^{(n-1)})^2-p_1'{}^2-i\epsilon_1}\,\frac{1}{(p_2^{(n-1)})^2-p_2'{}^2-i\epsilon_2}$$
$$\times\langle p_1p_2\,|\,V\,|\,p_1^{(1)}p_2^{(1)}\rangle\langle p_1^{(1)}p_2^{(1)}\,|\,V\,|\,p_1^{(2)}p_2^{(2)}\rangle\cdots\langle p_1^{(n-1)}p_2^{(n-1)}\,|\,V\,|\,p_1'p_2'\rangle+\cdots\Bigg]\;. \qquad (4.13)$$

This expansion may be represented in terms of Feynman diagrams.

The structure of Eqs. (4.12) and (4.13) is similar to that found in the perturbation expansions of quantum field theory, involving a product of propagators of the form (4.9).

Equations (4.7) and (4.13) for the S matrix have been derived in two different ways from the same initial equations, and they must therefore be equal. The equivalence of the two expressions is demonstrated explicitly, in each order of perturbation theory, in the Appendix.

It is noteworthy that the comparison of Eq. (4.13) with quantum field theory leads to the conclusion that the fundamental four-point function (playing the role of two vertices and a propagator for the carrier of interaction) is here characterized by V and not Φ, contrary to what might have been conjectured.

We further remark that the diagrams associated with (4.13) are topologically ladder diagrams even though, e.g., in the space-time picture $t(\tau_1') > t(\tau_1'')$ can occur together with $t(\tau_2') < t(\tau_2'')$. Topologically crossed diagrams can arise by iteration of an equation constructed from a suitable linear combination of the equations obtained by all semi-infinite limits of Eq. (3.30) excluding only the rectangle with the outstate $|\infty\infty\rangle = |\psi_{\rm out}\rangle$ in one corner. These possibilities will not be discussed here.

V. TIME-INDEPENDENT SOLUTION FOR THE TWO-BODY PROBLEM

In this section, we shall derive the analog of the Lippmann-Schwinger equation for the single-time and two-time versions of the two-body constraint dynamics. The result is formally similar to the nonrelativistic case in the single-time theory. For the two-time version, we obtain a generalization that can be extended to the many-body case.

Let us consider the scattering equation (4.1) in the Schrödinger picture. The substitution

$$|\tau\rangle = U_0^\dagger(\tau)\,|\Psi_S(\tau)\rangle = e^{iK^0\tau}\,|\Psi_S(\tau)\rangle \ , \tag{5.1}$$

where $K^0 = (K_1^0 + K_2^0)/2$ and $\tau = \tau_1 + \tau_2$ transforms (4.1) to

$$|\Psi_S(\tau)\rangle = U_0(\tau)\,|\psi_{\rm in}\rangle - i\int_{-\infty}^{\tau} d\tau' e^{iK^0(\tau'-\tau)}\Phi\,|\Psi_S(\tau')\rangle \ . \tag{5.2}$$

This equation will serve as the starting point for the derivation of the relativistic Lippmann-Schwinger equation in the single-time version of the two-body theory. Since that is an equation for "continuum eigenfunctions," we shall assume that the in states are sharp (generalized eigenstates[8]), i.e.,

$$K_i^0\,|\psi_{\rm in}(k_1k_2)\rangle = (k_i^2 + m_i^2)\,|\psi_{\rm in}(k_1k_2)\rangle \quad (i=1,2) \ . \tag{5.3}$$

Although this makes the calculation formal, one can justify the results rigorously.[9]

The sharp in states imply sharp Heisenberg states,

$$K_i\,|\Psi_H^{(\pm)}(k_1k_2)\rangle = (k_i^2 + m_i^2)\,|\Psi_H^{(\pm)}(k_1k_2)\rangle \tag{5.4}$$

as follows from the intertwining properties of the wave operators (see I),

$$\begin{aligned} K_i\,|\Psi_H^{(\pm)}(k_1k_2)\rangle &= K_i\Omega_\pm\,|\psi_{\rm in}(k_1k_2)\rangle \\ &= \Omega_\pm K_i^0\,|\psi_{\rm in}(k_1k_2)\rangle \\ &= (k_i^2+m_i^2)\,|\Psi_H^{(\pm)}(k_1k_2)\rangle \ . \end{aligned} \tag{5.5}$$

Because of (2.2), the scattering equation (5.2) can be written in the Heisenberg picture as

$$e^{-i(k_1^2+m_1^2+k_2^2+m_2^2)\tau/2}\,|\Psi_H(k_1k_2)\rangle = e^{-i(k_1^2+m_1^2+k_2^2+m_2^2)\tau/2}\,|\psi_{\rm in}\rangle - i\int_{-\infty}^{\tau} d\tau' e^{iK^0(\tau'-\tau)}\Phi e^{-i(k_1^2+m_1^2+k_2^2+m_2^2)\tau'/2}\,|\Psi_H(k_1k_2)\rangle \ .$$

When one brings the exponentials on the left over to the right-hand side, one can introduce $s=\tau'-\tau$ as a new variable in the integral, which now extends from $-\infty$ to 0. The condition of asymptotically outgoing waves is included in the standard fashion by factors $e^{\epsilon s}$ and the limit $\epsilon \to 0_+$. The result is

$$|\Psi_H^{(+)}(k_1k_2)\rangle = |\psi_{\rm in}(k_1k_2)\rangle - \frac{1}{K^0 - \frac{1}{2}(k_1^2+k_2^2+m_1^2+m_2^2) - i\epsilon}\Phi\,|\Psi_H^{(+)}(k_1k_2)\rangle \ . \tag{5.6}$$

This relativistic form of the Lippmann-Schwinger equation can also be derived by postponing the assumption of continuum eigenstates to the very end. Consider[9,10]

$$|\psi_{\rm in}\rangle = \Omega_+^\dagger\Omega_+\,|\psi_{\rm in}\rangle = \lim_{\tau\to-\infty} U_0(\tau)^{-1}U(\tau)\,|\Psi_H^{(+)}\rangle \ . \tag{5.7}$$

The derivative of $U_0^{-1}(\tau)U(\tau)$ is

$$\frac{d}{d\tau}[U_0(\tau)^{-1}U(\tau)] = -ie^{iK^0\tau}\Phi e^{-iK\tau} \ , \tag{5.8}$$

and integrating between $0,\tau$ we obtain

$$U_0^{-1}(\tau)U(\tau) - 1 = -i\int_0^{\tau} e^{iK^0\tau'}\Phi e^{-iK\tau'}d\tau' \ . \tag{5.9}$$

We now act with this result on $|\Psi_H^{(+)}\rangle$ and take the limit $\tau\to-\infty$ to obtain

$$|\Psi_H^{(+)}\rangle=|\psi_{\rm in}\rangle-i\int_{-\infty}^{0}d\tau' e^{iK^0\tau'}\Phi e^{-iK\tau'}|\Psi_H^{(+)}\rangle\ . \tag{5.10}$$

In the limit that $|\psi_{\rm in}\rangle$ becomes sharp in K_1^0,K_2^0, one obtains precisely the expression (5.6).

The scattering integral equation in the two-time version, Eq. (3.34), can be treated in a way parallel to the single-time analysis given above. In the Schrödinger picture, Eq. (3.34) becomes

$$|\Psi_S(\tau_1\tau_2)\rangle=U_0(\tau_1\tau_2)|\psi_{\rm in}\rangle-\int_{-\infty}^{\tau_1}d\tau_1'\int_{-\infty}^{\tau_2}d\tau_2' e^{iK_1^0(\tau_1'-\tau_1)}e^{iK_2^0(\tau_2'-\tau_2)}V|\Psi_S(\tau_1'\tau_2')\rangle\ . \tag{5.11}$$

Again, taking the in state to be sharp in K_1^0,K_2^0, we find that Eq. (5.11) can be written in the Heisenberg picture as

$$\begin{aligned}e^{-i(k_1{}^2+m_1{}^2)\tau_1}e^{-i(k_2{}^2+m_2{}^2)\tau_2}|\Psi_H(k_1k_2)\rangle\\ &=e^{-i(k_1{}^2+m_1{}^2)\tau_1}e^{-i(k_2{}^2+m_2{}^2)\tau_2}|\psi_{\rm in}(k_1k_2)\rangle\\ &\quad-\int_{-\infty}^{\tau_1}d\tau_1'\int_{-\infty}^{\tau_2}d\tau_2' e^{iK_1^0(\tau_1'-\tau_2)}e^{iK_2^0(\tau_2'-\tau_2)}Ve^{-i(k_1{}^2+m_1{}^2)\tau_1'}e^{-i(k_2{}^2+m_2{}^2)\tau_2'}|\Psi_H(k_1k_2)\rangle\ .\end{aligned}$$

When one brings the exponentials on the left over to the right-hand side, one can introduce $s_1=\tau_1'-\tau_1$, $s_2=\tau_2'-\tau_2$ as new variables in the integrals, which now extend from $-\infty$ to 0. We include the outgoing wave condition with the factors $e^{\epsilon_i s_i}$ $(i=1,2)$ and the limit $\epsilon_i\to 0_+$. The result is now

$$|\Psi_H^{(+)}(k_1k_2)\rangle=|\psi_{\rm in}(k_1k_2)\rangle+\frac{1}{K_1^0-(k_1{}^2+m_1{}^2)-i\epsilon_1}\,\frac{1}{K_2^0-(k_2{}^2+m_2{}^2)-i\epsilon_2}V|\Psi_H^{(+)}(k_1k_2)\rangle\ . \tag{5.12}$$

