Quarkonium spectra and quantum chromodynamics

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$c\bar{c}$ and $b\bar{b}$ spectra, including fine and hyperfine structures, are investigated with the use of a recently derived quantum-chromodynamic potential supplemented by a phenomenological long-range confining potential. Our theoretical results for all energy levels below the charm and bottom thresholds as well as their leptonic widths are in excellent agreement with experiments. We also give unambiguous theoretical predictions for unobserved $b\bar{b}$ energy levels, and present reasonable estimates for energy levels of $t\bar{t}$. Our values of the strong coupling constant α_s and the QCD scale parameter Λ are consistent with those extracted from high-energy positron-electron annihilation.

(i) Introduction. Recently the quark-antiquark potential to the fourth order in perturbative quantum chromodynamics was obtained[1] with special emphasis on a precise determination of the spin-dependent interaction terms. We shall now investigate the $c\bar{c}$, $b\bar{b}$, and $t\bar{t}$ spectra, including their fine and hyperfine structures, with the use of this quantum-chromodynamic potential supplemented by a phenomenological long-range confining potential.[2]

The confining potential was originally taken to be a spin-independent linear potential.[3] But it was subsequently pointed out[4–6] that relativistic corrections to the linear potential will give rise to spin-dependent terms, whose nature depends on whether we regard the confining potential as a vector or a scalar exchange. Our approach is based on the belief that the quark-antiquark potential at short distances can be determined fairly accurately by perturbative quantum chromodynamics, and therefore the confining terms should not unduly affect the potential at short distances, which requires the confining potential to be a scalar exchange.

It is well known that results of physical interest obtained from perturbative quantum chromodynamics are renormalization-scheme dependent. We shall here use the Gupta-Radford scheme, whose advantages have been discussed in an earlier paper.[7] Moreover, we shall treat the strong coupling constant α_s in a consistent manner by using the same value of α_s in the Coulomb term and the spin-dependent terms resulting from perturbative quantum chromodynamics, and our values of α_s for different quarkonia will satisfy the quantum-chromodynamic transformation relation.

As we shall see, our theoretical results for all energy levels below the charm and bottom thresholds[8] will be in excellent agreement with experiments,[9] and our values of α_s and the QCD scale parameter Λ will be consistent with those extracted from high-energy positron-electron annihilation.[10,11] We believe that this extensive agreement between theoretical and experimental results fully establishes the validity of perturbative quantum chromodynamics at short distances or large momentum transfers.

(ii) Quark-antiquark system in quantum chromodynamics. Our total quark-antiquark potential consists of a perturbative part $\mathcal{V}_p(\vec{r})$ and a confining part $\mathcal{V}_c(\vec{r})$. The Fourier transform of $\mathcal{V}_p(\vec{r})$, obtained from quantum chromodynamics, is given by[12]

$$\begin{aligned}\mathcal{V}_p(\vec{k}) = &-\frac{16\pi\alpha_s}{3}\left[\frac{1}{\vec{k}^2}+\frac{\vec{p}^2}{m^2\vec{k}^2}-\frac{1}{2m^2}-\frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{6m^2}+\frac{\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2-\frac{1}{3}\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2}{4m^2\vec{k}^2}+\frac{3i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{k}\times\vec{p}}{4m^2\vec{k}^2}\right]\\ &\times\left[1-\frac{\alpha_s}{12\pi}\left[(33-2n_f)\ln\frac{\vec{k}^2}{\mu^2}+18\right]\right]\\ &-\frac{2\alpha_s^2}{3m^2}\left[\frac{14\pi^2 m}{3|\vec{k}|}+\left[\frac{8}{9}+\ln 2-\frac{7}{2}\ln\frac{\vec{k}^2}{m^2}\right]\vec{\sigma}_1\cdot\vec{\sigma}_2+\left[\frac{17}{3}+3\ln\frac{\vec{k}^2}{m^2}\right]\frac{\vec{k}\cdot\vec{\sigma}_1\vec{k}\cdot\vec{\sigma}_2-\frac{1}{3}\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2}{\vec{k}^2}\right.\\ &\left.+\left[\frac{8}{3}+3\ln\frac{\vec{k}^2}{m^2}\right]\frac{2i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{k}\times\vec{p}}{\vec{k}^2}\right],\end{aligned}$$

