Feynman propagators and particle creation in linearly expanding Bianchi type-I universes

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Scalar particle creation in the linearly expanding Bianchi type-I universes is studied using the Feynman propagator technique. Explicit expressions for the propagators corresponding to an arbitrary "in" vacuum and the "out" vacuum defined by the WKB positive-frequency solutions are derived. The initial conditions are then singled out by requiring the square integrability of the analytically continued kernels of the propagators considered. It is shown that for each particular model there is only one Riemannian kernel which satisfies this condition. Furthermore, it is proved that all the kernels selected this way admit a well-defined path-integral representation defined on the Riemannian domains of physically allowed values of coordinates. This result confirms the assumption of Chitre and Hartle that the propagator they found in the isotropic model (which is precisely that singled out by the square-integrability condition) can be represented by a path integral defined on the domain to the future of the initial singularity. The initial conditions corresponding to the selected propagators are analyzed. It is shown that they give rise to the creation of pairs with spectrum, which at high energies resembles that of blackbody radiation in one, two, or three directions, depending on the background geometry. Finally, the conceptual and technical problems associated with the complexified spacetime path-integral method, as applied to cosmological models, are discussed.

I. INTRODUCTION

One of the major problems of quantum field theory in curved spacetime is the definition of the particle states. In general, there is no natural way to determine a preferred set of positive-frequency solutions of the field equations, or, equivalently, to choose the Feynman propagator. The difficulty is especially severe in the case of singular cosmological spacetimes, where no "in" regions having particular symmetries exist. In these situations initial conditions are either postulated "*ad hoc*" or selected according to some mathematical or physical criteria.

An important approach to the problem of defining the initial particle states in singluar universes was proposed by Chitre and Hartle.[1] It focuses on the Feynman propagator and represents a natural extension of the complexified spacetime path-integral method, applied earlier to the black-hole geometries.[2] Within this method the *ad hoc* specification of initial conditions is replaced by the requirement that only paths located to the future of the initial singularity contribute to the path integral. The latter has to be evaluated by analytic continuation to the Riemannian domain (with positive-definite metric), so that the integrals involved have proper meaning.

In attempting to apply this approach to a particular model, a linearly expanding, spatially flat Robertson-Walker universe, Chitre and Hartle replaced actual evaluation of the path integral by solution of the differential equation for the analytically continued kernel of the Feynman propagator, subject to certain boundary conditions. It was suggested that the propagator found by this procedure might be represented in the form of a Riemannian path integral.

It was recently shown that the boundary conditions used by Chitre and Hartle are not sufficient to determine the propagator uniquely.[3] Methods to strengthen the boundary conditions were also proposed. In particular, it was found that the propagator of Chitre and Hartle is the only one for which the analytically continued kernel has a well-defined Fourier transform. A covariant generalization of this requirement, namely, the square integrability of the Riemannian kernel, was proposed.

The present work is a further continuation and development of the analysis given in Refs. 1 and 3. The Riemannian method is applied in a unified way to the problem of defining the initial particle states in linearly expanding Bianchi type-I universes, which include the Robertson-Walker model, mentioned above, as a particular case. It is explicitly shown that in each case considered there is only one Feynman propagator, the Riemannian kernel of which may be represented by the Fourier integral defined over an infinite domain of the coordinates involved. All the propagators selected this way admit a well-defined path-integral representation in the domain of physically allowed values of coordinates. This result confirms the assumption of Chitre and Hartle that the propagator they found in the isotropic case can be represented by the path integral defined on the domain to the future of the initial singularity.

All of the Feynman propagators, singled out by the above requirement, correspond to the initial conditions which give rise to the particle creation in the course of cosmological expansion. At sufficiently high energies the spectrum of created quanta is thermal in one, two, or three dimensions, according to the background geometries involved.

The propagator obtained using the above complexification procedure in the case of the degenerate Kasner universe (which is flat) is found to be different from the standard Minkowski propagator. The origin of this difference is clarified by showing that nonequivalent Riemannian algorithms do not select the same propagator.

The organization of the paper is as follows. In Sec. II the complexified spacetime path-integral method is reviewed, and various boundary conditions associated with Riemannian kernels are discussed. In Sec. III the most general form of the Feynman propagator for the scalar fields in the linearly expanding Bianchi type-I universes is derived. The analytically continued kernels are obtained in Sec. IV. Using the existence of the Fourier integral as a selection principle we further analyze corresponding initial conditions and physical quantities, associated with the created quanta. In Sec. V the selected Riemannian kernels are discussed from the viewpoint of path integrals. Section VI is devoted to concluding remarks. In the Appendix, some mathematical expressions used in the text are derived.

II. PROPAGATORS, PATH INTEGRALS, AND BOUNDARY CONDITIONS

Feynman propagators are the central building blocks of quantum field theory in curved spacetime. Knowledge of the propagator enables one to evaluate all other physically interesting quantities.[1–5] In this section, the complexified spacetime path-integral method of evaluation of the Feynman propagator as applied to cosmological spacetimes is reviewed emphasizing the technical and conceptual difficulties of this approach. The discussion is restricted to the case of a neutral scalar field which satisfies the wave equation

$$(-\nabla_\mu \nabla^\mu + \xi R + m^2)\phi(x) = 0 \ . \qquad (2.1)$$

Here ∇_μ denotes the covariant derivative with respect to the background metric $g_{\mu\nu}$, R is the Ricci scalar, m is the mass of the field, and ξ is the nonminimal coupling constant.

In the Schwinger-DeWitt method[4–6] the Feynman propagator is defined by the integral representation

$$G(xx') = i \int ds\, e^{-im^2 s} \langle xs \mid x'0 \rangle \ , \qquad (2.2)$$

where m^2 is taken to have a small negative imaginary part and the kernel satisfies the Schrödinger-type equation

$$i \frac{\partial}{\partial s} \langle xs \mid x'0 \rangle = (-\nabla_\mu \nabla^\mu + \xi R) \langle xs \mid x'0 \rangle \ , \qquad (2.3)$$

with "initial" condition

$$\lim_{s \to 0} \langle xs \mid x'0 \rangle = [-g(x)]^{-1/2} \delta(xx') \ . \qquad (2.4)$$

Here $g(x)$ is the determinant of $g_{\mu\nu}$. The kernel $\langle xs \mid x'0 \rangle$ can be regarded as the probability amplitude for the fictitious particle to propagate on a four-dimensional hypersurface from point x'^μ at "time" $s=0$ to x^μ at "time" s.

In the limt $s \to 0^+$ the kernel can be represented as

$$\langle xs \mid x'0 \rangle \underset{s \to 0^+}{\sim} -\frac{i}{(4\pi s)^2} \left\{ \exp\left[\frac{i\sigma(xx')}{2s} \right] \right\} \Delta^{1/2} [1 + isf_1(x,x') + (is)^2 f_2(x,x') + \cdots] + F(x,x';is) \ . \qquad (2.5)$$

Here $\sigma(xx')$ is half of the proper distance squared (minus proper time squared) along the spacelike (timelike) geodesic connecting the points x'^μ and x^μ, Δ denotes the biscalar

$$\Delta(x,x') = -[-g(x)]^{1/2} \det[-\partial_\mu \partial_\nu \sigma(xx')][-g(x')]^{-1/2} \ , \qquad (2.6)$$

$f_i(x,x')$ are the so-called DeWitt-Hadamard coefficients, and $F(x,x';is)$ is some function which has an essential singularity at $s=0$.[5] It is present usually in those situations where the particle creation takes place. The function $F(x,x';is)$ cannot be specified uniquely by the initial condition (2.4), so that further nonlocal boundary conditions are required for its determination.

Using the similarity between Eqs. (2.3) and (2.4) and the standard one-particle quantum mechanics one can represent the kernel $\langle xs \mid x'0\rangle$ as a functional integral

$$\langle xs \mid x'0\rangle = \int d[x(s')]\exp\left[i\int_0^s ds' L\left[x,\frac{dx}{ds}\right]\right], \tag{2.7}$$

where the integration is carried out over all paths $x(s)$ which connect the point x'^μ at "time" $s=0$ with the point x^μ at "time" s. More concretely, the functional integral can be represented as the iterated integral

$$\langle xs \mid x'0\rangle = \lim_{N\to\infty}\int d^4x_N\sqrt{-g(x_N)}\cdots\int d^4x_1\sqrt{-g(x_1)}\langle xs \mid x_N s-\epsilon\rangle\cdots\langle x_1\epsilon \mid x'0\rangle, \tag{2.8}$$

where the "short-time" kernels have the form

$$\langle xs \mid x'0\rangle \sim Y(\epsilon,xx')\exp\left[\frac{i}{4}\int_0^\epsilon L\left[x,\frac{dx}{ds'}\right]ds'\right]. \tag{2.9}$$

Here $Y(\epsilon,xx')$ is an appropriate weight, which depends upon the coupling constant ξ.[5] If the classical motion of a fictitious particle is unrestricted, the corresponding Lagrangian can be taken in the form

$$L = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}. \tag{2.10}$$

When the spacetime is singular, incomplete, or compact, the motion of fictitious particle changes radically and the path integral has to reflect this effect, which can be viewed as the boundary conditions imposed on the kernel $\langle xs \mid x'0\rangle$. For compact manifolds the problem is quite clear and the path integral can be constructed by taking the sum of contributions from all geodesics, which connect the corresponding spacetimes points.[6] The situation is much more complicated when the manifolds are singular or incomplete, because in these cases the boundary conditions are, in general, unknown. For cosmological models with initial singularities Chitre and Hartle suggested that the effect of boundary conditions can be incorporated in the path integral by restricting the paths to those which are located to the future of the initial singularity. The iterated integral (2.8) is then defined over the domain of physically allowed coordinates. However, an explicit method of path-integral evaluation of the kernels which incorporates only those restricted paths has not been developed yet.

