PHYSICAL REVIEW D

PARTICLES AND FIELDS

THIRD SERIES, VOLUME 26, NUMBER 12 15 DECEMBER 1982

Singularities and evolution of the Szekeres cosmological models

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(Received 17 May 1982)

We present a new formulation of the two classes of Szekeres solutions of the Einstein field equations, which unifies the solutions as regards their dynamics, and relates them to the Friedmann-Robertson-Walker (FRW) cosmological models in a particularly transparent way. This reformulation enables us to give a general analysis of the scalar polynomial curvature singularities of the solutions, and of their evolution in time. In particular, the solutions which are close to an FRW model near the initial singularity, or in the late stages of evolution, are identified.

I. INTRODUCTION

The most interesting spatially inhomogeneous solutions of the nonvacuum Einstein field equations that have been published to date are those given by Szekeres.[1] These solutions are general in that they admit no Killing vector fields.[2] They were discovered by solving the Einstein field equations with irrotational dust as source, for the line element

$$ds^2=-dt^2+e^{2A}(dx^2+dy^2)+e^{2B}dz^2 ,$$

relative to comoving coordinates. Subsequently these solutions were invariantly characterized using the rate of shear tensor of the matter, and the spatial curvature tensor,[3] and also by using the Weyl conformal curvature tensor.[4] This revealed that the Szekeres solutions are considerably more specialized than the absence of Killing vectors might suggest.

The Szekeres solutions fall naturally into two classes, which are referred to as classes I and II. The class-I solutions are usually presented in a way that is formally analogous to the Tolman-Bondi spherically symmetric solutions, which they generalize. This class of solutions has primarily been used to model nonspherical collapse of an inhomogeneous dust cloud.[5–7] The class-II solutions are usually considered as generalizations of the Kantowski-Sachs[8] and Friedmann-Robertson-Walker (FRW) solutions and have primarily been studied as cosmological models.[9–11]

Despite the fact that the usual forms of the class I and II solutions are quite different, we have noticed that *the time evolution along a particular fluid world line is the same in both classes.* In fact *this time evolution is governed by precisely the same functions which govern the evolution of density perturbations in FRW dust models* (see, for example, Refs. 12–14 and references therein). We thus give a new formulation of the Szekeres solutions which unifies the two classes from the point of view of time evolution, and which relates the solutions to the FRW solutions in a particularly transparent way. In this reformulation two of the arbitrary functions appearing in the solutions (which we label $\beta_{\pm}$) can be linked to the increasing and decreasing density perturbation modes which arise in discussions of spatially inhomogeneous perturbations of FRW dust solutions (see, for example, Refs. 12–16). We show that these functions directly determine the (local) nature of the initial singularity and the future evolution of the solutions.

The new form of the solutions is presented in Sec. II together with expressions for various geometrical quantities of interest. The possible types of initial singularity that may arise are discussed in detail in Sec. III and the future evolution of the solutions is considered in Sec. IV. The role of the spatial curvature is discussed in Sec. V, and Sec. VI contains the concluding remarks. The relationship between the new and old forms of the Szekeres solutions is given in an appendix. Our sign conventions as regards the Riemann and Ricci tensors are those of Refs. 21 and 32, and we use geometrized units, so that $8\pi G=1=c$.

II. A NEW FORMULATION OF THE SZEKERES SOLUTIONS

The line element for both classes of the Szekeres solutions can be written in the following general form:

$$ds^2 = -dt^2 + S^2[e^{2\nu}(dx^2+dy^2)+H^2W^2dz^2] \,, \quad (2.1)$$

where

$$H = A - \beta_+ f_+ - \beta_- f_- \,, \quad (2.2)$$

and it is assumed that H, S, and W are positive functions. The coordinates are comoving and synchronous, so that the four-velocity of the dust is

$$u = \frac{\partial}{\partial t} \,, \quad (2.3)$$

and the hypersurfaces orthogonal to u are t=const. These hypersurfaces will be referred to as the "slices."

The evolution of the solutions in time is governed by S, f_+, and f_-, which are the only functions that depend on the time coordinate t. These functions are, however, independent of the x and y coordinates. The function S satisfies the (generalized) Friedmann equation

$$\dot{S}^2 = -k + \frac{2M}{S} \,, \quad (2.4)$$

where M is independent of t, x, and y, and $k = +1$, -1, or 0. The functions f_+ and f_- are, respectively, the increasing and decreasing solutions of the *linear* differential equation

$$\ddot{F} + 2\frac{\dot{S}}{S}\dot{F} = \frac{3M}{S^3}F \,. \quad (2.5)$$

Here and elsewhere, an overdot denotes $\partial/\partial t$.

The matter density for both classes of solutions is given by

$$\mu = \frac{6M}{S^3}\left[1 + \frac{\beta_+ f_+ + \beta_- f_-}{H}\right] , \quad (2.6)$$

or, equivalently, using (2.2),

$$\mu = \frac{6MA}{S^3 H} \,. \quad (2.7)$$

Remarks.

(i) The function S corresponds to the expansion factor in the FRW models in the sense that for each value of z it satisfies the Friedmann equation.

(ii) In discussions of linear perturbations of the FRW solutions with dust as source it is shown that there are two modes of density fluctuations called the growing and decaying modes (see, for example, Refs. 14 and 15 and references therein). It is remarkable that in the case when the perturbed solution also has zero pressure, the differential equation determining these modes is, in fact, Eq. (2.5). This is implicit in the previous references, although the authors do not write down the equation explicitly. It has been discussed, for example, by Weinberg[17] and Peebles.[18] A coordinate-independent derivation of (2.5) in the linearized theory is given by Raychaudhuri.[19] The same equation also governs the density fluctuations arising in Newtonian theory.[17]

The two classes of Szekeres solutions differ as regards spatial dependence of the metric. Before giving the spatial dependence, which is rather complicated, it is convenient to give the explicit forms of the time-dependent functions S, f_+, and f_-, which are the same for the two classes. It is first necessary to impose a restriction on the function M in Eq. (2.4), which is motivated as follows. When one specializes the Szekeres solutions to the FRW solutions (by setting $\beta_+ = 0 = \beta_-$, although this is not immediately obvious), it turns out that the sign of the matter density is determined by the sign of M. Since we are primarily interested in relating the Szekeres solutions to FRW solutions with positive matter density we assume $M > 0$. It now follows from (2.4) that when $k=0$ or -1, $\dot{S}$ cannot change sign. For these values of k, we assume without essential loss of generality, that $\dot{S} > 0$ ($\dot{S} < 0$ simply leads to models which are the time reverse of the models we consider). With these assumptions the solutions of (2.4) can be written implicitly in terms of an auxiliary variable η as

$$S = Mh'(\eta), \quad \text{with } t - T = Mh(\eta) \,, \quad (2.8)$$

where

$$h(\eta) = \begin{cases} \eta - \sin\eta, & k = +1 \,, \\ \sinh\eta - \eta, & k = -1 \,, \\ \frac{1}{6}\eta^3, & k = 0 \,. \end{cases} \quad (2.9)$$

Here a prime denotes $d/d\eta$ and T is a "constant" of integration, which is independent of t, x, and y.