Equations (5.6) and (5.12) for the Heisenberg state are equivalent. To show this, we first remark, from the definition (3.29) for V, that

$$\begin{aligned}V|\Psi_H^{(+)}(k_1k_2)\rangle&=([\Phi,K_1^0]+\Phi)^2|\Psi_H^{(+)}(k_1k_2)\rangle=(\Phi K_1-K_1^0\Phi)|\Psi_H^{(+)}(k_1k_2)\rangle\\ &=-[K_1^0-(k_1{}^2+m_1{}^2)]\Phi|\Psi_H^{(+)}(k_1k_2)\rangle\ ,\end{aligned}$$

and therefore Eq. (5.12) reduces to

$$|\Psi_H^{(+)}(k_1k_2)\rangle=|\psi_{\rm in}(k_1k_2)\rangle-\frac{1}{K_2^0-(k_2{}^2+m_2{}^2)-i\epsilon_2}\Phi|\Psi_H^{(+)}(k_1k_2)\rangle\ . \tag{5.13}$$

To show that Eqs. (5.6) and (5.13) are equivalent, it suffices to prove the validity of the relation

$$[K_1^0-(k_1{}^2+m_1{}^2)]\Phi|\Psi_H^{(+)}(k_1k_2)\rangle=[K_2^0-(k_2{}^2+m_2{}^2)]\Phi|\Psi_H^{(+)}(k_1k_2)\rangle\ , \tag{5.14}$$

in which case the denominator of the second term in (5.6) becomes identical to that of (5.13). The relation (5.14) follows directly from the first-class constraint condition (3.3):

$$[K_1^0-K_2^0,\Phi]|\Psi_H(k_1k_2)\rangle=\{K_1^0-K_2^0-[(k_1{}^2+m_1{}^2)-(k_2{}^2+m_2{}^2)]\}\Phi|\Psi_H(k_1k_2)\rangle=0\ . \tag{5.15}$$

With this result, we have shown that (5.6) and (5.12) are equivalent. However, Eq. (5.12) is in a form which as we shall see in later sections, can be generalized to the N-body problem, whereas (5.6) is, in the framework of constraint Hamiltonian dynamics, special to the two-body case.

It is instructive to show that the generalized form (5.12) of the Lippmann-Schwinger equation can also be derived by postponing the assumption of "continuum eigenstates" to the very end. To do this, we rewrite Eq. (5.7) in terms of τ_1,τ_2, and consider the second derivative:

$$\frac{\partial^2}{\partial\tau_1\partial\tau_2}[U_0{}^{-1}(\tau_1\tau_2)U(\tau_1\tau_2)]=-e^{iK_2^0\tau_2}e^{iK_1^0\tau_1}Ve^{-iK_1\tau_1}e^{-iK_2\tau_2}\ . \tag{5.16}$$

Integrating this equation, we obtain the identity

$$e^{iK_2^0\tau_2}e^{-iK_2\tau_2}+e^{iK_1^0\tau_1}e^{-iK_1\tau_1}-1=U_0{}^{-1}(\tau_1\tau_2)U(\tau_1\tau_2)+\int_0^{\tau_1}d\tau_1'\int_0^{\tau_2}d\tau_2' e^{iK_2^0\tau_2'}e^{iK_1^0\tau_1'}Ve^{-iK_1\tau_1'}e^{-iK_2\tau_2'}\ . \tag{5.17}$$

We now act with this result on $|\Psi_H^{(+)}\rangle$ and take the limit $\tau_1, \tau_2 \to -\infty$ to obtain

$$|\Psi_H^{(+)}\rangle = |\psi_{\rm in}\rangle - \int_{-\infty}^{0} d\tau_1' \int_{-\infty}^{0} d\tau_2' e^{iK_2^0\tau_2'} e^{iK_1^0\tau_1'} V e^{-iK_1\tau_1'} e^{-iK_2\tau_2'} |\Psi_H^{(+)}\rangle . \tag{5.18}$$

In the limit that $|\psi_{\rm in}\rangle$ becomes sharp in K_1^0, K_2^0, one obtains the expression (5.12).

By taking the adjoint of the equations leading to Eqs. (5.10) and (5.18), one observes an interesting "duality": the equations remain valid under the interchanges

$$K_i^0 \rightleftarrows K_i , \quad V \rightleftarrows V^\dagger , \quad \Phi \rightleftarrows -\Phi , \quad |\psi_{\rm in}\rangle \rightleftarrows |\Psi_H\rangle . \tag{5.19}$$

Thus, one obtains instead of (5.10),

$$|\Psi_H\rangle = |\psi_{\rm in}\rangle - i \int_{-\infty}^{0} e^{iK\tau'} \Phi e^{-iK^0\tau'} d\tau' |\psi_{\rm in}\rangle \tag{5.20}$$

and, instead of (5.18),

$$|\Psi_H\rangle = |\psi_{\rm in}\rangle + \int_{-\infty}^{0} d\tau_1' \int_{-\infty}^{0} d\tau_2' e^{iK_1\tau_1'} e^{iK_2\tau_2'} V^\dagger e^{-iK_2^0\tau_2'} e^{-iK_1^0\tau_1'} |\psi_{\rm in}\rangle . \tag{5.21}$$

When $|\psi_{\rm in}\rangle$ becomes sharp, these reduce to the "dual" to the Lippmann-Schwinger equations:

$$|\Psi_H^{(+)}(k_1k_2)\rangle = |\psi_{\rm in}(k_1k_2)\rangle - \frac{1}{K - \frac{1}{2}(k_1{}^2+m_1{}^2+k_2{}^2+m_2{}^2) - i\epsilon} \Phi |\psi_{\rm in}(k_1k_2)\rangle \tag{5.22}$$

and

$$|\Psi_H^{(+)}(k_1k_2)\rangle = |\psi_{\rm in}(k_1k_2)\rangle - \frac{1}{K_1 - (k_1{}^2+m_1{}^2) - i\epsilon_1} \frac{1}{K_2 - (k_2{}^2+m_2{}^2) - i\epsilon_2} V^\dagger |\psi_{\rm in}(k_1k_2)\rangle . \tag{5.23}$$

We shall now verify that Eqs. (5.6), (5.12), (5.22), and (5.23) are the scattering wave solutions of the spectral conditions (generalized time-independent Schrödinger eigenfunction equations) on $|\Psi_H^{(+)}(k_1k_2)\rangle$ and $|\psi_{\rm in}(k_1k_2)\rangle$. Operating on Eq. (5.12) with $K_1^0 - (k_1{}^2+m_1{}^2)$, we obtain

$$\begin{aligned}
[K_1^0 - (k_1{}^2+m_1{}^2)]\,|\Psi_H^{(+)}(k_1k_2)\rangle &= \frac{1}{K_2^0 - (k_2{}^2+m_2{}^2) - i\epsilon_2} V |\Psi_H^{(+)}(k_1k_2)\rangle \\
&= \frac{1}{K_2^0 - (k_2{}^2+m_2{}^2) - i\epsilon_2} (\Phi K_2 - K_2^0\Phi) |\Psi_H^{(+)}(k_1k_2)\rangle \\
&= -\Phi |\Psi_H^{(+)}(k_1k_2)\rangle ,
\end{aligned}$$

from which we obtain the spectral condition

$$[K_1 - (k_1{}^2+m_1{}^2)]\,|\Psi_H^{(+)}(k_1k_2)\rangle = 0 . \tag{5.24}$$

Similarly, operating on (5.23) with $K_1 - (k_1{}^2+m_1{}^2)$, we obtain

$$\begin{aligned}
[K_1 - (k_1{}^2+m_1{}^2)]\,|\Psi_H^{(+)}(k_1k_2)\rangle &= [K_1 - (k_1{}^2+m_1{}^2)]\,|\psi_{\rm in}(k_1k_2)\rangle - \frac{1}{K_2 - (k_2{}^2+m_2{}^2) - i\epsilon_2} V^\dagger |\psi_{\rm in}(k_1k_2)\rangle \\
&= \Phi |\psi_{\rm in}(k_1k_2)\rangle - \frac{1}{K_2 - (k_2{}^2+m_2{}^2) - i\epsilon_2} (K_2\Phi - \Phi K_2^0) |\psi_{\rm in}(k_1k_2)\rangle = 0 .
\end{aligned}$$

Operating on Eq. (5.6) with $K_1^0 - (k_1{}^2+m_1{}^2)$, we find

$$[K_1^0 - (k_1{}^2+m_1{}^2)]\,|\Psi_H^{(+)}(k_1k_2)\rangle = -[K_1^0 - (k_1{}^2+m_1{}^2)] \frac{1}{K^0 - \frac{1}{2}(k_1{}^2+m_1{}^2+k_2{}^2+m_2{}^2) - i\epsilon} \Phi |\Psi_H^{(+)}(k_1k_2)\rangle .$$

With the help of the relation (5.14), this reduces to (5.24). Finally, operating on (5.22) with $K_1 - (k_1{}^2+m_1{}^2)$, we obtain

$$[K_1-(k_1{}^2+m_1{}^2)]\,|\,\Psi_H^{(+)}(k_1k_2)\rangle=\Phi\,|\,\psi_{\rm in}(k_1k_2)\rangle-[K_1-(k_1{}^2+m_1{}^2)]\times\frac{1}{K-\frac{1}{2}(k_1{}^2+m_1{}^2+k_2{}^2+m_2{}^2)-i\epsilon}\Phi\,|\,\psi_{\rm in}(k_1k_2)\rangle\ . \tag{5.25}$$

With the help of a relation similar to (5.15), i.e.,

$$0=[K_1^0-K_2^0,\Phi]\,|\,\psi_{\rm in}(k_1k_2)\rangle=[(K_1-K_2)\Phi-\Phi(K_1^0-K_2^0)]\,|\,\psi_{\rm in}(k_1k_2)\rangle$$
$$=\{[K_1-(k_1{}^2+m_1{}^2)]-[K_2-(k_2{}^2+m_2{}^2)]\}\,|\,\psi_{\rm in}(k_1k_2)\rangle$$

the numerator and denominator in the second term of (5.25) cancel, and we again obtain (5.24). These arguments can be repeated for $K_2-(k_2{}^2+m_2{}^2)$, and we therefore see that the Lippmann-Schwinger equations we have obtained are solutions of the spectral conditions with appropriate asymptotic behavior.