Usually the iterated integral (2.8) is ill defined because it involves terms which do not fall off rapidly enough when the separation between the points χ^μ and χ'^μ tends to infinity. In Minkowski-spacetime quantum field theory it is customary to evaluate the functional integrals by "Wick rotation" to imaginary time and deal with the Euclidean quantum field theory. According to the Euclidean postulate the physical quantities are obtained by analytic continuation back to Minkowski spacetime. The analog for the general relativistic field theory is to construct the theory in spacetimes with positive-definite Riemannian metric $\gamma_{\mu\nu}$. The Riemannian version of the theory is obtained by rotating some of the coordinates x^μ in the complex plane and simultaneously rotating the "time" coordinate s by $-\pi/2$, writing $s=-i\Omega$. The major difficulty of the method is its nonuniqueness, which is due to the fact that the spacetime might be complexified by using nonequivalent Riemannian algorithms. Consequently one obtains nonequivalent field theories. This difficulty is present already in the flat spacetime if one starts with a non-Minkowskian coordinate system as in the case of the Rindler space.[7] A similar situation occurs for the degenerate Kasner universe, which is also a part of the Minkowski spacetime. This problem is discussed in detail in the forthcoming sections. Which Riemannian algorithm to choose cannot be decided within the theory itself. One of the criteria for such a choice could be the operational interpretation of corresponding field theories.

For the Riemannian space with coordinates χ^μ and metric $\gamma_{\mu\nu}$ the analytically continued kernel $\langle \chi\Omega \mid \chi'0\rangle$ is a solution of the parabolic equation

$$\frac{\partial}{\partial\Omega}\langle \chi\Omega \mid \chi'0\rangle = (\tilde{\nabla}_\mu\tilde{\nabla}^\mu - \xi\tilde{R})\langle \chi\Omega \mid \chi'0\rangle, \tag{2.11}$$

with the "initial" condition

$$\lim_{\Omega\to 0^+} \langle \mathcal{X}\Omega \mid \mathcal{X}'0\rangle = \eta[\gamma(\mathcal{X})]^{1/2}\delta(\mathcal{X},\mathcal{X}') . \quad (2.12)$$

Here $\widetilde{\nabla}_\mu$ and $\widetilde{R}$ denote the covariant derivative and the Ricci scalar evaluated with respect to the positive-definite metric $\gamma_{\mu\nu}$, and η is a possible phase factor. The nonlocal boundary conditions have to be further imposed to single out the propagator uniquely. The most natural boundary condition could be the requirement that the kernel $\langle \mathcal{X}\Omega \mid \mathcal{X}'0\rangle$ has to be represented by a well-defined iterated Riemannian integral

$$\langle \mathcal{X}\Omega \mid \mathcal{X}'0\rangle = \lim_{\substack{N\to\infty \\ (1+N)\bar{\epsilon}=\Omega}} \eta \int d^4\mathcal{X}_N[\gamma(\mathcal{X}_N)]^{1/2} \cdots \int d^4\mathcal{X}_1[\gamma(\mathcal{X}_1)]^{1/2}\langle \mathcal{X}\Omega \mid \mathcal{X}_N\Omega-\bar{\epsilon}\rangle \cdots \langle \mathcal{X}_1\bar{\epsilon} \mid \mathcal{X}'0\rangle , \quad (2.13)$$

where the integration is carried out over the physically allowed domain of coordinates $\mathcal{X}^\mu$. As applied to the cosmological case Chitre and Hartle[1] required this domain to be that obtained by analytic continuation of a physical domain, lying to the future of the initial singularity. Actual implementation of this approach is too difficult, and one might consider more explicit boundary conditions that ensure the existence of the Riemannian path integral, or equivalently, the iterated integral (2.13). One such condition was proposed recently in Ref. 3. For open spacetimes it requires the square integrability of the kernel $\langle \mathcal{X}\Omega \mid \mathcal{X}'0\rangle$, namely,

$$\int |\langle \mathcal{X}\Omega \mid \mathcal{X}'0\rangle|^2\gamma^{1/2}d^4x \text{ is convergent for each allowed value of } \mathcal{X}' . \quad (2.14)$$

Here the integration is performed over the domain of physically allowed values of coordinates. This condition seems to be a necessary one, because the condition proposed earlier by Chitre and Hartle,[1] which required the kernel just to vanish for infinitely separated points $\mathcal{X}$ and $\mathcal{X}'$, does not uniquely determine the propagator.[3]

In the following sections the boundary condition (2.14) will be used to single out the kernels for each of the cosmological models considered. It will be shown that these kernels can be represented by well-defined Riemannian path integrals. Furthermore, it will be shown that at least some other kernels that do not satisfy the boundary condition (2.14) do not admit the path-integral representation, provided that the integrals involved have infinite limits of integration required by the Riemannian schemes developed below.

III. SCALAR FIELDS IN LINEARLY EXPANDING BIANCHI TYPE-I UNIVERSES

In this section we develop the conventional field-theoretical analysis of conformally coupled massive scalar fields propagating on the classical background of linearly expanding Bianchi type-I universes. The metric of these cosmological models can be written in the form

$$ds^2 = -dt^2 + t^2dx^2 + t^{2p_1}dy^2 + t^{2p_2}dz^2 , \quad (3.1)$$

where p_1 and p_2 are constant parameters equal to 1 or 0. All these cosmological models do not have particle horizons. The case $p_1 = p_2 = 1$ is a particular Robertson-Walker universe. Both this model as well as the anisotropic model with $p_1 = 1$, $p_2 = 0$ have a curvature singularity at $t=0$. The case $p_1 = p_2 = 0$ corresponds to the degenerate Kasner universe.[8] It is flat and singularity free. It can be written in the usual Minkowski form with coordinates T,X,Y,Z related to t,x,y,z as

$$X = t\sinh x, \quad T = t\cosh x , \qquad Y = y, \quad Z = z . \quad (3.2)$$

This transformation is singluar at $t=0$. The degenerate Kasner metric represents that part of the flat spacetime which is located within the region $|T| > |X|$. For the background metrics (3.1) the conformally coupled ($\xi = \frac{1}{6}$) field equation (2.1) can be written explicitly as

$$\left[t^{-(1+p_1+p_2)}\frac{\partial}{\partial t}\left[t^{(1+p_1+p_2)}\frac{\partial}{\partial t}\right] - t^{-2}\frac{\partial^2}{\partial x^2} - t^{-2p_1}\frac{\partial^2}{\partial y^2} - t^{-2p_2}\frac{\partial^2}{\partial z^2} + m^2 + \gamma t^{-2}\right]\phi(\vec{x},t) = 0 , \quad (3.3)$$

where γ is the constant parameter determined by the exponents p_1 and p_2:

$$\gamma=\begin{cases}1, & p_1=p_2=1\ ;\\ \frac{1}{3}, & p_1=1,\ p_2=0\ ;\\ 0, & p_1=p_2=0\ .\end{cases} \tag{3.4}$$

The general solution of Eq. (3.3) can be written as

$$\phi(t,\vec{x})=\int d^3k\,[A_{\vec{k}}f_{\vec{k}}(t,\vec{x})+A^{\dagger}_{\vec{k}}f^*_{\vec{k}}(t,\vec{x})]\ , \tag{3.5}$$

where the basis functions $f_{\vec{k}}(t,\vec{x})$ are given by

$$f_{\vec{k}}(t,\vec{x})=(2\pi)^{-3/2}\psi_{\vec{k}}(t)\exp(i\vec{k}\cdot\vec{x})\ , \tag{3.6}$$

$$\psi_{\vec{k}}(t)=B_{\vec{k}}H^{(1)}_{i\nu}(\mu t)+C_{\vec{k}}H^{(2)}_{i\nu}(\mu t)\ , \tag{3.7}$$

$$\psi^*_{\vec{k}}(t)=B^*_{\vec{k}}H^{(2)}_{i\nu}(\mu t)e^{\pi\nu}+C^*_{\vec{k}}H^{(1)}_{i\nu}(\mu t)e^{-\pi\nu}\ . \tag{3.8}$$

In Eqs. (3.7) and (3.8) $H^{(1)}_{i\nu}$ and $H^{(2)}_{i\nu}$ are the Hankel functions of the first and second kind with imaginary index $i\nu$, which satisfy the equation $H^{(2)*}_{i\nu}=H^{(1)}_{i\nu}\exp(-\nu\pi)$. The parameters ν and μ are given by

$$\nu=\begin{cases}|\vec{k}|, & p_1=p_2=1\ ;\\ (k_x{}^2+k_y{}^2+\frac{1}{12})^{1/2}, & p_1=1,\ p_2=0\ ;\\ k_x, & p_1=p_2=0\ ;\end{cases} \tag{3.9}$$

and

$$\mu=\begin{cases}m, & p_1=p_2=1\ ;\\ (m^2+k_z{}^2)^{1/2}, & p_1=1,\ p_2=0\ ;\\ (m^2+k_z{}^2+k_y{}^2)^{1/2}, & p_1=p_2=0\ .\end{cases} \tag{3.10}$$

Introducing now the conserved Klein-Gordon scalar product

$$(g,h)=-i\int_{\Sigma}d\Sigma^{\mu}g^*\overleftrightarrow{\partial}_{\mu}h\ , \tag{3.11}$$

where Σ is a spacelike hypersurface, one can verify that the basis functions $f_{\vec{k}}$ and $f^*_{\vec{k}}$ satisfy the following conditions:

$$\begin{aligned}&(f_{\vec{k}},f_{\vec{k}'})=-(f^*_{\vec{k}},f^*_{\vec{k}'})=\delta(\vec{k}-\vec{k}')\ ,\\ &(f_{\vec{k}},f^*_{\vec{k}'})=0\ ,\end{aligned} \tag{3.12}$$

if and only if the coefficients $B_{\vec{k}}$ and $C_{\vec{k}}$ are restricted by the Wronskian condition

$$|C_{\vec{k}}|^2e^{-\nu\pi}-|B_{\vec{k}}|^2e^{\nu\pi}=\pi/4\ . \tag{3.13}$$

This condition also guarantees that the operators $A_{\vec{k}}$ and $A^{\dagger}_{\vec{k}}$ obey the standard commutation relations for the annihilation and creation operators.