Equation (2.5) can now be solved implicitly yielding the following expressions[20] for $f_\pm$:

$$f_+ = \begin{cases} \frac{6M}{S}[1 - (\eta/2)\cot\eta/2] - 1 \,, & k = +1 \,, \\ \frac{6M}{S}[1 - (\eta/2)\coth\eta/2] + 1 \,, & k = -1 \,, \\ \frac{1}{10}\eta^2, & k = 0 \,, \end{cases} \quad (2.10)$$

$$f_- = \begin{cases} \frac{6M}{S}\cot\eta/2 \,, & k = +1 \,, \\ \frac{6M}{S}\coth\eta/2 \,, & k = -1 \,, \\ \frac{24}{\eta^3} \,, & k = 0 \,. \end{cases} \quad (2.11)$$

When $k=0$ one can set $M=\frac{2}{9}$ without loss of generality in both classes.

The two classes of solutions arise as follows. The class-I solutions are the general solutions, while the class-II solutions are obtained if we impose the further restriction that $S_z=0$ (the subscript z denotes differentiation with respect to z). This implies that $f_\pm$, M, and T are also independent of z. In addition in class II the parameter η in (2.8) depends only on t, whereas, in general, it depends on t and z. Explicitly, the two classes of solutions are determined as follows:

Class I: $S=S(t,z),\ S_z\neq 0,\ f_\pm=f_\pm(t,z),\ T=T(z),\ M=M(z)$:

$$e^{\nu}=f(z)[a(z)(x^2+y^2)+2b(z)x+2c(z)y+d(z)]^{-1}, \tag{2.12}$$

where

$$ad-b^2-c^2=\tfrac{1}{4}\epsilon,\quad \epsilon=0,\pm 1, \tag{2.13}$$

$$A=f\nu_z-k\beta_+,\quad W^2=(\epsilon-kf^2)^{-1}, \tag{2.14}$$

$$\beta_+=-kfM_z/(3M),\quad \beta_-=fT_z/(6M). \tag{2.15}$$

Class II: $S=S(t),\ f_\pm=f_\pm(t),\ T=\text{const},\ M=\text{const}$:

$$e^{\nu}=[1+\tfrac{1}{4}k(x^2+y^2)]^{-1},\quad W=1, \tag{2.16}$$

$$A=\begin{cases} e^{\nu}\{a(z)[1-\tfrac{1}{4}k(x^2+y^2)]+b(z)x+c(z)y\}-k\beta_+, & k=\pm 1,\\ a(z)+b(z)x+c(z)y-\tfrac{1}{2}\beta_+(x^2+y^2), & k=0. \end{cases} \tag{2.17}$$

The relationship between the above form and the usual form of the Szekeres solutions is given in Appendix A.

Remarks.

(i) In class I the functions a,b,c,d,f,M,T are required to satisfy (2.13) and be sufficiently smooth functions of z, but are otherwise arbitrary. Since there is coordinate freedom of the form $z\to g(z)$, there are five essential arbitrary functions of z in this class.

(ii) In class II the functions $a,b,c,\beta_\pm$ are arbitrary but sufficiently smooth functions of z. The coordinate freedom allows us to specify one of these, and so there are four essential arbitrary functions in this class.

(iii) In both classes of solutions the two-surfaces $t,z=\text{const}$, are spaces of constant curvature. The sign of the curvature is determined by ϵ in class I [cf. Eq. (2.13)] and k in class II [cf. Eq. (2.16)].

Despite the rather complicated spatial dependence of the metrics, the kinematical quantities, Weyl tensor and spatial curvature (i.e., curvature of the slices) have the same remarkably simple form in both classes of solutions. We use the natural orthonormal basis for the line element (2.1), viz.,

$$w^{(0)}=dt,\quad w^{(1)}=Se^{\nu}dx,\quad w^{(2)}=Se^{\nu}dy,\quad w^{(3)}=SHW\,dz. \tag{2.18}$$

Then, using the orthonormal-frame formalism as described by MacCallum,[21] we obtain the following results. (Further details are given in Appendix B.) The rate of expansion scalar for the matter is

$$\theta=\frac{3\dot S}{S}-\frac{\beta_+\dot f_++\beta_-\dot f_-}{H}, \tag{2.19}$$

and the nonzero components of the rate of shear tensor are

$$2\sigma_{11}=2\sigma_{22}=-\sigma_{33}=\frac{2}{3}\left[\frac{\beta_+\dot f_++\beta_-\dot f_-}{H}\right]. \tag{2.20}$$

The acceleration and vorticity of the matter are zero.

The nonzero components of the electric part of the Weyl conformal curvature tensor relative to (2.18) are

$$2E_{11}=2E_{22}=-E_{33}=-\frac{2M}{S^3}\left[\frac{\beta_+f_++\beta_-f_-}{H}\right] \tag{2.21}$$

while the magnetic part $H_{\alpha\beta}$ is zero.[3] The Ricci scalar and trace-free Ricci tensor of the slices are, respectively,

$$R^* = \frac{6}{S^2}\left[k + \frac{2}{3H}[\beta_+(1+kf_+)+k\beta_- f_-]\right], \tag{2.22}$$

$$2S^*_{11} = 2S^*_{22} = -S^*_{33} = -\frac{2}{3S^2H}[\beta_+(1+kf_+)+k\beta_- f_-] . \tag{2.23}$$

Note that when $k=0$ in class I, the slices are flat, as follows from (2.15).