For sufficiently weak potentials, Eq. (5.6) and (5.12) admit a perturbation solution for $|\,\Psi_H^{(+)}(k_1k_2)\rangle$. We give the expansion (Born series) of Eq. (5.12) explicitly. Let

$$G_a(k_a{}^2)\equiv[K_a^0-(k_a{}^2+m_a{}^2)-i\epsilon_a]^{-1}=(p_a{}^2+k_a{}^2-i\epsilon_a)^{-1}\ . \tag{5.26}$$

Then, an iteration of Eq. (5.12) yields

$$|\,\Psi_H^{(+)}(k_1k_2)\rangle=[1+G_1(k_1{}^2)G_2(k_2{}^2)V+G_1(k_1{}^2)G_2(k_2{}^2)VG_1(k_1{}^2)G_2(k_2{}^2)V+\cdots]\,|\,\psi_{\rm in}(k_1k_2)\rangle\ . \tag{5.27}$$

A similar expansion exists for $|\,\Psi_H^{(-)}(k_1'k_2')\rangle$. The S matrix can then be found as

$$\langle k_1'k_2'\,|\,S\,|\,k_1k_2\rangle=\langle\Psi_H^{(-)}(k_1'k_2')\,|\,\Psi_H^{(+)}(k_1k_2)\rangle\ . \tag{5.28}$$

In a manner similar to nonrelativistic scattering theory,[11] one can also obtain expressions for the T matrix. Let us first consider the single-time version of the theory. Consider the difference, from Eq. (5.6),

$$|\,\Psi_H^{(+)}(k_1k_2)\rangle-|\,\Psi_H^{(-)}(k_1k_2)\rangle=-\left[\frac{1}{K-\frac{1}{2}(k_1{}^2+m_1{}^2+k_2{}^2+m_2{}^2)-i\epsilon}-\frac{1}{K-\frac{1}{2}(k_1{}^2+m_1{}^2+k_2{}^2+m_2{}^2)+i\epsilon}\right]\Phi\,|\,\psi_0(k_1k_2)\rangle\ , \tag{5.29}$$

where we have called the free sharp wave function $|\,\psi_0(k_1k_2)\rangle$ and used the fact that $|\,\Psi_H^{(-)}(k_1k_2)\rangle$ differs from $|\,\Psi_H^{(+)}(k_1k_2)\rangle$ only in the sign of ϵ. Formally we may write (5.29) as

$$|\,\Psi_H^{(+)}(k_1k_2)\rangle-|\,\Psi_H^{(-)}(k_1k_2)\rangle=-2\pi i\delta(K-\tfrac{1}{2}(k_1{}^2+m_1{}^2+k_2{}^2+m_2{}^2))\Phi\,|\,\psi_0(k_1k_2)\rangle\ , \tag{5.30}$$

from which we obtain the two equivalent relations

$$\langle\psi_0(k_1'k_2')\,|\,S\,|\,\psi_0(k_1k_2)\rangle=\langle\Psi_H^{(-)}(k_1'k_2')\,|\,\Psi_H^{(+)}(k_1k_2)\rangle$$
$$=\delta^4(k_1'-k_1)\delta^4(k_2'-k_2)-2\pi i\delta(\tfrac{1}{2}(k_1{}^2+k_2{}^2-k_1'^2-k_2'^2))\langle\psi_0(k_1'k_2')\,|\,\Phi\,|\,\Psi_H^{(+)}(k_1k_2)\rangle \tag{5.31}$$
$$=\delta^4(k_1'-k_1)\delta^4(k_2'-k_2)-2\pi i\delta(\tfrac{1}{2}(k_1{}^2+k_2{}^2-k_1'^2-k_2'^2))\langle\Psi_H^{(-)}(k_1'k_2')\,|\,\Phi\,|\,\psi_0(k_1k_2)\rangle\ . \tag{5.32}$$

Defining the T matrix by the relation[4]

$$S=1-2\pi iT\ , \tag{5.33}$$

we obtain

$$\langle\psi_0(k_1'k_2')\,|\,T\,|\,\psi_0(k_1k_2)\rangle=\delta(\tfrac{1}{2}(k_1{}^2+k_2{}^2-k_1'^2-k_2'^2))\langle\psi_0(k_1'k_2')\,|\,\Phi\,|\,\Psi_H^{(+)}(k_1k_2)\rangle$$
$$=\delta(\tfrac{1}{2}(k_1{}^2+k_2{}^2-k_1'^2-k_2'^2))\langle\Psi_H^{(-)}(k_1'k_2')\,|\,\Phi\,|\,\psi_0(k_1k_2)\rangle\ . \tag{5.34}$$

We now turn to the two-time version of the theory, and make use of Eq. (5.23). The difference $|\Psi_H^{(+)}(k_1k_2)\rangle - |\Psi_H^{(-)}(k_1k_2)\rangle$ contains the operator

$$-\left[\frac{1}{K_1-(k_1{}^2+m_1{}^2)-i\epsilon_1}\,\frac{1}{K_2-(k_2{}^2+m_2{}^2)-i\epsilon_2}-\frac{1}{K_1-(k_1{}^2+m_1{}^2)+i\epsilon_1}\,\frac{1}{K_2-(k_2{}^2+m_2{}^2)+i\epsilon_2}\right]V^\dagger$$

$$=-2\pi i\left[\delta(K_1-(k_1{}^2+m_2{}^2))P\frac{1}{K_2-(k_2{}^2+m_2{}^2)}+\delta(K_2-(k_2{}^2+m_2{}^2))P\frac{1}{K_1-(k_1{}^2+m_1{}^2)}\right]V^\dagger$$

$$\equiv -2\pi i\mathscr{V}(k_1{}^2,k_2{}^2)\ . \tag{5.35}$$

We may therefore write

$$|\Psi_H^{(+)}(k_1k_2)\rangle - |\Psi_H^{(-)}(k_1k_2)\rangle = -2\pi i\mathscr{V}(k_1{}^2,k_2{}^2)\,|\psi_0(k_1k_2)\rangle\ , \tag{5.36}$$

and hence

$$\begin{aligned}\langle\psi_0(k_1'k_2')\,|\,T\,|\,\psi_0(k_1k_2)\rangle &= \left[\delta(k_1'^2-k_1{}^2)P\frac{1}{k_2'^2-k_2{}^2}+\delta(k_2'^2-k_2{}^2)P\frac{1}{k_1'^2-k_1{}^2}\right]\\ &\quad\times\langle\Psi_H^{(-)}(k_1'k_2')\,|\,V^\dagger\,|\,\psi_0(k_1k_2)\rangle\\ &= \left[\delta(k_1'^2-k_1{}^2)P\frac{1}{k_2'^2-k_2{}^2}+\delta(k_2'^2-k_2{}^2)P\frac{1}{k_1'^2-k_1{}^2}\right]\\ &\quad\times\langle\psi_0(k_1'k_2')\,|\,V\,|\,\Psi_H^{(+)}(k_1k_2)\rangle\ .\end{aligned} \tag{5.37}$$

The equivalence of Eqs. (5.37) and the single-time form of (5.34) is easily established by utilizing the definition of V, e.g., for the second term of (5.37),

$$\begin{aligned}\langle\psi_0(k_1'k_2')\,|\,[\Phi,K_1^0]+\Phi^2\,|\,\Psi_H^{(+)}(k_1k_2)\rangle &= \langle\psi_0(k_1'k_2')\,|\,\Phi K_1-K_1^0\Phi\,|\,\Psi_H^{(+)}(k_1k_2)\rangle\\ &=(k_1{}^2-k_1'^2)\langle\psi_0(k_1'k_2')\,|\,\Phi\,|\,\Psi_H^{(+)}(k_1k_2)\rangle\ .\end{aligned} \tag{5.38}$$

The second of Eqs. (5.37) then becomes

$$\langle\psi_0(k_1'k_2')\,|\,T\,|\,\psi_0(k_1k_2)\rangle=\delta(k_2{}^2-k_2'^2)\langle\psi_0(k_1'k_2')\,|\,\Phi\,|\,\Psi_H^{(+)}(k_1k_2)\rangle\ . \tag{5.39}$$

From Eq. (5.15), it follows that

$$[(k_1'^2-k_2'^2)-(k_1{}^2-k_2{}^2)]\langle\psi_0(k_1'k_2')\,|\,\Phi\,|\,\Psi_H(k_1k_2)\rangle=0\ ,$$

so that the matrix element in (5.39) is proportional to $\delta(k_1'^2-k_2'^2-(k_1{}^2-k_2{}^2))$; the result therefore coincides precisely with the first of Eqs. (5.34).

VI. SCATTERING FOR $N\geq 3$

A simplification of the type which occurs in Eq. (3.8) can occur for $N\geq 3$ only if[3] $\Phi_1=\Phi_2=\cdots=\Phi_N$; however, this seems to be too strong a restriction.[12] We shall therefore follow a more general procedure.

A. The case $N=3$

1. $|\tau_1\tau_2\tau_3\rangle$ *in perturbation expansion*

We first study the three-body case in some detail. Following the procedure leading to Eq. (3.28) for the three-body case, we find

$$i^3\frac{\partial^3}{\partial\tau_1\partial\tau_2\partial\tau_3}\,|\,\tau_1\tau_2\tau_3\rangle=V_{123}(\tau_1\tau_2\tau_3)\,|\,\tau_1\tau_2\tau_3\rangle \tag{6.1}$$

where

$$V_{123}(\tau_1\tau_2\tau_3)=[[\phi_1,K_2^0],K_3^0]+[\phi_1,K_3^0]\phi_2+\phi_1[\phi_2,K_3^0]+[\phi_1,K_2^0]\phi_3+\phi_1\phi_2\phi_3\ . \tag{6.2}$$

The right-hand side of Eq. (6.2) can equivalently contain any permutation of the indices 1,2,3 corresponding to the interchangeability of the partial derivatives in Eq. (6.1). The validity of this symmetry (effective on the set of states $\{\,|\,\tau_1\tau_2\tau_3\rangle\}$) follows from the (weak) integrability conditions, Eq. (3.10) of I, and is shown in Appendix 1 of I for any N.