The Wronskian constraint (3.13) does not determine the basis functions uniquely. Consequently one can decompose the field operator $\phi(\vec{x},t)$ with respect to other basis functions $p_{\vec{k}}(\vec{x},t)$:

$$\phi(\vec{x},t)=\int d^3k\,[a_{\vec{k}}p_{\vec{k}}(\vec{x},t)+a^{\dagger}_{\vec{k}}p^*_{\vec{k}}(\vec{x},t)]\ , \tag{3.14}$$

where

$$p_{\vec{k}}(\vec{x},t)=(2\pi)^{-3/2}\psi'_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}\ , \tag{3.15}$$

$$\psi'_{\vec{k}}(t)=B'_{\vec{k}}H^{(1)}_{i\nu}(\mu t)+C'_{\vec{k}}H^{(2)}_{i\nu}(\mu t)\ , \tag{3.16}$$

and the coefficients $B'_{\vec{k}}$ and $C'_{\vec{k}}$ are subject to the condition (3.13). The basis functions $p_{\vec{k}},p^*_{\vec{k}}$ are orthonormal in the sense of Eq. (3.12), and the operators $a_{\vec{k}}, a^{\dagger}_{\vec{k}}$ satisfy the same commutation relations as the operators $A_{\vec{k}}$ and $A^{\dagger}_{\vec{k}}$.

The basis functions $\psi_{\vec{k}}(t)$ and $\psi'_{\vec{k}}(t)$ are related by the Bogoliubov transformations[9]

$$\begin{aligned}\psi_{\vec{k}}(t)&=\alpha_{\vec{k}}\psi'_{\vec{k}}(t)+\beta_{\vec{k}}\psi'^{*}_{\vec{k}}(t)\ ,\\ \psi^{*}_{\vec{k}}(t)&=\alpha^{*}_{\vec{k}}\psi'^{*}_{\vec{k}}(t)+\beta^{*}_{\vec{k}}\psi'_{\vec{k}}(t)\ ,\end{aligned} \tag{3.17}$$

with

$$|\alpha_{\vec{k}}|^2-|\beta_{\vec{k}}|^2=1\ . \tag{3.18}$$

Consequently the operators $A_{\vec{k}}, A^{\dagger}_{\vec{k}}$ and $a_{\vec{k}}, a^{\dagger}_{\vec{k}}$ are related as[9]

$$\begin{aligned}a_{\vec{k}}&=\alpha_{\vec{k}}A_{\vec{k}}+\beta^{*}_{\vec{k}}A^{\dagger}_{-\vec{k}}\ ,\\ a^{\dagger}_{\vec{k}}&=\alpha^{*}_{\vec{k}}A^{\dagger}_{\vec{k}}+\beta_{\vec{k}}A_{-\vec{k}}\ .\end{aligned} \tag{3.19}$$

We can now define the Feynman propagator $G(x,x')$ by the Schwinger average[1-6]

$$G_F(x,x')=i\frac{\langle 0_{\text{out}}\,|\,T\{\phi(x)\phi(x')\}\,|\,0_{\text{in}}\rangle}{\langle 0_{\text{out}}\,|\,0_{\text{in}}\rangle}\ , \tag{3.20}$$

where T denotes the time-ordered product of the field operators, and $|\,0_{\text{out}}\rangle$ and $|\,0_{\text{in}}\rangle$ are the "out" and "in" vacuums. Referring to the basis (3.6) as the "in" basis and (3.15) as the "out" basis, and using Eq. (3.19) one obtains

$$G_F(x,x')=i\int\frac{d^3k}{\alpha^{*}_{\vec{k}}}[\theta(t-t')p_{\vec{k}}(\vec{x},t)f^{*}_{\vec{k}}(\vec{x}',t')+\theta(t'-t)p_{\vec{k}}(\vec{x}',t')f^{*}_{\vec{k}}(\vec{x},t)]\ . \tag{3.21}$$

Here $\theta(t)$ is the Heaviside function

$$\theta(t)=\begin{cases}1, & t>0\ ,\\ 0, & t<0\ .\end{cases} \tag{3.22}$$

For large $t\,(t\gg m^{-1})$ the expansion is sufficiently slow and one can single out the out basis functions $\psi'_{\vec{k}}$ using the positive-frequency WKB solutions of Eq. (2.3). Making use of the asymptotic form of the Hankel functions for the large argument[10] one obtains

$$\psi'_{\vec{k}}=\pi^{1/2}2^{-1}t^{-(p_1+p_2)/2}\exp\left[\frac{\pi\nu}{2}-\frac{i\pi}{4}\right]H^{(2)}_{i\nu}(\mu t)\ . \tag{3.23}$$

The Bogoliubov coefficients $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ connecting the in and out bases are then given by

$$\begin{aligned}\alpha_{\vec{k}}&=2\pi^{-1/2}\exp\left[\frac{-\pi\nu}{2}+\frac{i\pi}{4}\right]C_{\vec{k}}\ ,\\ \beta_{\vec{k}}&=2\pi^{-1/2}\exp\left[\frac{\pi\nu}{2}-\frac{i\pi}{4}\right]B_{\vec{k}}\ .\end{aligned} \tag{3.24}$$

The Feynman propagator corresponding to the out basis (3.23) and the most general in basis is then given by Eq. (3.21) as

$$G_F(x,x')=\frac{i\pi}{4(2\pi)^3}(t_>t_<)^{-(p_1+p_2)}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}H^{(2)}_{i\nu}(\mu t_>)\left[H^{(1)}_{i\nu}(\mu t_<)+\frac{B^{*}_{\vec{k}}}{C^{*}_{\vec{k}}}e^{2\pi\nu}H^{(2)}_{i\nu}(\mu t_<)\right]\ . \tag{3.25}$$

Using the relationships between Hankel and Bessel functions,[10] one can write the latter equation as

$$G_F(x,x')=i\pi 2^{-1}(2\pi)^{-3}(t_>t_<)^{-(p_1+p_2)}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left[\left[\frac{e^{2\pi\nu}B^*_{\vec{k}}}{C^*_{\vec{k}}}-1\right](e^{2\pi\nu}-1)^{-1}H^{(2)}_{-i\nu}(\mu t_>)\mathscr{J}_{-i\nu}(\mu t_<)\right.$$
$$\left.+e^{2\pi\nu}(e^{2\pi\nu}-1)^{-1}\left[1-\frac{B^*_{\vec{k}}}{C^*_{\vec{k}}}\right]H^{(2)}_{i\nu}(\mu t_>)\mathscr{J}_{i\nu}(\mu t_<)\right]. \tag{3.26}$$

Making use of representations of products of Bessel and Hankel functions as given in Ref. 10, p. 439, Eq. (3.26) can be written in the Schwinger-DeWitt form (2.2) with the kernel $\langle xs\,|\,x'0\rangle$ given by

$$\langle xs\,|\,x'0\rangle=i\exp\left[\frac{t^2+t'^2}{4is}\right][(2s)^{-1}(2\pi)^{-3}(tt')^{-(p_1+p_2)/2}]$$
$$\times\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-is(\mu^2-m^2)}\left[(1-e^{-2\pi\nu})^{-1}\left[1-\frac{B^*_{\vec{k}}}{C^*_{\vec{k}}}I\right]I_{i\nu}\left[\frac{itt'}{2s}\right]\right.$$
$$\left.+(e^{2\pi\nu}-1)\left[\frac{e^{2\pi\nu}B^*_{\vec{k}}}{C^*_{\vec{k}}}-1\right]I_{-i\nu}\left[\frac{itt'}{2s}\right]\right]. \tag{3.27}$$

Here $I_\lambda(x)$ is the Bessel function of imaginary argument with index λ.

When the in and out bases are the same, i.e., $B_{\vec{k}}=0$, one can perform the Fourier integration in Eq. (3.25) for both the isotropic model and the degenerate Kasner universe. The resulting expression is found to be

$$G_F^{(0)}(x,x')=-(8\pi)^{-1}m^2\Delta^{1/2}(-2m^2\sigma)^{-1/2}H_1^{(2)}[(-2m^2\sigma)^{1/2}], \tag{3.28}$$

where

$$\sigma=\begin{cases}-2^{-1}(t^2+t'^2-2tt'\cosh r), & r=|\vec{x}-\vec{x}'|,\ p_1=p_2=1,\\ -2^{-1}[t^2+t'^2+(z-z')^2+(y-y')^2-2tt'\cosh(x-x')], & p_1=p_2=0,\end{cases} \tag{3.29}$$

is half of the proper distance squared (minus proper time squared) along the geodesics between x and x', and Δ is the biscalar, defined by Eq. (2.6). Its values are

$$\Delta^{1/2}=\begin{cases}r^{-1}\sinh r, & p_1=p_2=1,\\ 1, & p_1=p_2=0.\end{cases} \tag{3.30}$$

The calculation for the isotropic case is given in Ref. 3. The evaluation of the propagator $G^{(0)}(x,x')$ for the degenerate Kasner model is presented in the Appendix.