Finally, the Cotton-York tensor of the slices is zero for both classes of solutions so that the slices are conformally flat[22,3] in general.

Remark. It follows from (2.20) that

$$\sigma_{ab}=0 \Longleftrightarrow \beta_\pm = 0 . \tag{2.24}$$

Since the models in general have zero acceleration and vorticity it follows that $\beta_\pm \equiv 0$ *are necessary and sufficient conditions for the Szekeres solutions to reduce to an FRW solution.*[23] When $\beta_\pm \equiv 0$, it follows from (2.21) and (2.23) that $S^*_{\alpha\beta}=0$ and $E_{\alpha\beta}=0$, as expected. In addition the density μ and spatial curvature scalar R^* assume the usual FRW forms

$$\mu_{\text{FRW}} = \frac{6M}{S^3}, \quad R^*_{\text{FRW}} = \frac{6k}{S^2} , \tag{2.25}$$

as follows from (2.6) and (2.22). Thus, to every Szekeres solution with $M>0$ there corresponds an FRW dust solution with positive density, obtained by setting $\beta_\pm \equiv 0$. From (2.25) we see that it is the parameter k that determines the spatial geometry of the corresponding FRW models. It should be noted, however, that the Szekeres line element does not assume one of the standard FRW forms when $\beta_\pm = 0$, since the spatial coordinates are not standard FRW spatial coordinates.

III. THE INITIAL SINGULARITY

In this and the next section, we discuss the nature and occurrence of scalar polynomial curvature singularities[24] in the Szekeres solutions. Such singularities occur when a polynomial curvature scalar becomes unbounded. In a general spacetime there are 14 independent polynomial curvature scalars.[25] Due to the simple form of the Weyl tensor in the Szekeres solutions and the fact that the source is dust, this number is in fact reduced to 2 (see Appendix C), namely, the density μ and the scalar $C_{abcd}C^{abcd}$, where C_{abcd} is the Weyl conformal curvature tensor. The latter is in fact a constant multiple of $(E_{11})^2$ (see Appendix C). It thus follows from (2.6) and (2.21) that a scalar polynomial curvature singularity occurs in a Szekeres solution if and only if $S\to 0^+$ or $H\to 0^+$. We refer to these two types of singularities as $S=0$ and $H=0$ singularities, respectively.

The *maximum* range for η along a fluid world line (abbreviated WL), as determined by the requirement $S>0$, is

$$0<\eta<+\infty, \quad k=0,-1$$

or

$$0<\eta<2\pi, \quad k=+1 ,$$

since $S=0$ at the finite end points, as follows from Eqs. (2.8) and (2.9). These ranges correspond via (2.8) and (2.9) to

$$T(z)<t<+\infty, \quad k=0,-1$$

or

$$T(z)<t<T(z)+2\pi M(z), \quad k=+1 .$$

Thus if we move into the past along a fluid WL starting at a time $t=t_0>T(z)$ we will encounter an $S=0$ curvature singularity *after a finite time.* It may happen, however, that for a particular fluid WL we encounter an $H=0$ singularity *before* we reach the $S=0$ singularity, with the specific value of t being determined by solving

$$H(t,x,y,z)=0 \tag{3.1}$$

for t as a function of x,y,z. The existence of solutions to (3.1) depends on the values of k, $A(x,y,z)$, and $\beta_\pm(z)$. In any case, each fluid WL, when extended into the past, necessarily encounters an initial singularity.

Remark. The $S=0$ singularities are the analogs in the Szekeres solutions of the simultaneous big-bang singularities which arise in the FRW models, and specialize to them when we set $\beta_\pm \equiv 0$. We note however, that in the Szekeres solutions (class I), the $S=0$ singularities are not, in general, simultaneous for all observers, due to the z dependence in $T(z)$.

We assume that the energy density of the fluid is positive. Since we have $H>0$ in general, and have restricted our considerations to $M>0$, it follows from (2.7) that

$$\mu > 0 \Longleftrightarrow A > 0 \ . \tag{3.2}$$

In order to determine whether the *initial* singularity on a particular fluid WL is an $S=0$ or an $H=0$ singularity we need the behavior of the functions $f_{\pm}$, as defined by Eqs. (2.10) and (2.11). This is given in Fig. 1. Since $\lim_{\eta \to 0^+} f_+ = 0$, the term $\beta_+ f_+$ in H is insignificant as $\eta \to 0^+$. On the other hand, since $H > 0$, $A > 0$, and $\lim_{\eta \to 0^+} f_- = +\infty$, it follows that as η decreases along a fixed fluid WL, H will reach zero for some positive value of η (i.e., before[26] $S=0$ at $\eta=0$) if and only if $\beta_- > 0$.

In cosmological models, or in general in any solution of the Einstein field equations whose source is a perfect fluid with nonzero rate-of-expansion tensor, the scalar polynomial curvature singularities can be classified by considering the dynamics of the fluid as the singularity is approached along the fluid WL's. To this end we introduce the length scales l_α, $\alpha=1,2,3$ in the eigendirections of the rate-of-expansion tensor of the dust. The l_α are positive scalars and are defined,[27] up to a scale change which is constant along the flow lines, by

$$\frac{\dot{l}_\alpha}{l_\alpha} = \theta_{\alpha\alpha}, \quad \alpha = 1,2,3, \text{ no } \sum \text{ on } \alpha \ ,$$

where an overdot denotes differentiation along the flow lines and $\theta_{\alpha\alpha}$ are the components of the rate-of-expansion tensor $\theta_{\alpha\beta}$ in its eigenframe. The overall length scale l is defined by

$$l = (l_1 l_2 l_3)^{1/3} \text{ or } \frac{\dot{l}}{l} = \tfrac{1}{3}\theta \ ,$$

where θ is the rate-of-expansion scalar. The singularities can be classified according to the behavior of the l_α as $l \to 0$, i.e., as one approaches the singularity along a fluid WL. The following terminology is standard[27]: A scalar polynomial curvature singularity in a solution of the Einstein field equations with a perfect fluid (dust) as source is said to be

(a) a *pointlike* singularity if all three $l_\alpha \to 0$,

(b) a *cigar* singularity if two of the $l_\alpha \to 0$ and the other approaches infinity,

(c) a *pancake* singularity if two of the l_α approach finite numbers and the other tends to zero, as the singularity is approached.