Integrating Eq. (6.1) between arbitrary limits, we obtain

$$\begin{aligned}&|\,\tau_1\tau_2\tau_3\rangle-|\,\sigma_1\tau_2\tau_3\rangle-|\,\tau_1\sigma_2\tau_3\rangle-|\,\tau_1\tau_2\sigma_3\rangle+|\,\tau_1\sigma_2\sigma_3\rangle+|\,\sigma_1\tau_2\sigma_3\rangle+|\,\sigma_1\sigma_2\tau_3\rangle-|\,\sigma_1\sigma_2\sigma_3\rangle\\&\qquad=i\int_{\sigma_1}^{\tau_1}d\tau_1'\int_{\sigma_2}^{\tau_2}d\tau_2'\int_{\sigma_3}^{\tau_3}d\tau_3'V_{123}(\tau_1'\tau_2'\tau_3')\,|\,\tau_1'\tau_2'\tau_3'\rangle\ .\end{aligned} \tag{6.3}$$

As remarked at the end of Sec. IV, one may add all of the versions of Eq. (6.3) obtained by taking all semi-infinite integrals excluding only the orthohedron (in Cartesian $\tau_1\tau_2\tau_3$ space) which has the out state $|\,\infty\,\infty\,\infty\,\rangle=|\,\psi_{\rm out}\rangle$ at one corner. On would obtain in this way an integral equation which contains all permissible crossed diagrams in its iterated expansion. We shall, however, restrict our attention in this section to the somewhat simpler equation obtained by taking $[\sigma]\to-\infty$:

$$\begin{aligned}&|\,\tau_1\tau_2\tau_3\rangle-|\,-\infty\,\tau_2\tau_3\rangle-|\,\tau_1-\infty\,\tau_3\rangle-|\,\tau_1\tau_2-\infty\,\rangle+|\,\tau_1-\infty-\infty\,\rangle+|\,-\infty\,\tau_2-\infty\,\rangle\,|\,-\infty-\infty\,\tau_3\rangle\\&\qquad=|\,\psi_{\rm in}\rangle+i\int_{-\infty}^{\tau_1}d\tau_1'\int_{-\infty}^{\tau_2}d\tau_2'\int_{-\infty}^{\tau_3}d\tau_3'V_{123}(\tau_1'\tau_2'\tau_3')\,|\,\tau_1'\tau_2'\tau_3'\rangle\ .\end{aligned} \tag{6.4}$$

There are two types of inhomogeneous terms other than $|\,\psi_{\rm in}\rangle$. For terms of the type $|\,\tau_1\tau_2-\infty\,\rangle$, a relation with $|\,\tau_1\tau_2\tau_3\rangle$ can be constructed which depends on a knowledge of the partial one-body wave operators[13] $\Omega_+^{(a)}$. Terms of the type $|\,\tau_1-\infty-\infty\,\rangle$ can be related to $|\,\tau_1\tau_2\tau_3\rangle$ with a knowledge of the two-body wave operators $\Omega_+^{(ab)}$. We shall construct these relations in the following.

From Eqs. (2.2) and (2.6), we obtain

$$|\,\tau_1\tau_2\tau_3\rangle=e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}e^{-iK_3\tau_3}e^{-iK_2\tau_2}e^{-iK_1\tau_1}\,|\,\Psi_H\rangle\ . \tag{6.5}$$

From this relation, we find

$$|\,\tau_1\tau_2-\infty\,\rangle=e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}\Omega_+^{(3)\dagger}e^{-iK_2\tau_2}e^{-iK_1\tau_1}\,|\,\Psi_H\rangle=U_0^{-1}(\tau_1\tau_2\tau_3)\Omega_+^{(3)\dagger}U_0(\tau_1\tau_2\tau_3)\,|\,\tau_1\tau_2\tau_3\rangle\ , \tag{6.6}$$

where we have used the intertwining property of $\Omega_+^{(3)}$. In a similar way, we find

$$\begin{aligned}|\,\tau_1-\infty-\infty\,\rangle&=\lim_{\tau_2\to-\infty}e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}\Omega_+^{(3)\dagger}e^{-iK_2\tau_2}e^{-iK_1\tau_1}\,|\,\Psi_H\rangle\\&=e^{iK_1^0\tau_1}\Omega_+^{(23)\dagger}e^{-iK_1\tau_1}\,|\,\Psi_H\rangle=U_0^{-1}(\tau_1\tau_2\tau_3)\Omega_+^{(23)\dagger}U_0(\tau_1\tau_2\tau_3)\,|\,\tau_1\tau_2\tau_3\rangle\ .\end{aligned} \tag{6.7}$$

Equation (6.4) can then be written as

$$W_+^\dagger(\tau_1\tau_2\tau_3)\,|\,\tau_1\tau_2\tau_3\rangle=|\,\psi_{\rm in}\rangle+i\int_{-\infty}^{\tau_1}d\tau_1'\int_{-\infty}^{\tau_2}d\tau_2'\int_{-\infty}^{\tau_3}d\tau_3'V_{123}(\tau_1'\tau_2'\tau_3')\,|\,\tau_1'\tau_2'\tau_3'\rangle\ , \tag{6.8}$$

where $W_+(\tau_1\tau_2\tau_3)$ is the interaction-picture form of

$$W_+=1-\Omega_+^{(1)}-\Omega_+^{(2)}-\Omega_+^{(3)}+\Omega_+^{(23)}+\Omega_+^{(31)}+\Omega_+^{(12)}\ . \tag{6.9}$$

Since the leading contribution to W_+ for small enough potentials is unity, it has an inverse, and we may define

$$V_{123}'=V_{123}W_+^{\dagger-1}\ . \tag{6.10}$$

For sufficiently small potentials, one may reasonably hope that the iterative expansion of Eq. (6.8),

$$W_+^\dagger(\tau_1\tau_2\tau_3)\,|\,\tau_1\tau_2\tau_3\rangle = |\,\psi_{\rm in}\rangle + i\int_{-\infty}^{\tau_1} d\tau_1'\int_{-\infty}^{\tau_2} d\tau_2'\int_{-\infty}^{\tau_3} d\tau_3' V_{123}'(\tau_1'\tau_2'\tau_3')\,|\,\psi_{\rm in}\rangle$$
$$+i^2\int_{-\infty}^{\tau_1} d\tau_1'\int_{-\infty}^{\tau_2} d\tau_2'\int_{-\infty}^{\tau_3} d\tau_3'\int_{-\infty}^{\tau_1'} d\tau_1''\int_{-\infty}^{\tau_2'} d\tau_2''\int_{-\infty}^{\tau_3'} d\tau_3'' V_{123}'(\tau_1'\tau_2'\tau_3') \times V_{123}'(\tau_1''\tau_2''\tau_3'')\,|\,\psi_{\rm in}\rangle\ ,$$
$$+\cdots \tag{6.11}$$

converges. The momentum-space representation of (6.11) is similar to Eq. (4.12) with, however, three Feynman propagators going with each factor V' and the matrix of $W_+^\dagger(\tau_1\tau_2\tau_3)^{-1}$ multiplying each term. In the limit $\tau_1\tau_2\tau_3\rightarrow+\infty$, $|\,\tau_1\tau_2\tau_3\rangle\rightarrow|\,\psi_{\rm out}\rangle$, and the series (6.11) serves as a perturbation expansion for the S matrix. Since $W_+(\infty,\infty,\infty)$ commutes with K_1^0,K_2^0, and K_3^0, the S matrix conserves the individual particle masses (the integrals with infinite upper limits also conserve individual particle masses).

2. $\Omega_+^{(a)}$ and $\Omega_+^{(ab)}$ for $\Phi_a\neq\Phi_b$

The utility of the perturbation expansion (6.11) depends on a knowledge of the one- and two-body wave operators. The special techniques applicable to the simple case of the two-body problem (for which $\Phi_1=\Phi_2$) do not apply to the computation of one- and two-body subproblems for $N\geq 3$. We shall therefore have to construct the one- and two-body wave operators in the framework of the three-body problem. It is convenient to do this by means of the Lippmann-Schwinger equations.

Let $|\,\psi_{\rm in}\rangle$ be an in state, and define the partial Heisenberg state

$$|\,\Psi_H^{12(+)}\rangle\equiv\Omega_+^{(12)}\,|\,\psi_{\rm in}\rangle\ . \tag{6.12}$$

Then, following the procedure of Sec. V, we study

$$|\,\psi_{\rm in}\rangle=\Omega_+^{(12)\dagger}\Omega_+^{(12)}\,|\,\psi_{\rm in}\rangle = \lim_{\tau_1,\tau_2\rightarrow-\infty} e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}e^{-iK_1\tau_1}e^{-iK_2\tau_2}\,|\,\Psi_H^{(12)(+)}\rangle\ . \tag{6.13}$$

Equation (5.17) provides a valid representation of the operator appearing on the right-hand side of (6.13). Taking the limits as indicated, we obtain

$$W_+^{12\dagger}\,|\,\Psi_H^{12(+)}\rangle=|\,\psi_{\rm in}\rangle+\int_{-\infty}^{0}d\tau_1\int_{-\infty}^{0}d\tau_2 e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}V_{12}e^{-iK_1\tau_1}e^{-iK_2\tau_2}\,|\,\Psi_H^{12(+)}\rangle\ , \tag{6.14}$$

where

$$W_+^{12}=\Omega_+^{(1)}+\Omega_+^{(2)}-1\ . \tag{6.15}$$

We now take the in state to be sharp in p_1p_2 at the values k_1,k_2 (but not necessarily sharp in p_3). By the intertwining property of $\Omega_+^{(12)}$, $|\,\Psi_H^{12(+)}\rangle$ is then also sharp in p_1p_2, and Eq. (6.14) becomes

$$W_+^{12\dagger}\,|\,\Psi_H^{12(+)}(k_1k_2)\rangle=|\,\psi_{\rm in}(k_1k_2)\rangle-\frac{1}{K_1^0-(k_1{}^2+m_1{}^2)-i\epsilon_1}\,\frac{1}{K_2^0-(k_2{}^2+m_2{}^2)-i\epsilon_1}V_{12}\,|\,\Psi_H^{12(+)}(k_1k_2)\rangle\ . \tag{6.16}$$

For small enough potentials, $(W_+^{12\dagger})^{-1}$ exists, and this equation may be iterated to obtain a perturbation expansion for $\Omega_+^{(12)}$. Note that this equation differs from Eq. (5.12) for the simple case of the isolated two-body problem in the factor $W_+^{12\dagger}$ on the left-hand side, and a sign on the right-hand side. For the isolated two-body problem, $\Omega_+^{(1)}=\Omega_+^{(2)}=\Omega_+$ (the two-body wave operator), and Eq. (6.16) would then reduce to (5.12).