In the isotropic case the above choice of initial conditions is unrealistic due to the nonvanishing curvature of spacetime and finite mass of the field considered.[9] On the other hand in the case of the degenerate Kasner universe one does not expect particle creation, so that $B_{\vec{k}}=0$ seems to be the most natural choice. In fact, the vacuum defined by the WKB solutions of the wave equation is the same as the usual Minkowski vacuum, as was shown by Fulling, Parker, and Hu.[11] This is also reflected in the corresponding Feynman propagator (3.28) which is the same as the standard Minkowski-space propagator of the massive scalar field.

The corresponding kernel $\langle xs\,|\,x'0\rangle^{(0)}$ is then given by[3,4]

$$\langle xs\,|\,x'0\rangle^{(0)}=\frac{-i}{(4\pi s)^2}\Delta^{1/2}(x,x')\times\exp\left[i\frac{\sigma(x,x')}{2s}\right]. \tag{3.31}$$

IV. SELECTION OF THE INITIAL CONDITIONS

In this section we apply the analytic continuation algorithm discussed in Sec. II in order to single out those propagators (3.25) for which the analytically continued kernels satisfy the square-integrability condition (2.14). The resulting expressions are further used to determine the Bogoliubov transformations between the WKB out basis (3.23) and the in basis singled out by the square-integrability requirement. Physical quantities associated with the latter Bogoliubov transformations are also discussed.

We begin our discussion by considering the analytic continuation of the spacetimes (3.1). Following the method of Chitre and Hartle[1] one rotates the time coordinate t and the spatial coordinates, along the axes of which expansion takes place, by an angle $\pi/2$ in the complex plane,

$$t=i\lambda;\quad x=i\chi^1;\quad y=\begin{cases} i\chi^2, & p_1=1\ ,\\ \chi^2, & p_1=0\ ;\end{cases}\quad z=\begin{cases} i\chi^3, & p_2=1\ ,\\ \chi^3, & p_2=0\ ,\end{cases} \tag{4.1}$$

to obtain the Riemannian metric $\gamma_{\mu\nu}$ defined by the line element

$$dl^2=d\lambda^2+\lambda^2(d\chi^1)^2+\lambda^{2p_1}(d\chi^2)^2+\lambda^{2p_2}(d\chi^3)^2\ . \tag{4.2}$$

The range of variables $\lambda,\chi^1,\chi^2,\chi^3$ is taken to be the same as the range of the coordinates of the physical spacetimes, i.e.,

$$\lambda>0,\quad +\infty>\chi^1>-\infty,\quad +\infty>\chi^2>-\infty,\quad +\infty>\chi^3>-\infty\ . \tag{4.3}$$

It is worthwhile mentioning that the analytic continuation of the degenerate Kasner model is different from that of the entire Minkowski space. The latter has the same metric, but the range of parameters λ and χ^1 is given by

$$0\le\lambda<+\infty,\quad 0\le\chi^1\le 2\pi\ . \tag{4.4}$$

The geodesic distance $\tilde{\sigma}$ between two points $\chi'(\lambda',\vec{\chi}\,')$ and $\chi(\lambda,\vec{\chi})$ of the Riemannian manifolds (4.2) is given by

$$2\tilde{\sigma}=\begin{cases} \lambda^2+\lambda'^2-2\lambda\lambda'\cos r, \quad r=|\vec{x}-\vec{x}\,'|, \quad p_1=p_2=1\ ;\\ \lambda^2+\lambda'^2-2\lambda\lambda'\cos\rho+(\chi^3-\chi^{3'})^2, \quad \rho^2=(\chi^1-\chi^{1'})^2+(\chi^2\chi'^2)^2, \quad p_1=1,\ p_2=0\ ;\\ \lambda^2+\lambda'^2+(\chi^2-\chi'^2)^2+(\chi^3-\chi'^3)^2-2\lambda\lambda'\cos(\chi^1-\chi'^1), \quad p_1=p_2=0\ .\end{cases} \tag{4.5}$$

and the biscalar $\tilde{\Delta}^{1/2}(\chi,\chi')$ is found to be

$$\tilde{\Delta}^{1/2}(\chi,\chi')=\begin{cases} \dfrac{\sin r}{r}, & p_1=p_2=1\ ,\\ \left[\dfrac{\sin\rho}{\rho}\right]^{1/2}, & p_1=1,\ p_2=0\ ,\\ 1, & p_1=p_2=0\ .\end{cases} \tag{4.6}$$

When $\lambda>0$, as required by Eq. (4.3), the expressions (4.5) are valid only when $r<\pi$, $\rho<\pi$, or $|\chi'-\chi'|<\pi$, respectively. Otherwise the points χ and χ' cannot be connected by a geodesic, which lies entirely in the domain $\lambda>0$. The latter corresponds to the domain of the physical spacetime located to the future of initial singularity (actual or fictitious).

The transformation (4.1) will be used now to derive the analytically continued kernels $\langle\chi\Omega\,|\,\chi'0\rangle$, where $\Omega=$ is as described in Sec. II. Using Eq. (3.27) for the physical kernels $\langle\chi s\,|\,\chi'0\rangle$ and rotating the corresponding components of the $\vec{k}$ vector in the complex plane by writing

$$k_x=-i\kappa_1;\quad k_y=\begin{cases} -i\kappa_2, & p_1=1\ ,\\ \kappa_2, & p_1=0\ ;\end{cases}\quad k_2=\begin{cases} -i\kappa_3, & p_2=1\ ,\\ \kappa_3, & p_2=0\end{cases} \tag{4.7}$$

one finds

$$\langle \chi\Omega \mid \chi' 0\rangle = i[(2\pi)^3 2\lambda\lambda'\Omega]^{-1}\exp\left[-\frac{(\lambda^2+\lambda'^2)}{4\Omega}\right]\int d^3\kappa e^{i\vec{\kappa}\cdot(\vec{\chi}-\vec{\chi}')}\left[\Lambda_{\vec{\kappa}}^{(+)} I_{\vec{\kappa}}\left[\frac{\lambda\lambda'}{2\Omega}\right]+\Lambda_{\vec{\kappa}}^{(-)} I_{-\kappa}\left[\frac{\lambda\lambda'}{2\Omega}\right]\right], \quad p_1=p_2=1 , \quad (4.8)$$

$$\langle \chi\Omega \mid \chi' 0\rangle = -i[(2\pi)^3 2(\lambda\lambda')^{1/2}\Omega]^{-1}\exp\left[-\frac{(\lambda^2+\lambda'^2)}{4\Omega}\right] \times \int d^3\kappa e^{i\vec{\kappa}\cdot(\vec{\chi}-\vec{\chi}')-\kappa_3{}^2\Omega}\left[\Lambda_{\vec{\kappa}}^{(+)} I_{(\kappa_1{}^2+\kappa_2{}^2-\frac{1}{12})^{1/2}}\left[\frac{\lambda\lambda'}{2\Omega}\right] + \Lambda_{\vec{\kappa}}^{(-)} I_{-(\kappa_1{}^2+\kappa_2{}^2-\frac{1}{12})^{1/2}}\left[\frac{\lambda\lambda'}{2\Omega}\right]\right], \quad p_1=1,\ p_2=0 , \quad (4.9)$$

$$\langle \chi\Omega \mid \chi' 0\rangle = i[(2\pi)^3 2\Omega]^{-1}\exp\left[-\frac{(\lambda^2+\lambda'^2)}{4\Omega}\right] \times \int d^3\kappa e^{i\vec{\kappa}\cdot(\vec{\chi}-\vec{\chi}')-(\kappa_3{}^2+\kappa_2{}^2)\Omega}\left[\Lambda_{\vec{\kappa}}^{(+)} I_{\kappa_1}\left[\frac{\lambda\lambda'}{2\Omega}\right]+\Lambda_{\vec{\kappa}}^{(-)} I_{-\kappa_1}\left[\frac{\lambda\lambda'}{2\Omega}\right]\right], \quad p_1=p_2=0 , \quad (4.10)$$

where

$$\Lambda_{\vec{\kappa}}^{(+)} = \begin{cases} (1-e^{2\pi i\kappa})^{-1}(1-B^*_{-i\kappa}/C^*_{-i\kappa}), & p_1=p_2=1 ; \\ \{1-\exp[2\pi i(\kappa_1{}^2+\kappa_2{}^2-\frac{1}{12})^{1/2}]\}(1-B^*_{-i\kappa_2,-i\kappa_1}/C^*_{-i\kappa_2,-i\kappa_1}), & p_1=1,\ p_2=0 ; \\ [1-\exp(2\pi_{i\kappa_1})](1-B^*_{-i\kappa_1}/C^*_{-i\kappa_1}), & p_1=p_2=0 \end{cases} \quad (4.11)$$

and

$$\Lambda_{\vec{\kappa}}^{(-)} = \begin{cases} [\exp(-2\pi i\kappa)-1]^{-1}\{-1+[\exp(-2\pi i\kappa)]B^*_{-i\kappa}/C^*_{-i\kappa}\}, & p_1=p_2=1 ; \\ \{\exp[-2\pi i(\kappa_1{}^2+\kappa_2{}^2-\frac{1}{12})]^{1/2}-1\}^{-1}(-1+\{\exp[-2\pi i(\kappa_1{}^2+\kappa_2{}^2-\frac{1}{12})^{1/2}]\} \\ \qquad \times B^*_{-i\kappa_2,-i\kappa_1}/C^*_{-i\kappa_2,-i\kappa_1}), & p_1=1,\ p_2=0 ; \\ [\exp(-2\pi i\kappa_1)-1]^{-1}\{-1+[\exp(-2\pi i\kappa_1)]B^*_{-i\kappa_1}/C^*_{-i\kappa_1}\}, & \alpha_1=\alpha_2=0 . \end{cases} \quad (4.12)$$