For the Szekeres solutions with $M > 0$, we obtain, after rescaling,

$$l_1 = l_2 = S \ , l_3 = SH = S(A - \beta_+ f_+ - \beta_- f_-) \ . \tag{3.3}$$

This implies that *all $H=0$ singularities are necessarily pancake singularities.* On the other hand, we shall show that *$S=0$ singularities are either pointlike or cigar singularities.* For all values of k and both classes of solutions we have

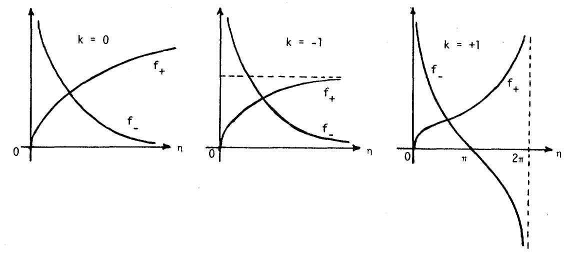

FIG. 1. Graphs of the increasing (f_+) and decreasing (f_-) solutions of the differential equation (2.5) in the three cases $k=0, -1, +1$.

$$l_1 = l_2 \sim \frac{M\eta^2}{2} \ , \qquad l_3 \sim \left[\frac{M\eta^2}{2}\right]\left[A - 24\frac{\beta_-}{\eta^3} - \frac{1}{10}\beta_+\eta^2\right] , \tag{3.4}$$

when $\eta \sim 0^+$, where $M = \frac{2}{9}$ if $k=0$, and the expressions are *exact* in both classes when $k=0$. It follows from (3.4) that an $S=0$ initial singularity, which will arise when $\beta_- \leq 0$, is a pointlike singularity if $\beta_- = 0$, and a cigar singularity if $\beta_- < 0$. On the other hand, the function β_+ has no effect on the classification of the initial singularity. The above results are summarized in the following.

Theorem 3.1 (initial singularities). The initial singularity in a Szekeres solution with $M > 0$ and positive density is (a) pointlike if and only if $\beta_- = 0$ ($S=0$ necessarily), (b) cigar if and only if $\beta_- < 0$ ($S=0$ necessarily), (c) pancake if and only if $\beta_- > 0$ ($H=0$ necessarily).

We now discuss the nature of the pointlike singularity, i.e., case (a) of theorem 3.1, in more detail. It is important to note that in this case the function T is a constant. This is always true for class II, and holds in class I on account of (2.15) and the fact that $\beta_- = 0$. This means that the singularity, which occurs when $t = T$, is simultaneous for all comoving observers, as in the FRW solutions. Indeed, we can redefine the origin of t so that $T=0$, and the singularity occurs when $t=0$. It follows from Eqs. (2.8), (2.9), and (3.4) that to first order in t, the line element can be written in the form

$$ds^2 = -dt^2 + t^{4/3}[g^{(0)}_{\alpha\beta} dx^\alpha dx^\beta + O(t^{2/3})] \ , \tag{3.5}$$

where $g^{(0)}_{\alpha\beta}, \alpha, \beta = 1,2,3$ *has only spatial dependence.* Thus in this case, the singularity is said to be

Friedmann-like,[28] and in the terminology of Eardley, Liang, and Sachs,[29] $g^{(0)}_{\alpha\beta}$ is the metric of the singularity, defined to be the three-manifold $t=0$.

The terminology "Friedmann-like singularity" may appear inappropriate when one notes that the rate of shear tensor and the Weyl conformal curvature tensor, both of which are zero in the FRW solutions, diverge as the singularity is approached:

$$\lim_{t\to 0^+} \sigma = +\infty ,$$

$$\lim_{t\to 0^+} C_{abcd} = +\infty ,$$

where the shear scalar σ is defined by $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$. However, the matter flow does approach isotropy at the singularity in the sense that the shear scalar diverges less rapidly than the expansion scalar, i.e.,

$$\lim_{t\to 0^+} \frac{\sigma}{\theta} = 0 . \tag{3.6}$$

Another manifestation of this limiting isotropy is that the three length scales l_α tend to zero at the same rate, as $t \to 0^+$, i.e.,

$$\lim_{t\to 0^+} \frac{l_\alpha}{l} = 1, \quad \alpha = 1,2,3 ,$$

as follows from (3.4), after rescaling l_3. Likewise the Weyl tensor does become negligible in the sense that it diverges less rapidly than the Ricci tensor (which is determined algebraically by the matter density μ), i.e.,

$$\lim_{t\to 0^+} \frac{C_{abcd}C^{abcd}}{\mu^2} = 0 . \tag{3.7}$$

A property which these Friedmann-like singularities do have in common with the initial singularity in an exact FRW dust model is the behavior of the matter density relative to the expansion scalar, viz.,[11]

$$\lim_{t\to 0^+} \frac{3\mu}{\theta^2} = 1 . \tag{3.8}$$

The limits (3.6)–(3.8) are a consequence of the formulas of Sec. II and the fact that

$$S \sim \tfrac{1}{2}M\eta^2, \quad \dot{S} \sim 2\eta^{-1} ,$$
$$f_+ \sim \tfrac{1}{10}\eta^2, \quad \dot{f}_+ \sim \frac{2}{5M}\eta^{-1} , \tag{3.9}$$

as $\eta \to 0^+$. Indeed it follows that any one of the limits (3.6)–(3.8) is equivalent to the vanishing of β_-. To summarize we have established[30] theorem 3.2.

Theorem 3.2 (Friedmann-like singularities). The initial singularity in a Szekeres solution with $M>0$ and positive density is Friedmann-like if and only if $\beta_- = 0$, and this situation is characterized by the validity of any one of the limits (3.6)–(3.8).

IV. EVOLUTION OF THE SOLUTIONS

In this section we analyze the evolution of the solutions as one follows a fluid world line into the future, starting at a time $t = t_0 > T(z)$. The main result is as follows. If the parameter k equals 1, a final singularity inevitably occurs, as in the corresponding FRW model. This final singularity will occur either when $S=0$ at $t = T + 2\pi M$, or when $H=0$ at an earlier time. On the other hand, if $k = 0, -1$, the model may expand indefinitely, as in the corresponding FRW models, or one may encounter a final singularity, if $H=0$ for some value of $t > t_0$. In the cases where a final singularity does not occur, we will discuss to what extent the solution approaches an FRW solution as $t \to +\infty$. The detailed analysis follows.