Taking the adjoint of Eq. (5.17) and acting on $|\,\psi_{\rm in}(k_1k_2)\rangle$, we obtain, in the limit $\tau_1,\tau_2\rightarrow-\infty$, the dual relation

$$|\,\Psi_H^{12(+)}(k_1k_2)\rangle=W_+^{12}\,|\,\psi_{\rm in}(k_1k_2)\rangle+\frac{1}{K_1-(k_1{}^2+m_1{}^2)-i\epsilon_1}\,\frac{1}{K_2-(k_2{}^2+m_2{}^2)-i\epsilon_2}V_{12}^\dagger\,|\,\psi_{\rm in}(k_1k_2)\rangle\ . \tag{6.17}$$

To obtain the single-particle wave operators of the three-body problem, we shall proceed in a way that yields relations similar to the single-time result (5.6). Transcribing Eq. (5.9) to the form

$$U_0^{-1}(\tau_1)U(\tau_1)-1=-i\int_0^{\tau} d\tau_1' e^{iK_1^0\tau_1'}\Phi_1 e^{-iK_1\tau_1'} \tag{6.18}$$

for particle 1, we define

$$|\Psi_H'^{(+)}\rangle=\Omega_+^{(1)}|\psi_{\rm in}\rangle \tag{6.19}$$

to obtain, in the case of $|\psi_{\rm in}\rangle$ sharp in p_1 with value k_1,

$$|\Psi_H'^{(+)}(k_1)\rangle=|\psi_{\rm in}(k_1)\rangle-\frac{1}{K_1^0-(k_1{}^2+m_1{}^2)-i\epsilon_1}\Phi_1|\Psi_H'^{(+)}(k_1)\rangle\ . \tag{6.20}$$

This equation may be iterated to obtain a perturbation expansion for $\Omega_+^{(1)}$. No additional inhomogeneous terms appear in the one-body subproblem. The dual relation to (6.20) is then found, as in Eq. (5.22), to be

$$|\Psi_H'^{(+)}(k_1)\rangle=|\psi_{\rm in}(k_1)\rangle-\frac{1}{K_1-(k_1{}^2+m_1{}^2)-i\epsilon}\Phi_1|\psi_{\rm in}(k_1)\rangle\ . \tag{6.21}$$

Equations (6.17) and (6.20) must be solved for particle pairs 12, 23, and 31 and the single particles 1, 2, and 3, respectively, in order to have available all of the terms in Eq. (6.9).

3. *The S matrix*

In order to compute the S matrix for the three-body problem from (6.11), we must study the limit

$$\lim_{\tau_1,\tau_2,\tau_3\to\infty} W_+(\tau_1\tau_2\tau_3)=W_+(\infty,\infty,\infty)\ .$$

Consider, in particular, the one-particle contribution

$$\begin{aligned}\Omega_+^{(1)}(\infty,\infty,\infty)&=\lim_{\tau_1,\tau_2,\tau_3\to\infty} e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}\Omega_+^{(1)}e^{-iK_1^0\tau_1}e^{-iK_2^0\tau_2}e^{-iK_3^0\tau_3}\\ &=\lim_{\tau_2,\tau_3\to\infty} e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}S^{(1)}e^{-iK_2^0\tau_2}e^{-iK_3^0\tau_3}\ ,\end{aligned} \tag{6.22}$$

where $\Omega_+^{(1)}(\tau_1\tau_2\tau_3)$ is the interaction picture form of $\Omega_+^{(1)}$, and $S^{(1)}=\Omega_-^{(1)\dagger}\Omega_+^{(1)}$ is the one-particle S operator. According to (6.20), $\Omega_+^{(1)}$ is a function of K_1^0,K_1. The limit (6.22) applied to K_1^0 is trivial, and applied to K_1 it results in

$$\begin{aligned}\lim_{\tau_2\tau_3\to\infty} e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}K_1e^{-iK_2^0\tau_2}e^{-iK_3^0\tau_3}&=\lim_{\tau_2\tau_2\to\infty} e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}e^{-iK_2\tau_2}e^{-iK_3\tau_3}K_1e^{iK_2\tau_2}e^{iK_3\tau_3}e^{-iK_2^0\tau_2}e^{-iK_3^0\tau_3}\\ &=\Omega_-^{(23)\dagger}K_1\Omega_-^{(23)}\ .\end{aligned} \tag{6.23}$$

We then obtain the formal relation

$$\Omega_+^{(1)}(\infty,\infty,\infty)=\lim_{\tau_2,\tau_3\to\infty} e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}S^{(1)}(K_1,K_1^0)e^{-iK_2^0\tau_2}e^{-iK_3^0\tau_3}=S^{(1)}(\Omega_-^{(23)\dagger}K_1\Omega_-^{(23)},K_1^0)\ , \tag{6.24}$$

with similar results for $\Omega_+^{(2)},\Omega_+^{(3)}$.

For the two-particle contribution $\Omega_+^{(12)}$, we obtain

$$\lim_{\tau_1\tau_2\tau_3\to\infty} e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}\Omega_+^{(12)}e^{-iK_1^0\tau_1}e^{-iK_2^0\tau_2}e^{-iK_3^0\tau_3}=\lim_{\tau_3\to\infty} e^{iK_3^0\tau_3}S^{(12)}e^{-iK_3^0\tau_3}\ . \tag{6.25}$$

According to Eq. (6.16), $S^{(12)}$ is a function at K_1,K_2,K_1^0,K_2^0 only. The action of the mapping (6.25) on K_1^0,K_2^0 is trivial, but

$$\lim_{\tau_3\to\infty} e^{iK_3^0\tau_3}K_ie^{-iK_3^0\tau_3}=\Omega_-^{(3)\dagger}K_i\Omega_-^{(3)}$$

for $i=1,2$, and hence we obtain the formal expression

$$\Omega_+^{(12)}(\infty,\infty,\infty)=S^{12}(\Omega_-^{(3)\dagger}K_1\Omega_-^{(3)},\Omega_-^{(3)\dagger}K_2\Omega_-^{(3)},K_1^0,K_2^0)\ . \tag{6.26}$$

The S operators with modified evolution operators can be obtained by calculating the associated modified wave operators using the perturbative techniques we have discussed above.

The existence of the modified wave operator $\Omega_+^{(1)}(\Omega_-^{(23)\dagger}K_1\Omega_-^{(23)},K_1^0)$ is assured if there is a dense set of ψ's for which

$$\int_{-\infty}^{0}||(\Omega_-^{(23)\dagger}K_1\Omega_-^{(23)}-K_1^0)e^{-iK_1^0\tau_1}\psi||d\tau_1<\infty\ . \tag{6.27}$$

Since $\Omega_-^{(23)}$ is unitary (we assume no discrete eigenvalues), (6.27) is equivalent to

$$\int_{-\infty}^{0}||(K_1\Omega_-^{(23)}-\Omega_-^{(23)}K_1^0)e^{-iK_1^0\tau_1}\psi||d\tau_1<\infty \tag{6.28}$$

which is the condition for the existence of $\Omega_{+--}^{(1\,2\,3)}$ (see I), where we indicate the sense of the τ_i limits in the same order as the sequence of evolution generators ($\Omega_{+--}^{(1\,2\,3)}=\Omega_{-+-}^{(2\,1\,3)}$). This result also follows by examining

$$\Omega_+^{(1)}(\Omega_-^{(23)\dagger}K_1\Omega_-^{(23)},K_1^0)=\lim_{\tau_1\to-\infty}e^{i\Omega_-^{(23)\dagger}K_1\Omega_-^{(23)}\tau_1}e^{-iK_1^0\tau_1}=\lim_{\tau_1\to-\infty}\Omega_-^{(23)\dagger}e^{iK_1\tau_1}\Omega_-^{(23)}e^{-iK_1^0\tau_1}$$
$$=\Omega_-^{(23)\dagger}\Omega_{+--}^{(1\,2\,3)}\ . \tag{6.29}$$

Similarly,

$$\Omega_+^{(12)}(\Omega_-^{(3)\dagger}K_1\Omega_-^{(3)},\Omega_-^{(3)\dagger}K_2\Omega_-^{(3)},K_1^0,K_2^0)=\lim_{\tau_1,\tau_2\to-\infty}e^{i\Omega_-^{(3)\dagger}K_1\Omega_-^{(3)}\tau_1}e^{i\Omega_-^{(3)\dagger}K_2\Omega_-^{(3)}\tau_2}e^{-iK_1^0\tau_1}e^{-iK_2^0\tau_2}$$
$$=\lim_{\tau_1,\tau_2\to-\infty}\Omega_-^{(3)\dagger}e^{iK_1\tau_1}e^{iK_2\tau_2}\Omega_-^{(3)}e^{-iK_1^0\tau_1}e^{-iK_2^0\tau_2}$$
$$=\Omega_-^{(3)\dagger}\Omega_{++-}^{(1\,2\,3)}\ . \tag{6.30}$$

The conditions for the existence of any of the eight three-body wave operators are essentially the same. Their existence assures the existence of the modified one- and two-body wave operators required for the calculation of $W_+(\infty,\infty,\infty)$.