Expressions (4.8)—(4.12) can be further simplified leading in each case to a single integral as follows:

$$\langle \chi\Omega \mid \chi' 0\rangle = i[(2\pi)^3 2\lambda\lambda'\Omega\rho]^{-1}\exp\left[-\frac{\lambda^2+\lambda'^2}{4\Omega}\right]4\pi \times \int_0^\infty \kappa\sin\kappa p\, dk\left[\Lambda_\kappa^{(+)} I_\kappa\left[\frac{\lambda\lambda'}{2\Omega}\right]+\Lambda_\kappa^{(-)} I_{-\kappa}\left[\frac{\lambda\lambda'}{2\Omega}\right]\right], \quad p_1=p_2=1 , \quad (4.13)$$

$$\langle \chi\Omega \mid \chi' 0\rangle = -i\sqrt{\pi}[(2\pi)^2(2\Omega)^{3/2}(\lambda\lambda')^{1/2}]^{-1}\exp\left[-\frac{(\chi^3-\chi'^3)^2+\lambda^2+\lambda'^2}{4\Omega}\right] \times \int_0^\infty u\mathscr{J}_0(u\rho)[\Lambda_u^{(+)} I_{(u^2-\frac{1}{12})^{1/2}}\left[\frac{\lambda\lambda'}{2\Omega}\right]+\Lambda_u^{(-)} I_{(u^2-\frac{1}{12})^{1/2}}\left[\frac{\lambda\lambda'}{2\Omega}\right]]du , \quad p_1=1,\ p_2=0 , \quad (4.14)$$

and

$$\langle \chi\Omega \mid \chi'0\rangle = i(4\pi\Omega)^{-2}\exp\left[-\frac{(\chi^2-\chi'^2)^2+(\chi^3-\chi'^3)^2+\lambda^2+\lambda'^2}{4\Omega}\right]$$
$$\times\int_{-\infty}^{+\infty} d\kappa_1 e^{i\kappa_1(\chi^1-\chi'^1)}\left[\Lambda^{(+)}_{\kappa_1} I_{+\kappa_1}\left[\frac{\lambda\lambda'}{2\Omega}\right]+\Lambda^{(-)}_{\kappa_1} I_{-\kappa_1}\left[\frac{\lambda\lambda'}{2\Omega}\right]\right],\quad p_1=p_2=0\,. \tag{4.15}$$

Making use of the integral representation of the Bessel functions $I_\kappa(z)$ given in Ref. 10, Eq. (3.11),

$$I_\kappa(z)=(2\pi i)^{-1}\int_{\infty-\pi i}^{\infty+\pi i} dw\,\exp(z\cosh w-\kappa w)\,, \tag{4.16}$$

one finds that the integrals in Eqs. (4.13) and (4.14) are convergent if and only if $\Lambda^{(-)}_{\vec{\kappa}}=0$. The contour of integration in Eq. (4.16) has to be located entirely in the domain Re$w>0$. In the case of the degenerate Kasner universe the kernel has a well-defined Fourier transform only when

$$\Lambda^{(+)}_{\vec{k}}=\theta(\kappa_1),\quad \Lambda^{(-)}_{\vec{k}}=\theta(-\kappa_1)\,. \tag{4.17}$$

We thus see that the requirement that the Riemannian kernel be expandable into a Fourier integral, or, equivalently, the requirement of square integrability of the Riemannian kernel given by Eq. (2.14), singles out a unique propagator in each of the models considered.

Let us determine now the initial conditions of the scalar field which correspond to the selected kernels. Both in the isotropic case, considered previously in Refs. 1 and 3 and in the anisotropic case with $p_1=1, p_2=0$ the condition $\Lambda^{(-)}_{\vec{k}}=0$ gives $B^*_\nu=C^*_\nu e^{-2\pi\nu}$. Using the Wronskian condition (3.13) one finds

$$|B_{\vec{k}}|=2^{-1}\pi^{1/2}\exp\left[-\frac{\pi\nu}{2}\right][\exp(2\pi\nu)-1]^{1/2}\,,$$
$$|C_{\vec{k}}|=2^{-1}\pi^{1/2}[\exp(2\pi\nu)-1]^{-1/2}\,. \tag{4.18}$$

The absolute values of the Bogoliubov coefficients follow from Eqs. (3.24) and (4.18):

$$|\alpha_{\vec{k}}|=\exp(\pi\nu)[\exp(2\pi\nu)-1]^{-1/2}\,,$$
$$|\beta_{\vec{k}}|=[\exp(2\pi\nu)-1]^{-1/2}\,. \tag{4.19}$$

Using the latter expressions for $|\alpha_{\vec{k}}|$ and $|\beta_{\vec{k}}|$ and the results of Ref. 9, the probability of detecting n particles in the $\vec{k}$th mode at late times is found to be

$$P_n(\vec{k})=\exp(-2\pi\nu n)[1-\exp(-2\pi\nu)]\,. \tag{4.20}$$

The propagator selected for the isotropic case is the same as that of Chitre and Hartle[3] and corresponds to the isotropic distribution of created quanta. For sufficiently large momenta of created particles the spectrum (4.20) resembles that of the blackbody radiation with the temperature

$$T=(\pi t k_B)^{-1}\,, \tag{4.21}$$

where k_B is the Boltzmann constant.

In the anisotropic case $p_1=1, p_2=0$ the initial condition (4.18) leads to production of particles with anisotropic momentum distribution. Particles are created with probability which is a function only of the transverse component of the physical momentum $p_\perp=(k_x{}^2+k_y{}^2)^{1/2}t$.

For sufficiently large $p_\perp$ the constant $\frac{1}{12}$ in the expression $\nu=(k_x{}^2+k_y{}^2+\frac{1}{12})^{1/2}$ can be neglected as well as the mass term in the energy of created quanta. Under this condition the spectrum of created particles resembles that of the two-dimensional blackbody radiation with the temperature given by Eq. (4.21). The initial conditions leading to this spectrum were first discussed by Nariai.[12]

Finally let us determine the initial conditions of the scalar field in the degenerate Kasner universe. From Eq. (4.17) it follows that

$$B_{\vec{k}}=\begin{cases} C_{\vec{k}}\exp(-2\pi|k_x|), & k_x\geq 0\,,\\ C_{\vec{k}}, & k_x<0\,.\end{cases} \tag{4.22}$$

The corresponding Feynman propagator is different from the standard Minkowski-space propagator (3.28) for which $B_{\vec{k}}=0$. This difference results from the fact that the complexification scheme used above is not equivalent to the standard Euclidean treatment of the Minkowski space. Using Eqs. (3.13) and (3.24) the coefficients of the Bogoliubov transformation between the in and out basis functions are found to be

$$|\alpha_{\vec{k}}|=\exp(\pi|k_x|)[\exp(2\pi|k_x|)-1]^{-1/2}\,,$$
$$|\beta_{\vec{k}}|=[\exp(2\pi|k_x|)-1]^{-1/2}\,. \tag{4.23}$$

For sufficiently large k_x, namely,

$k_x{}^2/t^2 \gg k_y{}^2+k_z{}^2+m^2$, the spectrum of the out particles resembles that of a unidirectional blackbody radiation with the temperature defined by Eq. (4.21). Integrating over all modes one finds that the total number of the out particles present in the in vacuum is infinite. Consequently the Fock representations built on the in and out vacuums are not unitarily equivalent.

As follows from Eq. (3.2) in the present case the operator $-i\partial/\partial x$ is the generator of the Lorentz boosts in the T-X plane. Since both in and out basis functions are the eigenfunctions of this operator they are invariant under these transformations. However, as was shown by Sommerfield[13] and Fulling, Parker and Hu[11] by using the Fourier integral representations of the Hankel functions, only the out particles can be represented by the wave packets constructed entirely from the Poincaré-invariant positive-frequency solutions. This fact is also reflected in the corresponding Feynman propagator (3.28).

The in vacuum, determined by the Bogoliubov transformation (4.23), was discussed by Sommerfield,[13] Berger,[14] and Davies and Fulling.[15] This vacuum can be obtained by diagonalizing the t-time dependent Hamiltonian at $t=0$.[13,14] As was pointed out by Parker,[9] from the operational point of view one would not expect creation of particles in the degenerate Kasner universe, since the space-time is flat and the clocks measuring the Kasner time are not accelerated.