In the case $k=0$, since f_- decreases monotonically to zero, and f_+ is positive and increases monotonically without bound (see Fig. 1), it follows from the expression (2.2) for H, viz.,

$$H = A - \beta_+ f_+ - \beta_- f_- ,$$

and the fact that $H>0$ and $A>0$, that H will reach zero for a finite $t > t_0$ if and only if $\beta_+ > 0$. When $k=-1$, it is convenient to write the expressions for A and H in the form

$$A = H_0 + \beta_+ , \tag{4.1}$$

$$H = H_0 - \beta_+(f_+ - 1) - \beta_- f_- , \tag{4.2}$$

where H_0 denotes the value of H when $\beta_\pm = 0$. The explicit form of H_0 can be obtained from Eqs. (2.14) and (2.17). Although H and A are required to be positive, H_0 can be zero or negative, and in fact the sign of H_0 plays a significant role as regards the evolution (when $k=-1$). It follows from (4.1) and $A>0$ that

$$H_0 \le 0 \text{ implies } \beta_+ > 0 .$$

The analysis when $k=-1$ again depends on the behavior of $f_\pm$. It follows from Fig. 1 that f_- decreases monotonically to zero, and f_+ is positive and increases monotonically to approach 1, as $\eta \to +\infty$. Thus if $H_0 \ge 0$, it follows from (4.2) that H will never equal zero, and thus there will be no final singularity.[31] On the other hand, if $H_0 < 0$ (and

$\beta_+ > 0$ necessarily), it follows from (4.2) that H will equal zero for some finite t, and a final singularity occurs.

Finally, consider the case $k = +1$. It follows from Eqs. (2.10) and (2.11) that we can write

$$\beta_+ f_+(\eta) + \beta_- f_-(\eta) = \beta_+ f_+(\tau) + (\pi\beta_+ - \beta_-) f_-(\tau) , \tag{4.3}$$

in terms of $\tau = 2\pi - \eta$. Thus, the possible singularity types as η increases can be inferred from the initial singularity types if we replace β_- by $\pi\beta_+ - \beta_-$. Thus, H will reach zero for some positive value of $\eta < 2\pi$ [i.e., before $S=0$ at $\tau=0$ ($\eta = 2\pi$)] if and only if $\pi\beta_+ - \beta_- > 0$. Further, the expressions for the length scales l_α as $\eta \to 2\pi$ can be obtained from (3.3) by replacing η by τ and β_- by $\pi\beta_+ - \beta_-$. It follows that an $S=0$ final singularity is pointlike if and only if $\pi\beta_+ - \beta_- = 0$.

The above discussion is summarized in the following theorem.

Theorem 4.1 (final singularities).

(i) $k=0$: There is a final singularity if and only if $\beta_+ > 0$. The singularity is a pancake ($H=0$ necessarily).

(ii) $k = -1$: There is a final singularity if and only if and only if $H_0 < 0$. The singularity is a pancake ($H=0$ necessarily).

(iii) $k = +1$: The final singularity is (a) pointlike if and only if $\pi\beta_+ - \beta_- = 0$ ($S=0$ necessarily), (b) cigar if and only if $\pi\beta_+ - \beta_- < 0$ ($S=0$ necessarily), (c) pancake if $\pi\beta_+ - \beta_- > 0$ ($H=0$ necessarily).

Remark. As with the initial singularity, a pointlike final singularity is Friedmann-like. It follows from theorems 3.1 and 4.1, however, that a fluid WL can begin and terminate at a Friedmann-like singularity if and only if $\beta_+ = 0 = \beta_-$, i.e., if and only if the solution is an FRW solution.

We now discuss the cases where the solution evolves indefinitely into the future, along a particular fluid world line. According to theorem 4.1, this occurs when $k=0$ and $\beta_+ \le 0$, and when $k = -1$ and $H_0 \ge 0$. We ask whether or not the solutions become close to an FRW solution as $t \to +\infty$, in the sense that the limits (3.6) and (3.7) hold. In other words, does the shear become negligible compared to the expansion, and does the Weyl tensor become negligible compared to the Ricci tensor? The answer is provided by the following theorem.

Theorem 4.2 (asymptotic evolution). If a Szekeres solution with $M > 0$ and positive density expands indefinitely, then

$$\lim_{t \to +\infty} \frac{\sigma}{\theta} = 0 \tag{4.4}$$

is valid if and only if (a) $H_0 > 0$, when $k = -1$, or (b) $\beta_+ = 0$, when $k=0$; and

$$\lim_{t \to +\infty} \frac{C_{abcd}C^{abcd}}{\mu^2} = 0 \tag{4.5}$$

is valid if and only if $\beta_+ = 0$, when $k = -1$ or 0.

Proof. This is a straightforward calculation using the formulas of Sec. II.

Remark. If condition (4.4) holds, we say that the solution is *asymptotically isotropic* as $t \to +\infty$. In this case, the asymptotic form of the line element as $t \to +\infty$, is the FRW form, as can be verified using the formulas of Sec. II. If the stronger condition (4.5) holds, then in addition, the asymptotic form of the energy density as $t \to +\infty$ is homogeneous and of the FRW form. In this case, we say that the solution is *asymptotically FRW* as $t \to +\infty$.

V. THE ROLE OF THE SPATIAL CURVATURE

In this section we discuss the influence of the spatial curvature (i.e., curvature of the slices t=const) on the dynamics of the solutions near the singularities.