4. *The Lippmann-Schwinger equation*

To complete our study of the three-body problem, we follow the procedures of Sec. V to obtain the Lippmann-Schwinger equations corresponding to Eq. (6.8). The third crossed derivative of $U_0{}^{-1}U$ is

$$\frac{\partial^3}{\partial\tau_1\partial\tau_2\partial\tau_3}(U_0{}^{-1}U)=(-i)^3e^{iK_1^0\tau_1}e^{iK_2^0\tau_2}e^{iK_3^0\tau_3}V_{123}e^{-iK_1\tau_1}e^{-iK_2\tau_2}e^{-iK_3\tau_3}\ . \tag{6.31}$$

Integrating this expression, we obtain

$$\begin{aligned}U_0{}^{-1}(\tau_1\tau_2\tau_3)U(\tau_1\tau_2\tau_3)-i\int_0^{\tau_1}d\tau_1'\int_0^{\tau_2}d\tau_2'\int_0^{\tau_3}d\tau_3'U_0{}^{-1}(\tau_1'\tau_2'\tau_3')V_{123}U(\tau_1'\tau_2'\tau_3')\\
=U_0^\dagger(\tau_1\tau_2)U(\tau_1\tau_2)+U_0^\dagger(\tau_1\tau_3)U(\tau_1\tau_3)+U_0^\dagger(\tau_2\tau_3)U(\tau_2\tau_3)\\
-U_0^\dagger(\tau_1)U(\tau_1)-U_0^\dagger(\tau_2)U(\tau_2)-U_0^\dagger(\tau_3)U(\tau_3)+1\ .\end{aligned} \tag{6.32}$$

Multiplying by $|\Psi_H^{(+)}\rangle$ and taking the limit as $\tau_1,\tau_2,\tau_3\to-\infty$, for $|\psi_{\rm in}\rangle$ sharp in all three momenta, one obtains the three-body relativistic Lippmann-Schwinger equation:

$$W_+^\dagger \,|\,\Psi_H^{(+)}(k_1k_2k_3)\rangle = |\,\psi_{\rm in}(k_1k_2k_3)\rangle - \frac{1}{K_1^0-(k_1{}^2+m_1{}^2)-i\epsilon_1}\,\frac{1}{K_2^0-(k_2{}^2+m_2{}^2)-i\epsilon_2} \times \frac{1}{K_3^0-(k_3{}^2-m_3{}^2)i\epsilon_3} V_{123}\,|\,\Psi_H^{(+)}(k_1k_2k_3)\rangle \ . \tag{6.33}$$

Taking the adjoint of Eq. (6.32) and operating on $|\,\psi_{\rm in}(k_1k_2k_3)\rangle$, we obtain, for $\tau_1,\tau_2,\tau_3 \to -\infty$, the dual relation

$$|\,\Psi_H^{(+)}(k_1k_2k_3)\rangle = W_+\,|\,\psi_{\rm in}(k_1k_2k_3)\rangle - \frac{1}{K_1-(k_1{}^2+m_1{}^2)-i\epsilon_1}\,\frac{1}{K_2-(k_2{}^2+m_2{}^2)-i\epsilon_2} \times \frac{1}{K_3-(k_3{}^2-m_3{}^2)-i\epsilon_3} V_{123}^\dagger\,|\,\psi_{\rm in}(k_1k_2k_3)\rangle \ . \tag{6.34}$$

An iterative solution of Eq. (6.33) can be used (with the corresponding expression for $|\,\Psi_H^{(-)}\rangle$) to define the S matrix, or, following the method of Sec. V, the T matrix. This solution has the form of Eq. (5.27), but with V' [defined in Eq. (6.10)] in place of V, three Feynman propagators following each action of V', and an overall factor of $(W_+^\dagger)^{-1}$.

B. The case $N>3$

We now turn to the general case $N>3$. Differentiating the equation

$$i\frac{\partial}{\partial\tau_a}\,|\,[\tau]\rangle = \phi_a([\tau])\,|\,[\tau]\rangle \tag{6.35}$$

an additional $N-1$ times, with respect to the other time parameters, we obtain

$$i^N\frac{\partial^N}{\partial\tau_1\cdots\partial\tau_N}\,|\,[\tau]\rangle = V_N\,|\,[\tau]\rangle \ , \tag{6.36}$$

where $V_N = V_{1,2,\ldots,N}$ is defined in Appendix 1 of I, and is shown there to be independent of the order of the indices. Integrating with respect to each τ_a, we obtain

$$|\,[\tau]\rangle - \sum_a |\,[\tau]\tau_a\to\sigma_a\rangle + \sum_{a>b} |\,[\tau]\tau_a\to\sigma_a,\tau_b\to\sigma_b\rangle - \cdots + (-1)^N\,|\,[\sigma]\rangle = (-1)^N\int_{\sigma_1}^{\tau_1}d\tau_1'\int_{\sigma_2}^{\tau_2}d\tau_2'\cdots\int_{\sigma_N}^{\tau_N}d\tau_N'\,V_N([\tau'])\,|\,[\tau']\rangle \ . \tag{6.37}$$

Again, one may add all of the versions of Eq. (6.37) obtained by taking all semi-infinite integrals excluding only the orthohedron (in Cartesian $\tau_1,\cdots,\tau_N$ space) which has the out state $|\,\infty,\cdots,\infty\rangle = |\,\psi_{\rm out}\rangle$ at one corner. The resulting integral equation would contain all permissible crossed diagrams in its iteration expansion. We shall, however, consider here only the equation obtained for $[\sigma]\to-\infty$:

$$|\,[\tau]\rangle - \sum_a |\,[\tau]\tau_a\to\tau_a\rangle + \cdots + (-1)^N\,|\,\psi_{\rm in}\rangle = (-1)^N\int_{-\infty}^{\tau_1}d\tau_1'\int_{-\infty}^{\tau_2}d\tau_2'\cdots\int_{-\infty}^{\tau_N}d\tau_N'\,V_N([\tau'])\,|\,[\tau']\rangle \ . \tag{6.38}$$

As in Eqs. (6.6) and (6.7), the inhomogeneous terms may all be related to $|\,[\tau]\rangle$ by means of wave operators for $j<N$ particle scattering processes:

$$|\,[\tau]\tau_a\to-\infty\rangle = e^{iK_1^0\tau_1}\cdots \underline{e^{iK_a^0\tau_a}}\cdots e^{iK_N^0\tau_N}\Omega_+^{(a)\dagger}e^{-iK_1\tau_1}\cdots \underline{e^{-iK_a\tau_a}}\cdots e^{-iK_N\tau_N}\,|\,\Psi_H^{(+)}\rangle \ , \tag{6.39}$$

where an underline indicates that the factor is to be deleted. Inserting the factors $e^{iK_a\tau_a}e^{-iK_a\tau_a}$ on the right of $\Omega_+^{(a)\dagger}$, and using its intertwining property, we obtain

$$|\,[\tau]\tau_a\to-\infty\rangle = \Omega_+^{(a)\dagger}([\tau])\,|\,[\tau]\rangle \ . \tag{6.40}$$

Similarly, we have

$$|[\tau]\tau_a\to-\infty,\tau_b\rangle=e^{iK_1^0\tau_1}\cdots\underline{e^{iK_a^0\tau_a}}\cdots\underline{e^{iK_b^0\tau_b}}\cdots e^{iK_N^0\tau_N}\times\Omega_+^{(ab)\dagger}e^{-iK_1\tau_1}\cdots\underline{e^{-iK_a\tau_a}}\cdots\underline{e^{-iK_b\tau_b}}\cdots e^{-iK_N\tau_N}|\Psi_H^{(+)}\rangle \tag{6.41}$$

and inserting the factors $e^{iK_a\tau_a}e^{-iK_a\tau_a}e^{iK_b\tau_b}e^{-iK_b\tau_b}$ to the right of the wave operator, we obtain

$$|[\tau]\tau_a\to-\infty,\tau_b\to-\infty\rangle=\Omega_+^{(ab)\dagger}([\tau])\,|[\tau]\rangle\ . \tag{6.42}$$

Representing all of the inhomogeneous terms in Eq. (6.38) in this way, we write it as

$${}^{(N)}W_+([\tau])^\dagger\,|[\tau]\rangle=|\psi_{\rm in}\rangle-i^N\int_{-\infty}^{\tau_1}d\tau_1'\cdots\int_{-\infty}^{\tau_N}d\tau_N'V_N([\tau'])\,|[\tau']\rangle\ , \tag{6.43}$$

where ${}^{(N)}W_+([\tau])$ is the Dirac-picture form of

$${}^{(N)}W_+=(-1)^{N+1}\left[1-\sum_a\Omega_+^{(a)}+\sum_{a<b}\Omega_+^{(ab)}-\sum_{a<b<c}\Omega_+^{(abc)}+\cdots\right] \tag{6.44}$$

including terms up to the $(N-1)$-body wave operators. Since each of the $\Omega_+^{(ab\cdots)}$ has unity as its dominant contribution for small potentials, a simple application of the binomial theorem indicates that ${}^{(N)}W_+$ also has dominant contribution unity [as in Eqs. (6.9) and (6.15)].

The iterative expansion of Eq. (6.43) has essentially the same form as Eq. (6.11), with N Feynman propagators multiplying each (N-body) amplitude, and

$$V'=V({}^{(N)}W_+^\dagger)^{-1}\ . \tag{6.45}$$

To calculate the S matrix from Eq. (6.43), the limit $[\tau]\to\infty$ must be studied. Since

$$[K_a^0,{}^{(N)}W_+([\infty])]=0 \tag{6.46}$$

for all $a=1,\ldots N$, the S matrix conserves the individual particle masses. Furthermore, since the leading contribution to ${}^{(N)}W_+$ is unity, the leading term in the expansion of the S matrix is $\delta_4(p_1-p_1')\cdots\delta_4(p_N-p_N')$. The operator ${}^{(N)}W_+([\infty])$ can be calculated perturbatively using an extension of the methods discussed earlier in this section for the three-body case. Only wave operators for the cases $n\le(N-1)$-body case are required for this calculation.