V. ANALYSIS OF THE KERNELS

Let us now examine in more detail the kernels selected in the previous section, which will be denoted by $\langle \chi\Omega \mid \chi' 0\rangle^{(1)}$:

$$\langle \chi\Omega \mid \chi' 0\rangle^{(1)}=i[(2\pi)^3 2\lambda\lambda'\Omega r]^{-1}\exp\left[-\frac{(\lambda^2+\lambda'^2)}{4\Omega}\right]4\pi\int_0^\infty k\,\sin krI_\kappa\left[\frac{\lambda\lambda'}{2\Omega}\right]dk,\quad p_1=p_2=1\,, \tag{5.1}$$

$$\langle \chi\Omega \mid \chi' 0\rangle^{(1)}=-i\sqrt{\pi}[(2\pi)^2 2\Omega^{3/2}(\lambda\lambda')^{1/2}]^{-1}\exp\left[-\frac{(\chi^3-\chi'^3)^2+\lambda^2+\lambda'^2}{4\Omega}\right]$$
$$\times\int_0^\infty u\mathscr{J}_0(u\rho)I_{(u^2-\frac{1}{12})^{1/2}}\left[\frac{\lambda\lambda'}{2\Omega}\right]du,\quad p_1=1, p_2=0\,, \tag{5.2}$$

$$\langle \chi\Omega \mid \chi' 0\rangle^{(1)}=i(4\pi\Omega)^{-2}\exp\left[-\frac{(\chi^3-\chi'^3)^2+(\chi^2-\chi'^2)^2+\lambda^2+\lambda'^2}{4\Omega}\right]$$
$$\times\int_0^\infty d\kappa_1\cos[\kappa_1(\chi^1-\chi'^1)]I_{\kappa_1}\left[\frac{\lambda\lambda'}{2\Omega}\right],\quad p_1=p_2=0\,. \tag{5.3}$$

For the Riemannian metrics (4.2) one can rewrite the parabolic equation (2.11) as

$$\frac{\partial}{\partial\Omega}\langle \chi\Omega \mid \chi' 0\rangle=\left[\lambda^{-(1+p_1+p_2)}\frac{\partial}{\partial\lambda}\left[\lambda^{1+p_1+p_2}\frac{\partial}{\partial\lambda}\right]+\lambda^{-2}\frac{\partial^2}{(\partial\chi^1)^2}+\lambda^{-2p_1}\frac{\partial^2}{(\partial\chi^2)^2}\right.$$
$$\left.+\lambda^{-2p_2}\frac{\partial^2}{(\partial\chi^3)^2}+\frac{\gamma}{\lambda^2}\right]\langle \chi\Omega \mid \chi' 0\rangle\,. \tag{5.4}$$

By substituting the kernels (5.1)–(5.3) into Eq. (5.4) and interchanging the orders of differentiation and integration one verifies that the kernels $\langle \chi\Omega \mid \chi' 0\rangle^{(1)}$ are solutions of the parabolic equations for all values of coordinates χ. The latter change of orders of differentiation and integration is justified, provided the contour of integration in Eq. (4.16) is taken to lie in the right half of the complex w plane.

Let us show now that $\langle \chi\Omega \mid \chi' 0\rangle^{(1)}$ satisfy also the initial condition (2.12). Using the large argument asymptotic expansions of function I_ν as given in Ref. 10, p. 203, and the Gaussian approximation of the δ function one obtains

$$\langle \chi\Omega \mid \chi' 0\rangle^{(1)}\underset{\Omega\to 0^+}{\sim} i\lambda^{-3}\delta(\chi,\chi')+i(\lambda\lambda')^{-3/2}2\pi^{-3}\delta(\lambda+\lambda')\int d^3\kappa e^{i\vec{\kappa}\cdot(\vec{\chi}-\vec{\chi}')\pm(\kappa+\frac{1}{2})\pi i},\quad p_1=p_2=1\,, \tag{5.5}$$

$$\langle \chi\Omega | \chi'0\rangle^{(1)} \underset{\Omega\to 0^+}{\sim} -i\lambda^{-2}\delta(\chi,\chi') - i(\lambda\lambda')^{-1}(2\pi)^{-2}\delta(\lambda+\lambda')\delta(\chi^3-\chi'^3) \times \int_{-\infty}^{+\infty} d\kappa_2 \int_{-\infty}^{+\infty} d\kappa_3 \exp\{i[\kappa_1(\chi^1-\chi'^1)+\kappa_2(\chi^2-\chi'^2)\pm\pi(\kappa_1{}^2+\kappa_2{}^2-\tfrac{1}{2}+\tfrac{1}{2})^{1/2}]\}\ ,\quad p_1=1,\ p_2=0\ , \tag{5.6}$$

$$\langle \chi\Omega | \chi'0\rangle^{(1)} \underset{\Omega\to 0^+}{\sim} i\lambda\delta(\chi,\chi') + i(\lambda\lambda')^{-1/2}(2\pi)^{-1}\delta(\lambda+\lambda')\delta(\chi^3-\chi'^3)\delta(\chi^2-\chi'^2) \times \int_{-\infty}^{+\infty} d\kappa_1 e^{i[\kappa_1(\chi^1-\chi'^1)\pm\pi(|\kappa_1|+1/2)]},\quad p_1=p_2=0\ . \tag{5.7}$$

In the domain $\lambda,\lambda'>0$ which corresponds to the region of the physical spacetime, located to the future of the singularity (physical in the cases $p_1\neq 0$ and fictitious in the degenerate Kasner model), the kernels (5.1)–(5.3) satisfy the initial conditions. However in order to fix the kernels uniquely one has to consider the domain $\lambda,\lambda'<0$ as well.[3] As Ω increases the effect of initial conditions in this extended domain diffuses into the region $\lambda>0$, $\lambda'>0$ and influences the behavior of the kernels considered.

In the particular cases $p_1=p_2=0$ and $p_1=p_2=1$ one can represent the kernels $\langle \chi\Omega | \chi'0\rangle^{(1)}$ in a more simple form. Let us consider first the case $p_1=p_2=0$. The contour of integration in the integral (4.16) might be always chosen as follows: it is a straight line from $w=\infty-\pi i$ to $w=\epsilon-\pi i$ ($\epsilon>0$), a straight line from $w=\epsilon-\pi i$ to $w=\epsilon+\pi i$, and, finally, a straight line from $w=\epsilon+\pi i$ to $w=\infty+\pi i$. Interchanging the orders of integration over κ_1 and w in Eq. (5.3) one obtains

$$\langle \chi\Omega | \chi'0\rangle^{(1)} = i(4\pi\Omega)^{-2}(2\pi i)^{-1}\left\{\int_{\infty-\pi i}^{\infty+\pi i} dw e^{(\lambda\lambda'/2\Omega)\cosh w}\left[\frac{1}{w+i(\chi^1-\chi'^1)}+\frac{1}{w-i(\chi^1-\chi'^1)}\right]\right\} \times\exp\left[-\frac{(\chi^2-\chi'^2)^2+(\chi^3-\chi'^3)^2\lambda^2+\lambda'^2}{4\Omega}\right],\quad p_1=p_2=0\ . \tag{5.8}$$

The integral in the latter equation might be further simplified:

$$\int_{\infty-\pi i}^{\infty+\pi i} dw e^{(\lambda\lambda'/2\Omega)\cosh w}\left[\frac{1}{\omega+i(\chi^1-\chi'^1)}+\frac{1}{\omega-i(\chi^1-\chi'^1)}\right]$$
$$=\int_{-\infty}^{-\epsilon} dw e^{-(\lambda\lambda'/2\Omega)\cosh w}\left\{\frac{1}{w+i[(\chi^1-\chi'^1)+\pi]}+\frac{1}{w+i[-(\chi^1-\chi'^1)+\pi]}\right\}$$
$$+\int_{\epsilon}^{\infty} dw e^{-(\lambda\lambda'/2\Omega\cosh w}\left[\frac{1}{w+i[\pi+(\chi^1-\chi'^1)]}+\frac{1}{w+i[\pi-(\chi^1-\chi'^1)]}\right]$$
$$+\int_{-\pi}^{\pi} dv e^{(\lambda\lambda'/2\Omega)\cos(v-i\epsilon)}\left[\frac{1}{v-i\epsilon+(\chi^1-\chi'^1)}+\frac{1}{v-i\epsilon(\chi^1-\chi'^1)}\right]. \tag{5.9}$$

In the limit $\epsilon\to 0$ one then finds

$$\langle \chi\Omega | \chi'0\rangle^{(1)} = \langle \chi\Omega | \chi'0\rangle^{(0)}\{\theta[\pi+(\chi^1-\chi'^1)]-\theta[-\pi+(\chi^1-\chi'^1)]\} - i(2\pi)^{-1}(4\pi\Omega)^{-2}\left[\int_{-\infty}^{+\infty} e^{-(\lambda\lambda'/2\Omega)\cosh w}\left[\frac{1}{w+i[\pi+(\chi^1-\chi'^1)]}+\frac{1}{w+i[\pi-(\chi^1-\chi'^1)]}\right]\right] \times\exp\left\{-\frac{(\chi^2-\chi'^2)^2+(\chi^3-\chi'^3)^2+\lambda^2+\lambda'^2}{4\Omega}\right\}, \tag{5.10}$$

where the kernel $\langle \chi\Omega | \chi'0\rangle^{(0)}$ for $p_1=p_2=0$ is given by

$$\langle \chi\Omega | \chi'0\rangle^{(0)} = i(4\pi\Omega)^{-2}\exp\left\{-\frac{[\lambda^2+\lambda'^2-2\lambda\lambda'\cos(\chi^1-\chi'^1)]+(\chi^2-\chi'^2)^2+(\chi^3-\chi'^3)^2}{4\Omega}\right\},\quad p_1=p_2=0\ . \tag{5.11}$$

Each term in Eq. (5.10) is discontinuous at $|\chi^1-\chi'^1|=\pi$. The discontinuity of the second term might be verified by using the identities[16]

$$\lim_{\epsilon\to 0}\frac{1}{w+i\epsilon}=-\pi i\delta(w)+P\left[\frac{1}{w}\right], \tag{5.12}$$

$$\lim_{\epsilon\to 0}\frac{1}{w-i\epsilon}=\pi i\delta(w)+P\left[\frac{1}{w}\right], \tag{5.13}$$

where $P[1/w]$ is the principal value of $1/w$. The discontinuities of both terms compensate each other, so that $\langle\chi\Omega\,|\,\chi'0\rangle^{(1)}$ is continuous and differentiable everywhere. Because of the discontinuities mentioned each term in Eq. (5.12) when taken separately violates the parabolic equation (5.4). It is worthwhile mentioning that the latter discontinuities appear exactly on the hypersurface, which separates points that can be connected by geodesics which do not cross the fictitious singularity at $\lambda=0$, from those points that cannot be connected by such geodesics.