Near a singularity in an FRW model, the dynamics is unaffected by the spatial curvature, i.e., it is independent of whether the constant curvature of the slices is positive, negative, or zero.[32] This is a consequence of the following first integral for irrotational perfect fluids[33]:

$$R^* = 2(\mu - \tfrac{1}{3}\theta^2 + \sigma^2) , \tag{5.1}$$

which reduces to the Friedmann equation in an FRW model ($\sigma^2=0$). On account of the limit (3.8), it follows that

$$\lim_{t \to 0^+} \frac{R^*}{\theta^2} = 0 , \tag{5.2}$$

i.e., that R^* does not affect the dynamics as the singularity is approached in an FRW model. This phenomena is also found in many (though not all) irrotational spatially homogeneous cosmologies, and in spherically symmetric and plane-symmetric spatially inhomogeneous cosmologies.[29] In these models, the spatial curvature is not necessarily isotropic so that $S^*_{\alpha\beta} \neq 0$, and (5.2) is augmented by

$$\lim_{t\to 0^+} \frac{S^*_{\alpha\beta}}{\theta^2}=0 . \tag{5.3}$$

This type of singularity, in which the spatial curvature of the slices orthogonal to the fluid flow becomes dynamically negligible as the singularity is approached, is referred to as a *velocity-dominated singularity*.[29]

One of the results of this section is that the singularities in the Szekeres solutions with $M>0$ are velocity dominated. In order to include all the types of singularities, we replace $t\to 0^+$ in (5.2) and (5.3) by $l\to 0$, where l denotes the overall length scale, as defined in Sec. III.

Theorem 5.1. All initial and final singularities in the Szekeres solutions with $M>0$ and positive density are velocity dominated, in the sense that

$$\lim_{l\to 0^+} \frac{R^*}{\theta^2}=0, \quad \lim_{l\to 0^+} \frac{S^*_{\alpha\beta}}{\theta^2}=0 .$$

Proof. For an $H=0$ singularity, the results follow immediately from Eqs. (2.19), (2.22), and (2.23), since R^* and $S^*_{\alpha\beta}$ have a factor of H in the denominator, while θ^2 has a factor of H^2 in the denominator. On the other hand, for an $S=0$ initial singularity a more detailed calculation is required. It follows from Eqs. (2.8)–(2.11), (2.19), (2.22), and (2.23) that in all cases the leading η dependence in θ^2 as $\eta\to 0^+$ is η^{-6}, while in R^* and $S^*_{\alpha\beta}$ it is η^{-4}, except in class II with $k=0$ and $\beta_-\neq 0$, in which case it is η^{-1}. On account of (4.3), an $S=0$ final singularity with $k=+1$ has the same behavior, and the theorem is proved.

Although all the singularities are velocity dominated, the spatial curvature does play an important role in the case of a Friedmann-like singularity. As one approaches such a singularity, both σ^2 and R^* become negligible compared to μ and θ^2, in the dynamical equation (5.1) [cf. (3.6)]. However, R^* is significant compared to σ^2 in the sense that

$$\lim_{t\to 0^+} \frac{\sigma^2}{R^*}=0 .$$

In addition the anisotropic spatial curvature $S^*_{\alpha\beta}$ has the same rate of growth as R^*, when $t\to 0^+$. This means that as one approaches a Friedmann-like singularity, the spatial geometry does not approach isotropy (unless the solution is an FRW solution). This is reflected in the fact that the three-metric $g^{(0)}_{\alpha\beta}$ of the Friedmann-like singularity [cf. (3.5)], is not a metric of constant curvature (unless the solution is FRW). Indeed, one can show that the curvature of the three-metric $g^{(0)}_{\alpha\beta}$ is given by

$$R^*_{(0)}=6\left[k+\frac{2\beta_+}{3A}\right],$$

$$2S^*_{(0)11}=2S^*_{(0)22}=-S^*_{(0)33}=-\frac{2\beta_+}{3A} .$$

Thus the three-metric of the Friedmann-like singularity plays an important role in determining the future evolution of the solutions, since it contains the functions β_+ and H_0.

The final point concerns the sign of the spatial curvature scalar R^*. For $k=0$, the expression (2.22) simplifies to

$$R^*=\frac{4\beta_+}{S^2H},$$

so that R^* has the sign of β_+. It thus follows from theorem 4.1 that recollapse occurs (along a particular fluid world line) if and only if $R^*>0$ on that world line. For $k=\pm 1$, the expression (2.22) can be written in the form

$$R^*=\frac{2k}{S^2H}(H+2H_0) .$$

Thus if $H_0\geq 0$, R^* has the sign of k. On the other hand, it follows from theorem 4.1 that if a future singularity occurs in a $k=-1$ model (i.e., $H_0<0$), then $R^*>0$ sufficiently near the singularity.

VI. CONCLUDING REMARKS

Partial results on the singularities and evolution of the Szekeres solutions have been given previously for the class-II solutions.[9,11] Our new formulation of the Szekeres solutions enabled us to discuss both classes of solutions simultaneously, and our main results hold without modification for both classes. We have given a complete description of the scalar polynomial curvature singularities in both classes and have established that they are velocity dominated. In addition, we have analyzed the asymptotic behavior in the distant future.

The key to the Szekeres solution lies in the fact that their evolution in time is governed by the differential equations (2.4) and (2.5). For fixed z, Eq. (2.4) is simply the well-known Friedmann equation, while the linear differential equation (2.5) is the equation which determines the two modes of density fluctuations in linear perturbations of the FRW solutions, as discussed in Sec. II. The role of this differential equation in the Szekeres solutions had not previously been recognized. This differential equation in fact arises directly from the well-known

Raychaudhuri equation for irrotational dust[33]:

$$\dot{\theta}+\tfrac{1}{3}\theta^2+2\sigma^2+\tfrac{1}{2}\mu=0\ , \tag{6.1}$$

as can easily be verified using (2.4), (2.6), (2.19), and (2.20). What is surprising is that even for these exact solutions the nonlinearities in (6.1) are eliminated by use of the Friedmann equation (2.4), and cancellation of terms. The term $\beta_+f_+ +\beta_-f_-$ which appears in the line element and density represents the general solution of (2.5) with $\beta_\pm(z)$ being the constants of integration. These two functions of z are the second key feature of the Szekeres solutions. Their vanishing is necessary and sufficient for the Szekeres solutions to specialize to FRW solutions, and we have shown that they play a dominant role in determining the nature of the singularities and the evolution of the solutions. For example, the vanishing of β_- is necessary and sufficient for the initial singularity to be Friedmann-like, while in those solutions that do not recollapse, the vanishing of β_+ is necessary and sufficient in order that the solution be asymptotically FRW in the distant future.