In the limit of sharp $|\psi_{\rm in}\rangle$, Eq. (6.43) reduces to the analog of the Lippmann-Schwinger equation for N-body scattering:

$${}^{(N)}W_+^\dagger\,|\Psi_H^{(+)}(k_1\cdots k_N)\rangle=|\psi_{\rm in}(k_1\cdots k_N)\rangle-\frac{1}{K_1^0-(k_1{}^2+m_1{}^2)-i\epsilon_1}\cdots\frac{1}{K_N^0-(k_N{}^2-m_N{}^2)-i\epsilon_N}V_N\,|\Psi_H^{(+)}(k_1\cdots k_N)\rangle\ . \tag{6.47}$$

An iterative solution of this equation has essentially the same form as that of Eq. (6.33), but each factor V' is accompanied by the product of N free Feynman propagators. With the corresponding expansion for $|\Psi_H^{(-)}\rangle$, the S matrix (or, following the method of Sec. V, the T matrix) can therefore be calculated perturbatively.

To obtain the direct formal solution of the scattering equation (6.47), consider the adjoint of the operator relation associated with Eq. (6.36):

$$\frac{\partial^N}{\partial\tau_1\cdots\partial\tau_N}(U^{-1}U_0)=i^NU^{-1}V_N^\dagger U_0\ . \tag{6.48}$$

Integrating this equation between $(0,[\infty])$, we may operate with the result on a sharp $|\psi_{\rm in}\rangle$ to obtain

$$\Omega_+^{(1,2,\ldots N)}|\psi_{\rm in}(k_1\cdots k_N)\rangle=|\Psi_H^{(+)}(k_1\cdots k_N)\rangle={}^{(N)}W_+\,|\psi_{\rm in}(k_1\cdots k_N)\rangle+(-1)^N\frac{1}{K_1-(k_1{}^2+m_1{}^2)-i\epsilon_1}\cdots\times\frac{1}{K_N-(k_N{}^2-m_N{}^2)-i\epsilon_N}V_N^\dagger\,|\psi_{\rm in}(k_1\cdots k_N)\rangle\ . \tag{6.49}$$

VII. CONCLUDING REMARKS

In the *classical* theory of relativistic direct-interaction dynamics it is not sufficient to specify the generalized mass shells K_a. The Hamiltonian $H=\sum_a \omega_a K_a$ also requires the specification of the ω_a in order to convert the many-time formalism to a single-time formalism. This conversion is a necessity on the classical level. It then leads to the problem of finding ω_a compatible with both the cluster property and the world-line condition.[5,12]

We have claimed in I that the conversion to a single-time formalism is not necessary for a quantum-mechanical scattering theory. The present paper supports this claim. In particular, we were able to prove (see Appendix) that for $N=2$ the two formulations are, in fact, equivalent and lead to identical S matrices; the expansions (4.7) and (4.13) can be shown to be equal. But it is the S matrix in the multitime form (4.13) that yields diagrams similar to the Feynman diagrams of quantum field theory.

For $N=2$ we also showed that the equivalence of the two formulations implies the asymptotic conservation of the particle masses. If the ${p'_a}^2$ of the incident wave packets are concentrated at ${p'_a}^2 \approx -{m_a}^2$, the ${p'_a}^2$ in the denominators of (4.13) can be replaced by the (negative) squared masses. The perturbation expansion then has a form closely analogous to that of quantum field theory with gauge-mediated interactions, as described by conventional Feynman diagrams.

In the time-independent version of scattering theory the two formulations lead to two different but equivalent Lippmann-Schwinger-type equations, (5.6) and (5.12), only the former having the form of the conventional nonrelativistic counterpart.

Turning to the many-body case, $N \geq 3$, we find that, in general, a single-time formulation does not emerge. The two-body subproblem, required for the construction of the inhomogeneous terms of the equation forming the basis for the perturbation expansion, does not have the symmetry of the two-body problem in the presence of one or more of the other particles, e.g., in the three-body case, $\phi_1 \neq \phi_2$ if particle 3 is present. Hence, the two-body subproblem of an ($N \geq 3$)-body problem must be solved in greater generality. In accordance with the cluster decomposition property, the symmetries of the ($n<N$)-particle subsystem are restored when the $N-n$ other particles are at large spacelike distances.

The problem of a quantum-mechanical formulation of the relativistic N-body problem (on the first-quantized level) is, of course, an old one. The extensive literature (much too large to be quoted here) emphasizes especially the two-body system. It is cast in a Hilbert space of $L^2(\mathrm{R}^3)$. Our manifestly covariant formulation, however, led us to a Hilbert space $L^2(\mathrm{R}^4)$ and a many-τ formulation. The physical significance of such a structure has been studied elsewhere.[14]

As we have shown here, the scattering theory of this formulation of N-particle relativistic quantum dynamics leads to an S matrix which is similar to a quantum-field-theoretic S matrix, in which the τ dependence has disappeared (all $|\tau_a| \to \infty$) and in which the difference between $L^2(\mathrm{R}^4)$ and $L^2(\mathrm{R}^3)$ has become trivial (effectively sharp mass shells of free particles). Our work can thus be compared with that on pseudopotentials[15] and on the eikonal approximation.[16] Also related is the multichannel relativistic scattering theory by Coester,[17] which points toward future extension of our work. A nice recent application within this general framework (though to a bound state) is the treatment by Crater and Van Alstine[18] of the triplet quarkonium system.

In I we have presented the basic equations of an N-particle relativistic quantum dynamics that has as its classical limit the generalized mass-shell dynamics (Ref. 6 of I). In the present paper we obtain the S matrix explicitly, show how it is obtained from Lippmann-Schwinger-type equations, and present its perturbation expansion.

ACKNOWLEDGMENTS

One of us (L.P.H.) wishes to thank K. Hepp and R. Jost for their kind hospitality during his visit at E.T.H. Zürich, and for stimulating discussions, and his co-author and J. N. Goldberg for their hospitality at Syracuse University where this work was begun. He also wishes to acknowledge partial support from the Binational Science Foundation, Jerusalem, Israel.

APPENDIX

In this appendix, we show that the expansions (4.7) and (4.13) for the two-body S matrix (for the case $\Phi_1=\Phi_2\equiv\Phi$) are equivalent in each order.

We first remark that the momentum matrix elements of Φ are of the form[4]

$$\langle p_1p_2 \,|\, \Phi \,|\, p_1'p_2' \rangle = 2\pi(-P^2)^{1/2}\delta(\Delta\cdot P)\delta^4(P-P')\tilde{\Phi}(\Delta_\perp)\,, \tag{A1}$$

where $\Delta = p_1 - p_1' = p - p'$ is the momentum transfer.

The single-τ perturbation expansion for the wave function [from Eq. (4.5)]

$$\begin{aligned}\psi(p_1p_2;\tau) = {}& \psi_{\rm in}(p_1p_2) - 2\int (dp_1')(dp_2')e^{i(p_1{}^2+p_2{}^2-p_1'{}^2-p_2'{}^2)\tau/2}\frac{\langle p_1p_2\,|\,\Phi\,|\,p_1'p_2'\rangle}{p_1{}^2+p_2{}^2-p_1'{}^2-p_2'{}^2-i\epsilon}\psi_{\rm in}(p_1'p_2')\\ &+4\int (dp_1^{(1)})(dp_2^{(1)})(dp_1')(dp_2')e^{i(p_1{}^2+p_2{}^2-p_1'{}^2-p_2'{}^2)\tau/2}\\ &\times\frac{\langle p_1p_2\,|\,\Phi\,|\,p_1^{(1)}p_2^{(1)}\rangle\langle p_1^{(1)}p_2^{(1)}\,|\,\Phi\,|\,p_1'p_2'\rangle}{(p_1{}^2+p_2{}^2-p_1'{}^2-p_2'{}^2-i\epsilon)(p_1^{(1)2}+p_2^{(1)2}-p_1'{}^2-p_2'{}^2-i\epsilon)}\psi_{\rm in}(p_1'p_2')+\cdots\end{aligned} \tag{A2}$$

then becomes

$$\begin{aligned}\psi(p_1p_2;\tau) = {}& \psi_{\rm in}(p_1p_2) - 2\pi(-P^2)^{1/2}\int (dp_1')(dp_2')\delta^4(P-P')\\ &\times\delta\!\left(\tfrac{1}{2}(p_1{}^2-p_1'{}^2)-\tfrac{1}{2}(p_2{}^2-p_2'{}^2)\right)\tilde{\Phi}((p_1-p_1')_\perp)\frac{e^{i(p_1{}^2-p_1'{}^2)\tau}}{p_1{}^2-p_1'{}^2-i\epsilon}\psi_{\rm in}(p_1'p_2')\\ &+(2\pi)^2(-P^2)\int (dp_1^{(1)})(dp_2^{(1)})(dp_1')(dp_2')\delta\!\left(\tfrac{1}{2}(p_1{}^2-p_1^{(1)2})-\tfrac{1}{2}(p_2{}^2-p_2^{(1)2})\right)\\ &\times\delta\!\left(\tfrac{1}{2}(p_1^{(1)2}-p_1'{}^2)-\tfrac{1}{2}(p_2^{(1)2}-p_2'{}^2)\right)\delta^4(P-P^{(1)})\delta^4(P^{(1)}-P')e^{i(p_1{}^2-p_1'{}^2)\tau}\\ &\times\frac{\tilde{\Phi}((p_1-p_1^{(1)})_\perp)\tilde{\Phi}((p_1^{(1)}-p_1')_\perp)}{(p_1{}^2-p_1'{}^2-i\epsilon)(p_1^{(1)2}-p_1'{}^2-i\epsilon)}\psi_{\rm in}(p_1'p_2')+\cdots\,,\end{aligned} \tag{A3}$$

where we have used the kinematical relation (for $P=P'$)

$$\Delta\cdot P = \tfrac{1}{2}(p_1{}^2-p_1'{}^2)-\tfrac{1}{2}(p_2{}^2-p_2'{}^2)\,.$$