In the case $p_1=p_2=1$ one obtains by a similar procedure

$$\langle\chi\Omega\,|\,\chi'0\rangle^{(1)}=\langle\chi\Omega\,|\,\chi'0\rangle^{(0)}_{p_1=p_2=1}\theta(\pi-r)$$
$$+\frac{ie^{(\lambda^2+\lambda'^2)/4\Omega}}{(2\pi)^3\Omega^2\rho}\int_{-\infty}^{+\infty}e^{-(\lambda\lambda'/2\Omega)\cosh v}\sinh v\left[\frac{1}{v+i(\pi-r)}-\frac{1}{v+i(\pi+r)}\right]dv\,, \tag{5.14}$$

where

$$\langle\chi\Omega\,|\,\chi'0\rangle^{(0)}=\frac{i}{(4\pi\Omega)^2}\frac{\sin r}{r}\exp\left[-\frac{\lambda^2+\lambda'^2-2\lambda\lambda'\cos r}{4\Omega}\right],\quad p_1=p_2=1\,. \tag{5.15}$$

When $\Omega\to 0$ the major contribution to $\langle\chi\Omega\,|\,\chi'0\rangle^{(1)}$ in Eqs. (5.10) and (5.14) comes from the terms $\langle\chi\Omega\,|\,\chi'0\rangle^{(0)}$, which are, in fact, the analytically continued kernels (3.31). The second terms in Eqs. (5.10) and (5.14) have an essential singularity at $\Omega=0$.

Making use of Eqs. (5.10) and (5.14) one can determine the behavior of kernels $\langle\chi\Omega\,|\,\chi'0\rangle^{(1)}$ in the limits $|\chi^1-\chi'^1|\to\infty$ $(p_1=p_2=0)$ and $r\to\infty$ $(p_1=p_2=1)$. In the first case the following inequality holds:

$$\left|\int_{-\infty}^{+\infty}e^{-(\lambda\lambda'/2\Omega)\cosh w}\left[\frac{1}{\{w+i[\pi+(\chi^1-\chi'^1)]\}}+\frac{1}{w+i[\pi-(\chi^1-\chi'^1)]}\right]dw\right|$$
$$<\frac{2\pi e^{-\lambda\lambda'/2\Omega}}{|\chi^1-\chi'^1|^2-\pi^2}\int_{-\infty}^{+\infty}e^{-(\lambda\lambda'/4\Omega)w^2}dw=\frac{4\pi^{3/2}\Omega}{\lambda\lambda'}\exp\left[-\frac{\lambda\lambda'}{2\Omega}\right][(\chi^1-\chi'^1)^2-\pi^2]^{-1}, \tag{5.16}$$

so that

$$|\langle\chi\Omega\,|\,\chi'0\rangle^{(1)}|<2^{-3}\pi^{-3/2}\Omega^{-3/2}(\lambda\lambda')^{-1}[(\chi^1-\chi'^1)^2-\pi^2]^{-1}$$
$$\times\exp\left[-\frac{(\lambda+\lambda')^2+(\chi^2-\chi'^2)^2+(\chi^3-\chi'^3)^2}{4\Omega}\right],\quad p_1=p_2=0\,. \tag{5.17}$$

Similarly one finds that in the case $p_1=p_2=1$,

$$|\langle\chi\Omega\,|\,\chi'0\rangle|^{(1)}<\pi^{-1}(r^2-\pi^2)^{-2}(\lambda\lambda')^{-3/2}(4\pi\Omega)^{-1/2}\exp\left[-\frac{(\lambda+\lambda')^2}{4\Omega}\right]. \tag{5.18}$$

Using Eqs. (5.17) and (5.18) one can explicitly see that the kernels $\langle\chi\Omega\,|\,\chi'0\rangle^{(1)}$ fall off rapidly enough when the separation between the points χ and χ' go to infinity, so that the square-integrability condition (2.16) is certainly satisfied. [This, obviously, follows also from the Fourier integral representations (5.1)–(5.3).]

Let us show now that the kernels (5.1)–(5.3) can be represented by well-defined iterated integrals (2.13), and therefore correspond to well-defined Riemannian path integrals, defined over the regions given by Eq.

(4.3).

First let us evaluate the integral

$$\int \langle \mathcal{X}''2\epsilon \,|\, \mathcal{X}'\epsilon\rangle^{(1)}\langle \mathcal{X}'\epsilon \,|\, \mathcal{X}0\rangle^{(1)}\gamma(\mathcal{X}'')^{1/2}d^4\mathcal{X}''$$

for the isotropic case. Substituting the kernels, given by Eq. (5.1) in the above expression, and performing the integration over the spatial variables $\mathcal{X}'$, one obtains

$$\int \langle \mathcal{X}''2\epsilon \,|\, \mathcal{X}'\epsilon\rangle^{(1)}\langle \mathcal{X}'\epsilon\rangle^{(1)}\langle \mathcal{X}'\epsilon \,|\, \mathcal{X}0\rangle^{(1)}[\gamma(\mathcal{X}')]^{1/2}d^4\mathcal{X}''$$
$$= -(2\pi)^3(2\epsilon)^{-2}(\lambda'\lambda)^{-1}\exp\left[-\frac{\lambda''^2+\lambda'^2}{4\epsilon}\right]\int_0^\infty d\lambda\lambda'\exp\left[-\frac{\lambda'^2}{2\epsilon}\right]$$
$$\times \int d^3\vec{\kappa}e^{i\vec{\kappa}\cdot(\vec{\mathcal{X}}-\vec{\mathcal{X}}')}I_\kappa\left[\frac{\lambda'\lambda''}{2\epsilon}\right]I_\kappa\left[\frac{\lambda'\lambda}{2\epsilon}\right]. \quad (5.19)$$

Using the integral identity (Ref. 10, p. 395)

$$\int_0^\infty e^{-\epsilon x^2}\mathcal{J}_k(\lambda' x)\mathcal{J}_k(\lambda'' x)x\,dx = 2\epsilon^{-1}\exp\left[-\frac{\lambda'^2+\lambda''^2}{4\epsilon}\right]I_k\left[\frac{\lambda'\lambda''}{2\epsilon}\right] \quad (5.20)$$

valid for $k > -1$, and the orthogonality of Bessel functions

$$\int_0^\infty x\,dx\mathcal{J}_k(\lambda'' x)\mathcal{J}_k(\lambda' x) = \frac{\delta(\lambda''-\lambda')}{\lambda''}, \quad (5.21)$$

one obtains

$$\int \langle \mathcal{X}''2\epsilon \,|\, \mathcal{X}'\epsilon\rangle^{(1)}\langle \mathcal{X}'\epsilon \,|\, \mathcal{X}0\rangle^{(1)}[\gamma(\mathcal{X}'')]^{1/2} = -\langle \mathcal{X}''2\epsilon \,|\, \mathcal{X}0\rangle^{(1)},\ p_1=p_2=1\,. \quad (5.22)$$

Making use of the latter result in the iterated integral (2.13) and adjusting the phase by introducing the weight factor $-i$ to be associated with each integration involved, one concludes that $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(1)}$ indeed admits path-integral representation. Similar results are obtained for remaining cases $p_1=1$, $p_2=0$ (with the phase factor i), and $p_1=p_2=0$ (the weight factor i).

The next problem one faces is to recover the functional integral form of the iterated integrals found, namely, the action associated with each path and the functional measure. This problem, which is a nontrivial one even for unbounded coordinates is especially severe when the boundaries are present, as in the models considered here. It is still unresolved and will be considered in the future.

Finally, let us compare the kernels selected above using the square-integrability condition with some other solutions of the parabolic equation (5.4). In the particular cases $p_1=p_2$ kernels (5.11) and (5.15) are solutions of Eq. (5.4) which cannot be expanded into a Fourier integral, or equivalently, do not satisfy the square-integrability condition (2.14). For the case $p_1=p_2=0$ the Riemannian kernel $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(0)}$ given by Eq. (5.11) corresponds to the Minkowski-space Feynman propagator. It is the heat kernel for periodic space ($\mathcal{X}$ identified with $\mathcal{X}^1+2\pi$), lifted to the covering space. This is quite similar to the well-known Rindler[7] and Schwarzschild[2] cases. For the isotropic model, i.e., for the case $p_1=p_2=1$, the corresponding kernel $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(0)}$ given by Eq. (5.15) is not periodic because of the factor r^{-1}. This kernel decreases when the separation between $\mathcal{X}$ and $\mathcal{X}'$ tends to infinity, and therefore it satisfies the Chitre-Hartle boundary condition mentioned in Sec. II. However, the kernel (5.15) falls off too slowly, so that the square-integrability condition (2.14) is not satisfied. It is easy to verify by direct substitution that the iterated integrals (2.13) which involve the kernels (5.11) and (5.15) are ill defined in the domain $\lambda > 0$, $-\infty < \mathcal{X}^1,\mathcal{X}^2,\mathcal{X}^3 < +\infty$ because of the oscillating character of the integrands. The one-parameter family of kernels

$$\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(Q)} = Q\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(1)} + (1-Q)\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(0)}, \quad (5.23)$$

the members of which satisfy the parabolic equation (5.4), also lead to ill-defined Riemannian path integrals in the infinite domain of coordinates involved when $Q\neq 1$.