In a subsequent paper we will show that the Szekeres solutions are of interest as realistic inhomogeneous cosmological models, since they can approximate the FRW dust models arbitrarily closely, at least over a finite time interval, when the arbitrary functions are suitably restricted. Again, the functions β_+ and β_- play a key role in that they determine the magnitude and sign of the increasing and decreasing modes, respectively, of the density fluctuations in the resulting "perturbed" FRW models. Although one might anticipate this on the basis of our earlier remarks concerning the linear differential equation (2.5) and the form (2.6) of the energy density, a full justification of this is quite involved, particularly in class I.

This interpretation of $\beta_\pm$ gives some physical insight into theorems 4.1 and 4.2. For example, theorem 4.1 (i) states that a $k=0$ model has a final singularity if and only if the increasing mode of the density fluctuation is positive (i.e., $\beta_+>0$). This is plausible, since in the $k=0$ FRW model the density is just small enough for the model to avoid recollapse, so that any local enhancement of the density would lead to recollapse, at least locally. On the other hand, in a $k=-1$ model, a positive density fluctuation does not necessarily lead to recollapse. The condition for recollapse [theorem 4.1 (ii)] is $H_0<0$, and we will show in a subsequent paper that this condition prevents the model from being close to an FRW model at any time. In other words, those $k=-1$ Szekeres solutions which are sufficiently close to an FRW model, expand indefinitely, and indeed, by theorem 4.2, also approach isotropy. However these models are asymptotically FRW as $t\to+\infty$, in the sense of (4.5) if and only if the increasing density perturbation is zero[34] (i.e., $\beta_+=0$). The $k=0$ solutions also differ from the $k=-1$ solutions as regards asymptotic isotropy. Even if a $k=0$ solution is close to an FRW solution at some time, the presence of the increasing density perturbation (i.e., $\beta_+\neq 0$, which is only possible in class II, when $k=0$) will inevitably lead to an anisotropic solution in the future, as follows from theorem 4.2.

Despite many similarities, however, the class-I and class-II solutions do differ as regards the role of the functions $\beta_\pm(z)$. In class II, the functions M and T, which determine the solution S of the Friedmann equation, are constants, and $\beta_\pm$ are arbitrary functions. On the other hand, in class I, additional constraints arise from the field equations, which relate β_+ and β_- to M and T according to Eq. (2.15). The functions M and T determine whether or not the $S=0$ singularities, which occur when $t=T(z)$ or when $t=T(z)+2\pi M(z)$, are simultaneous for all comoving observers, i.e., are given by $t=$const. Thus in class-II solutions, all $S=0$ singularities are simultaneous, while in class-I solutions, the density perturbation functions $\beta_\pm$ determine, via (2.15), whether or not an $S=0$ singularity is simultaneous. For example if $\beta_+=0$ in class I, an $S=0$ initial singularity is nonsimultaneous (unless the solution is an FRW solution). An example of this phenomenon in a spherically symmetric model has previously been noted by Silk.[35]

The restrictive feature of our analysis in this paper is that our results apply only to the behavior of the solutions along the world line of a particular comoving observer, or in some neighborhood of such a world line. In other words we have studied the solutions from a local point of view. In a subsequent paper, we will discuss the extent to which the Szekeres solutions are globally defined in the sense of admitting a slice $t=t_0$ on which the metric is regular, and the density positive and finite. This will enable us to investigate under what conditions the Szekeres solutions approximate an exact FRW solution over a finite time interval.

ACKNOWLEDGMENTS

This was supported in part by the Natural Sciences and Engineering Research Council of Canada under Grant No. A7229. Part of this research was carried out while one of us (J.W.) was on sabbatical leave in the Department of Applied Mathematics at the University of the Witwatersrand, whose hospitality and financial support are gratefully acknowledged.

APPENDIX A

In this appendix, we establish the relationship between the form the Szekeres solutions given in Sec. II, and the form given in the literature. For the class-I solutions, we start with the form given by Bonnor *et al.*[2] The line element is given by

$$ds^2=-dt^2+\left[\frac{\phi}{P}\right]^2(dx^2+dy^2)+P^2\left[\frac{\partial}{\partial z}\left[\frac{\phi}{P}\right]\right]^2\frac{dz^2}{W^2}, \quad (A1)$$

where

$$P=a(z)(x^2+y^2)+2b(z)x+2c(z)y+d(z)$$

and

$$ad-b^2-c^2=\tfrac{1}{4}\epsilon,\ \ \epsilon=\pm 1,0 .$$

The only remaining field equation, assuming $\dot\phi\neq 0$, is

$$\dot\phi^2=W^2-\epsilon+\frac{Q(z)}{\phi}, \quad (A2)$$

$Q(z)$ being an arbitrary function of z. The density is given by

$$\mu=\frac{PQ_z-3QP_z}{\phi^2(P\phi_z-\phi P_z)} . \quad (A3)$$

We change notation as follows:

$$W^2=\epsilon-kf^2 ,$$

$$Q=2Mf^3 ,$$

$$P=fe^{-\nu} ,$$

$$\phi=Sf .$$

It then follows that (A1) assumes the form

$$ds^2=-dt^2+S^2\left[e^{2\nu}(dx^2+dy^2)+\frac{H^2}{\epsilon-kf^2}dz^2\right] \quad (A4)$$

with

$$H=f\nu_z+f\frac{S_z}{S} . \quad (A5)$$

Equations (A2) and (A3) become, respectively,

$$\dot S^2=-k+\frac{2M}{S} \quad (A6)$$

and

$$\mu=\frac{2}{S^3H}(3Mf\nu_z+fM_z) . \quad (A7)$$

From Eq. (A7) we see that $M\equiv 0$ implies $\mu\equiv 0$, and since we are interested in nonvacuum spacetimes we assume $M(z)\neq 0$. Thus we can write (A7) in the form

$$\mu=\frac{6M}{S^3H}\left[f\nu_z+\frac{fM_z}{3M}\right] . \quad (A8)$$

For $M>0$, the solution to (A6) can be written parametrically in the form (2.8) and (2.9). From Eq. (2.8) we obtain

$$\eta_z=-\frac{1}{Mh'}(T_z+M_zh) ,$$

so that

$$\frac{S_z}{S}=\frac{1}{S}\left[M_z\left[h'-\frac{hh''}{h'}\right]-\frac{h''}{h'}T_z\right] .$$

Since $M_z=0$ when $k=0$, we can use (2.15) to write the previous expression as

$$\frac{S_z}{S}=-\frac{6M}{Sf}\left[\tfrac{1}{2}k\left[h'-\frac{hh''}{h'}\right]\beta_++\frac{h''}{h'}\beta_-\right] . \quad (A9)$$

Then, using the explicit form (2.9) for h, it follows, somewhat surprisingly, that this expression can be written in the form

$$f\frac{S_z}{S}=-[\beta_+(f_++k)+\beta_-f_-] , \quad (A10)$$

where $f_\pm$ are given by Eqs. (2.10) and (2.11). Substitution of Eq. (A10) into (A5) yields the desired expression (2.2) for H. The density (A8) also assumes the desired form (2.7).