The two-τ perturbation expansion of the wave function [from Eq. (4.12)],

$$\begin{aligned}\psi(p_1p_2;\tau_1\tau_2) = {}& \psi_{\rm in}(p_1p_2) + \int (dp_1')(dp_2')e^{i(p_1{}^2-p_1'{}^2)\tau_1}e^{i(p_2{}^2-p_2'{}^2)\tau_2}\\ &\times\frac{\langle p_1p_2\,|\,V\,|\,p_1'p_2'\rangle}{(p_1{}^2-p_1'{}^2-i\epsilon_1)(p_2{}^2-p_2'{}^2-i\epsilon_2)}\psi_{\rm in}(p_1'p_2')\\ &+\int (dp_1^{(1)})(dp_2^{(1)})(dp_1')(dp_2')e^{i(p_1{}^2-p_1'{}^2)\tau_1}e^{i(p_2{}^2-p_2'{}^2)\tau_2}\\ &\times\frac{\langle p_1p_2\,|\,V\,|\,p_1^{(1)}p_2^{(1)}\rangle\langle p_1^{(1)}p_2^{(1)}\,|\,V\,|\,p_1'p_2'\rangle}{(p_1{}^2-p_1'{}^2-i\epsilon_1)(p_2{}^2-p_2'{}^2-i\epsilon_2)(p_1^{(1)2}-p_1'{}^2-i\epsilon_1)(p_2^{(1)2}-p_2'{}^2-i\epsilon_2)}\psi_{\rm in}(p_1'p_2')\\ &+\cdots\,,\end{aligned} \tag{A4}$$

can be compared to (A3) if we recall the definition (3.29) of V, for which we obtain

$$\begin{aligned}\langle p_1p_2\,|\,V\,|\,p_1'p_2'\rangle = {}& (p_1'{}^2-p_2{}^2)\langle p_1p_2\,|\,\Phi\,|\,p_1'p_2'\rangle\\ &+\int (dp_1^{(1)})(dp_2^{(1)})\langle p_1p_2\,|\,\Phi\,|\,p_1^{(1)}p_2^{(1)}\rangle\langle p_1^{(1)}p_2^{(1)}\,|\,\Phi\,|\,p_1'p_2'\rangle\,.\end{aligned} \tag{A5}$$

The "Born term" of (A4), therefore, contains a contribution which is quadratic in Φ, and must be combined with the part of the second-order term which is second order in Φ (this term contains third and fourth order in Φ as well). Carrying out these steps, we obtain for (A4), using again (A1),

$$
\begin{aligned}
\psi(p_1p_2;\tau_1\tau_2) = \psi_{\text{in}}(p_1p_2) &- 2\pi(-P^1)^{1/2}\int (dp'_1)(dp'_2)\delta^4(P-P')\delta\!\left(\tfrac{1}{2}(p_1{}^2-p'_1{}^2)-\tfrac{1}{2}(p_2{}^2-p'_2{}^2)\right) \\
&\times \frac{e^{i(p_1{}^2-p'_1{}^2)(\tau_1+\tau_2)}}{p_1{}^2-p'_1{}^2-i\epsilon}\widetilde{\Phi}((p_1-p'_2)_\perp)\psi_{\text{in}}(p'_1p'_2) \\
&+ (2\pi)^2(-P^2)\int (dp_1^{(1)})(dp_2^{(1)})(dp'_1)(dp'_2)\delta\!\left(\tfrac{1}{2}(p_1{}^2-p_1^{(1)2})-\tfrac{1}{2}(p_1{}^2-p_2^{(1)2})\right) . \\
&\times \delta\!\left(\tfrac{1}{2}(p_1^{(1)2}-p'_1{}^2)-\tfrac{1}{2}(p_2^{(1)2}-p'_2{}^2)\right)\delta^4(P-P^{(1)})\delta^4(P^{(1)}-P') \\
&\times e^{i(p_1{}^2-p'_1{}^2)(\tau_1+\tau_2)}\frac{\widetilde{\Phi}((p_1-p_1^{(1)})_\perp)\widetilde{\Phi}((p_1^{(1)}-p'_1)_\perp)}{(p_1{}^2-p'_1{}^2-i\epsilon)^2} \\
&\times \left[1-\frac{p_1^{(1)2}-p_1{}^2}{p_1^{(1)2}-p'_1{}^2-i\epsilon}\right]\psi_{\text{in}}(p'_1p'_2)+\cdots \ . \qquad \text{(A6)}
\end{aligned}
$$

The two terms in the large brackets in the second-order part arise, respectively, from the "Born term" in the expansion (A4) and the part of the second-order term which is second order in Φ; they combine to cancel one factor of $p_1{}^2-p'_1{}^2-i\epsilon$ in the denominator. The resulting series then precisely coincides with the single-τ expansion (A3) for $\tau=\tau_1+\tau_2$ up to second order. The comparison of higher-order terms proceeds in a similar way. The equivalence of the wave-function expansions order by order in perturbation theory, for every $\tau=\tau_1+\tau_2$, then implies the perturbative equivalence of the S-matrix expansions (4.7) and (4.13).

---

[*]On sabbatical leave from Tel Aviv University.

[1]L. P. Horwitz and F. Rohrlich, Phys. Rev. D 24, 1528 (1981). This paper will be referred to as I.

[2]The reader is referred to references given in I as indicative of the rapidly increasing literature on this subject. See, in particular, Refs. 3 and 5 below.

[3]Ph. Droz-Vincent, Nuovo Cimento 58A, 355 (1980). For the classical case see I. Todorov, Lectures at ICTP, Trieste, 1980 (unpublished).

[4]L. P. Horwitz, Y. Lavie, and A. Soffer, in *Group Theoretical Methods in Physics,* proceedings of the VIII Colloquium on Group Theoretical Methods in Physics, Kiriat Anavim, Israel, 1979, edited by L. Horowitz and Y. Ne'eman (Israel Phys. Soc., Haifa, 1980), Vol. 3; L. P. Horwitz and Y. Lavie, Phys. Rev. D 26, 819 (1982).

[5]For the classical treatment see F. Rohrlich, Phys. Rev. D 23, 1305 (1981), 25, 2576 (1982).

[6]See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Springer, New York, 1976), (A6-1), p. 464.

[7]V contains a commutator of K_1^0 with Φ as well as a contribution of the form Φ^2, which may restrict the set of $|\psi_{\text{in}}\rangle$ for which these statements are applicable.

[8]The notation $|\psi_{\text{in}}(k_1k_2)\rangle$ for a (improper) *state* with sharp values of momenta p_1p_2 at k_1,k_2, is to be distinguished from a *wave function,* which corresponds to a state in some given representation. In momentum space, for example, a state is represented as a set of amplitudes for values p_1,p_2 in the spectrum of the momentum operators P_1,P_2, i.e. as the wave function $\langle p_1p_2|\psi_{\text{in}}\rangle$. In particular, $\langle p_1p_2|\psi_{\text{in}}(k_1k_2)\rangle = \delta^4(p_1-k_1)\delta^4(p_2-k_2)$. In the present section, we shall work almost entirely with operators and state vectors. We shall not need to compute matrix elements or wave functions up to Eq. (5.28). Our formulas, in terms of state vectors, may be alternatively interpreted as equations for wave functions in an arbitrary representation.

[9]See, for example, M. Reed and B. Simon, *Methods of Mathematical Physics, III. Scattering Theory* (Academic, New York, 1979), p. 98, for a treatment of the nonrelativistic Lippmann-Schwinger equation using a method of this type. One can, alternatively, use spectral theory.

[10]W. O. Amrein, J. M. Jauch, and K. B. Sinha [*Scattering Theory in Quantum Mechanics* (Benjamin, Reading, Mass., 1977), Chap. 6] use this method from the point of view of spectral representation.

[11]R. G. Newton, *Scattering Theory of Particles and Waves* (McGraw-Hill, New York, 1966), p. 186.

[12]E. C. G. Sudarshan, N. Mukunda, and J. N. Goldberg, Phys. Rev. D 23, 2218 (1981) have studied the case $\Phi_1=\Phi_2=\cdots=\Phi_N\equiv\Phi$. Their solution violates the cluster property: the potential corresponds to an N-body force which vanishes when one particle goes

to spacelike ∞; all the particles become free when just one particle moves away.

[13]We use the general notation of I in this section:

$$\Omega_+^{(a)} = \lim_{\tau_a \to -\infty} e^{iK_a \tau_a} e^{-iK_a^0 \tau_a} ,$$

$$\Omega_+^{(ab)} = \lim_{\substack{\tau_a \to -\infty \\ \tau_b \to -\infty}} e^{iK_a \tau_a} e^{iK_b \tau_b} e^{-iK_a^0 \tau_a} e^{-iK_b^0 \tau_b} .$$

[14]Section I, Refs. 8 and 9; L. P. Horwitz and F. C. Rotbart, Phys. Rev. D 24, 2127 (1981) and references therein.

[15]V. A. Rizov, I. T. Todorov, and B. L. Aneva, Nucl. Phys. B98, 447 (1975); I. T. Todorov, Phys. Rev. D 3, 2351 (1971).

[16]A. P. Logunov and A. N. Tavkhelidze, Nuovo Cimento 29, 380 (1963); R. Fong and J. Sucher, J. Math. Phys. 5, 456 (1964); M. Levy and J. Sucher, Phys. Rev. 186, 1656 (1969); I. T. Todorov, Phys. Rev. D 3, 2351 (1971). This list is of course very incomplete.

[17]F. Coester, in *Lecture Notes in Physics,* No. 130 (Springer, Berlin, 1980), p. 190.

[18]H. P. Crater and P. Van Alstine, Phys. Lett. 100B, 166 (1981).