Let us examine the difference between the kernels $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(1)}$ and $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(Q)}$ ($Q\neq 1$) from the point of view of the initial condition (2.12). Following Chitre and Hartle[1] one requires this condition to hold in the domain in which the points $\mathcal{X}$ and $\mathcal{X}'$ can be connected by a single geodesic that does not cross the singularity (actual for $p_1=p_2=1$ or fictitious for $p_1=p_2=0$) at $\lambda=0$. For the case $p_1=p_2=0$ the above domain is defined by $\lambda>0$, $|\mathcal{X}^1-\mathcal{X}'^1|<\pi$, and for the case $p_1=p_2=1$ it is given by $\lambda>0$, $r<\pi$. In these domains each of the kernels $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(Q)}$ satisfies the initial condition (2.12). One can then consider whether the latter kernels satisfy the initial condition (2.12) globally, i.e., in the entire region $\lambda>0, -\infty<\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3<+\infty$ corresponding to the Riemannian schemes chosen in this paper. According to Eqs. (5.5) and (5.7) the kernels $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(1)}$, selected by the square-integrability requirement, indeed satisfy the initial condition (2.12). One the other hand the kernels $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(0)}$, and consequently $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(Q\neq 1)}$, violate this condition in the regions $|\mathcal{X}^1-\mathcal{X}'^1|\geq\pi$ for $p_1=p_2=0$ or $r>\pi$ for $p_1=p_2=1$. In the case $p_1=p_2=0$ the kernel $\langle \mathcal{X}\Omega \,|\, \mathcal{X}'0\rangle^{(0)}$ given by Eq. (5.11) is periodic, so that the δ function at $\mathcal{X}=\mathcal{X}'$ is repeated at points $\mathcal{X}_m=(\lambda',\mathcal{X}'^1+2\pi m,\mathcal{X}'^2,\mathcal{X}'^3)$ for all $m=\pm1,\pm2,\ldots$. In the case $p_1=p_2=1$ the corresponding kernel (5.15) is not periodic, but as $\Omega\rightarrow 0^+$ it also evolves into δ functions repeated periodically with reduced strength. For the models considered here and in Ref. 3 the above analysis suggests that the "true" propagator can be singled out by imposing the initial condition (2.12) in the entire domain (4.3) corresponding to the Riemannian schemes chosen. The resulting kernel is the one which is selected by the nonlocal boundary condition of square integrability and admits a well-defined Riemannian path-integral representation.

VI. SUMMARY AND DISCUSSION

The path-integral formulation of quantum field theory in curved spacetimes faces serious difficulties when the background geometry is singular or incomplete. As applied to the singular cosmological models the important idea of Chitre and Hartle to restrict the paths to those located to the future of the initial singularity does not resolve the problem completely, since it is unclear what is the action to be associated with such paths and what is the corresponding functional measure.

The complexification of spacetimes is not so much a method for evaluating formally divergent expressions, but rather a way to establish the form of the kernels of the Feynman propagators. The major difficulty of this approach is its noncovariance. More specifically, one applies the analytic continuation to a certain coordinate system so that different Riemannian schemes lead to nonidentical field theories. A particular scheme seems to be justified if the operational meaning can be given to corresponding particle states.

Once the Riemannian section is chosen there is a problem of finding the propagator corresponding to this choice. Because of the lack of knowledge of both the action and the functional measure of a path integral when the manifold is singular or incomplete, one is forced to solve the parabolic equation and to choose the appropriate boundary conditions. It is at this stage that the path integral plays an important role, being the selection principle for the solutions of the above equation. I conjecture that there is only one Riemannian kernel that admits a path-integral representation defined on the domain specified by the complexification scheme chosen. The existence of the Riemannian path integral, or more precisely, of the iterated integral, is a boundary condition which is too implicit, but it certainly requires the integrands of the iterated integral to fall off rapidly when the separation between the arguments of the kernels involved tends to infinity. This condition may be embodied in the square integrability of the Riemannian kernel, proposed in Ref. 3 and explicitly used in this paper, although it is still not clear whether these conditions are equivalent.

In the present work all of these problems were not touched in their full generality, but rather a restricted class of linearly expanding Bianchi type-I models was discussed. For all of them the square integrability singles out only one propagator within the Riemannian algorithms used. However, as was explicitly shown in the case of the degenerate Kasner model, the condition of square integrability is not totally invariant, because the limits of integration are determined by the Riemannian scheme chosen. Furthermore, it was shown that all the kernels selected this way can be represented by well-defined iterated integrals, although the problem of explicit functional integral representation of the results still remains to be solved. It was also shown that the kernels selected by the square-integrability condition satisfy the initial condition at $\Omega\rightarrow 0^+$ in the entire domain chosen by the Riemannian schemes used. Explicit expressions were found for some other heat kernels. Within the infinite domains of coordinates involved the latter kernels violate the initial condition at $\Omega\rightarrow 0^+$ and do not admit well-defined path-integral representations.

Finally, all of the selected propagators correspond to particle creation with spectra that at higher energies resemble those of one-, two-, or three-dimensional blackbody radiation. How general this result is is also a problem which has to be solved in the future.

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APPENDIX

Let us consider the propagator $G^{(0)}(xx')$ for the degenerate Kasner universe, which is given by the Fourier integral

$$G_F^{(0)}(xx')=\frac{\pi i}{4(2\pi)^3}\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}H_{ik_x}^{(2)}(\mu t_>)H_{ik_x}^{(1)}(\mu t_<)\ . \tag{A1}$$

Using the result of Ref. 3, the integral over k_x is found to be

$$\int_{-\infty}^{+\infty} e^{ik_x(x-x')}H_{ik_x}^{(2)}(\mu t_>)H_{ik}^{(1)}(\mu t_<)=\begin{cases}-2iH_0^{(2)}((-2\mu^2\tilde{\sigma}_1)^{1/2}), & \sigma_1<0\ ,\\ \dfrac{4}{\pi}K_0((2\mu^2\tilde{\sigma}_1)^{1/2}), & \sigma_1>0\ ,\end{cases} \tag{A2}$$

where

$$2\tilde{\sigma}_1=-[t^2+t'^2-2tt'\cosh(z-z')]\ , \tag{A3}$$

and $K_0(x)$ is the Hankel function of the imaginary argument of order zero.

Let us consider first the case $\sigma_1>0$. Using the integral representation of the Bessel function (Ref. 10, p. 359)

$$\int_0^{2\pi} e^{iu\rho\cos\phi}d\phi=2\pi\mathscr{J}_0(u\rho)\ , \tag{A4}$$

one can rewrite the integral in Eq. (A1) as

$$G_F^{(0)}(xx')=\frac{i}{(2\pi)^2}\int_0^{\infty} d\lambda\mathscr{J}_0(\lambda\rho)K_0([2\tilde{\sigma}_1(\lambda^2+m^2)]^{1/2})\ , \tag{A5}$$

where

$$\rho^2=(y-y')^2+(z-z')^2\ . \tag{A6}$$

Using the standard integral given in Ref. 17, p. 706, this leads to

$$G_F^{(0)}(xx')=\frac{i}{4\pi^2}\frac{m}{\sqrt{2\sigma}}K_1((2m^2\sigma)^{1/2})=-\frac{m^2}{8\pi}\frac{H_1^{(2)}((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}}\ , \tag{A7}$$

where σ is given by Eq. (3.29). In the case $\sigma_1<0$,

$$G^{(0)}(xx')=\frac{\pi}{2(2\pi)^3}\int_{-\infty}^{+\infty}dk_2e^{ik_2(z-z')}\int_{-\infty}^{+\infty}dk_ye^{ik_y(y-y')}H_0^{(2)}((-2\mu^2\sigma_1)^{1/2})\ . \tag{A8}$$

Using the integral representation of the Hankel function $H_0^{(2)}$ as given in Ref. 10, p. 180 the latter expression can be written as

$$G^{(0)}(xx')=\frac{i}{2(2\pi)^3}\int_{-\infty}^{+\infty}dk_2e^{ik_2(z-z')}\int_{-\infty}^{+\infty}dk_ye^{ik_y(y-y')}\int_0^{\infty}\frac{du}{u}\exp\left[\left[i\frac{\sigma_1}{u}-i\frac{m^2u}{2}\right]-\frac{ik_x{}^2u}{2}-\frac{ik_y{}^2u}{2}\right]\ . \tag{A9}$$

Introducing small convergence factors and interchanging the orders of integration leads to the following expression:

$$G^{(0)}(xx')=\frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s^2}\exp\left[-im^2s+\frac{i\sigma}{s}\right], \tag{A10}$$

where σ is defined by Eq. (3.29). Using the result of Ref. 4 one finds that in this case once again

$$G^{(0)}(xx')=-\frac{m^2}{8\pi}\frac{H_1^{(2)}((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}} . \tag{A11}$$

Therefore in both cases, $\sigma_1<0$ and $\sigma_1>0$, the resulting propagator is given by Eq. (A11).

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