Equation (A10) is, in fact, also valid when $M<0$ (with $k=-1$ necessarily), and for completeness we give the expressions for h and $f_\pm$ in this case, viz.,

$$h=-(\sinh\eta+\eta) ,$$

$$f_+=\frac{6M}{S}\left[1-\frac{\eta}{2}\tanh\frac{\eta}{2}\right]+1 ,$$

$$f_-=\frac{6M}{S}\tanh\frac{\eta}{2} .$$

For the class-II solutions we refer to Bonnor and Tomimura.[9] The models with $M>0$ correspond to those labeled PI ($k=0$), HI ($k=-1$), and EI ($k=+1$) by Bonnor and Tomimura. The present form of the Szekeres solutions corresponds to that given in the previous reference, provided that we make the following identifications:

$$R=S\ ,$$

$$A+\frac{T}{R}=H\ ,$$

$$k=M\ ,$$

$$\beta=-\frac{1}{2}\beta_+,\ \ \mu=-\beta_-,\ \ \text{when } k=0\ ,$$

$$\beta=6M\beta_+,\ \ \mu=-6M\beta_-,\ \ \text{when } k=\pm 1\ .$$

A possible point of confusion is that the function A in Ref. 9 corresponds to $A+k\beta_+$ in our notation. The difference in presentation of the two forms of the solutions now reduces to the use of two different systems of coordinates to describe a two-space of constant curvature, $e^{2\nu}(dx^2+dy^2)$ in our notation, and $(dy^2+h^2dz^2)$ in Ref. 9. Note that the arbitrary functions depend on x in Ref. 9 but on z in our presentation.

We note that the Szekeres solutions with $M\equiv 0$ are not included in our new formulation since in this case $f_\pm$ do not satisfy Eq. (2.5). There are no nonvacuum solutions with $M\equiv 0$ in class I, and the class II solutions with $M\equiv 0$ correspond to those labeled PII ($k=0$) and $HIII$ ($k=-1$) in Ref. 9.

APPENDIX B

We outline the derivation of the expressions (2.19)–(2.23). We assume that the reader is familiar with the orthonormal-frame formalism as described by MacCallum.[21] The expressions (2.19) and (2.20) are obtained by applying the commutators to the orthonormal basis defined by (2.18). Equation (2.22) can then be obtained directly from the first integral (5.1).

The expressions for $E_{\alpha\beta}$ and $S^*_{\alpha\beta}$ are derived using the following formulas:

$$E_{\alpha\beta}=-\partial_0\sigma_{\alpha\beta}-\sigma^\gamma_\alpha\sigma_{\gamma\beta}-\tfrac{2}{3}\theta\sigma_{\alpha\beta}+\tfrac{2}{3}\sigma^2\delta_{\alpha\beta}\ ,$$

$$S^*_{\alpha\beta}=-\partial_0\sigma_{\alpha\beta}-\theta\sigma_{\alpha\beta}\ .$$

The first of these is given by Ellis.[33] The second is obtained by taking the trace-free part of the equation for $R^*_{\alpha\beta}$ given by MacCallum,[21] and specializing it to the case where $e_{(0)}$ is tangent to irrotational geodesics, and the $e_{(\alpha)}$ are Fermi propagated along $e_{(0)}$. These conditions are satisfied[3] by the frame (2.18), which is a shear eigenframe. Since $e_{(0)}=\partial/\partial t$, the differential operator ∂_0 is simply $\partial/\partial t$.

It follows, using (2.19) and (2.20) that

$$E_{33}=\frac{2}{3H}\left[\beta_+\left[\ddot f_+ +\frac{2\dot S}{S}\dot f_+\right]+\beta_-\left[\ddot f_- +\frac{2\dot S}{S}\dot f_-\right]\right],$$

$$S^*_{33}=\frac{2}{3H}\left[\beta_+\left[\ddot f_+ +\frac{3\dot S}{S}\dot f_+\right]+\beta_-\left[\ddot f_- +\frac{3\dot S}{S}\dot f_-\right]\right].$$

Since $f_\pm$ satisfy (2.5), the first of these immediately yields (2.21). To verify that the second reduces to (2.23), we need the fact that (2.4) and (2.5) admit a first integral of the form

$$S\dot S\dot F-\left[k-\frac{3M}{S}\right]F=\alpha\ ,$$

where $\alpha=1$ when $F=f_+$ and $\alpha=0$ when $F=f_-$, as is easily verified.

APPENDIX C

We establish that there are only two independent polynomial curvature scalars in the Szekeres solutions. In Ref. 36, the standard set of 14 independent polynomial curvature scalars I_1-I_{14} is expressed in terms of the Newman-Penrose[37] complex tetrad components of the Weyl and Ricci tensors, which are denoted by ψ_A, ϕ_{AB}, Λ. For the Szekeres solutions, relative to a null tetrad based on the repeated principal null directions, the only nonzero Weyl tensor component is[4] ψ_2 and the only nonzero Ricci components satisfy[4]

$$\phi_{00}=2\phi_{11}=\phi_{22}=6\Lambda\ .$$

It is easily verified, using the formulas in Ref. 36, that in this case, a set of independent scalars is provied by I_1 and I_5, the rest being either zero or algebraically dependent on these two. The scalar I_5 is a constant multiple of the energy density μ, and I_1 is a constant multiple of $(\psi_2)^2$. Finally, it follows using the relationship[38] between the ψ_A and the electric and magnetic components $E_{\alpha\beta}$, $H_{\alpha\beta}$ that E_{11} is a constant multiple of ψ_2.

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