so that

$$\mathcal{V}_p(\vec{r}) = -\frac{4\alpha_s}{3r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E]\right]+\frac{4\alpha_s}{3m^2 r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E]\right]\nabla^2$$
$$+\frac{8\pi\alpha_s}{3m^2}\left[\left(1-\frac{3\alpha_s}{2\pi}\right)\delta(\vec{r})-\frac{\alpha_s}{24\pi^2}(33-2n_f)\nabla^2\left(\frac{\ln(\mu r)+\gamma_E}{r}\right)\right]-\frac{14\alpha_s{}^2}{9mr^2}$$
$$+\frac{32\pi\alpha_s}{9m^2}\vec{s}_1\cdot\vec{s}_2\left[\left(1-\frac{\alpha_s}{12\pi}(26+9\ln 2)\right)\delta(\vec{r})-\frac{\alpha_s}{24\pi^2}(33-2n_f)\nabla^2\left(\frac{\ln(\mu r)+\gamma_E}{r}\right)+\frac{21\alpha_s}{16\pi^2}\nabla^2\left(\frac{\ln(mr)+\gamma_E}{r}\right)\right]$$
$$+\frac{4\alpha_s}{m^2}\frac{\vec{s}_1\cdot\hat{r}\,\vec{s}_2\cdot\hat{r}-\frac{1}{3}\vec{s}_1\cdot\vec{s}_2}{r^3}\left[1+\frac{4\alpha_s}{3\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E-\tfrac{4}{3}]-\frac{3\alpha_s}{\pi}[\ln(mr)+\gamma_E-\tfrac{4}{3}]\right]$$
$$+\frac{2\alpha_s}{m^2}\frac{\vec{L}\cdot\vec{S}}{r^3}\left[1-\frac{11\alpha_s}{18\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E-1]-\frac{2\alpha_s}{\pi}[\ln(mr)+\gamma_E-1]\right] . \quad (2.1)$$

The confining part $\mathcal{V}_c(\vec{r})$ is taken as

$$\mathcal{V}_c(\vec{r}) = Ar - \frac{A}{2m^2}\frac{\vec{L}\cdot\vec{S}}{r}+C , \quad (2.2)$$

which corresponds to a scalar-exchange interaction in the quasistatic approximation and a constant C. The complete Hamiltonian for the quark-antiquark system is

$$\mathcal{H} = 2m+\vec{p}^2/m-\vec{p}^4/4m^3+\mathcal{V}_p(\vec{r})+\mathcal{V}_c(\vec{r}) , \quad (2.3)$$

and we shall briefly describe our procedure for solving the Schrödinger equation for this system.

We found it most convenient to obtain the radial wave functions by assuming them to be of the form

$$R(r) = P(r/a)e^{-(r/a)^{3/2}}, \quad a \equiv \left(\frac{9}{4mA}\right)^{1/3} , \quad (2.4)$$

where $P(r/a)$ is a polynomial, and the exponential factor corresponds to the asymptotic solution of the Schrödinger equation. The coefficients in $P(r/a)$ were determined by the variational technique of minimizing the expectation value of the unperturbed Hamiltonian $\mathcal{H}_0$, given by

$$\mathcal{H}_0 = 2m+\frac{\vec{p}^2}{m}-\frac{4\alpha_s}{3r}\left[1-\frac{3\alpha_s}{2\pi}+\frac{\alpha_s}{6\pi}(33-2n_f)[\ln(\mu r)+\gamma_E]\right]+Ar+C , \quad (2.5)$$

and then the contribution of $\mathcal{H}'=\mathcal{H}-\mathcal{H}_0$ to the energy levels was included by first-order perturbation theory. Our computation program for the variational technique was also tested by verification of the virial theorem.[13]

The perturbative potential is not only renormalization-scheme dependent but also renormalization-scale dependent. Besides using the renormalization scheme of Ref. 7, we chose μ so as to minimize the effect of higher-order terms which would be generated by renormalization-group improvement of our potential. These terms involve higher powers of

$$\frac{\alpha_s}{12\pi}(33-2n_f)\ln\frac{\vec{k}^2}{\mu^2} ,$$

whose expectation value is not very sensitive to variations in μ for quarkonium states other than the S states. We therefore required that for all S states

$$|\xi| \ll 1 , \quad (2.6)$$

where

$$\xi = \frac{(\alpha_s/12\pi)(33-2n_f)\langle\ln(\vec{k}^2/\mu^2)\rangle}{\langle 1\rangle}$$
$$= \frac{\alpha_s(33-2n_f)}{24\pi^2|\psi(0)|^2}$$
$$\times\int d\vec{r}\,\frac{\ln(\mu r)+\gamma_E}{r}\nabla^2[\psi^*(\vec{r})\psi(\vec{r})] , \quad (2.7)$$

and $\psi(\vec{r})$ is the quarkonium wave function.

(iii) Quarkonium spectra. Following the procedure described in Sec. (ii), we varied the parameters in the potential so as to bring the theoretical values for energy levels as close as possible to the observed results. We also ensured that the condition (2.6) is fulfilled, and that the quantum-chromodynamic transformation relation

$$\alpha_s' = \frac{\alpha_s}{1+(\alpha_s/12\pi)(33-2n_f)\ln(\mu'^2/\mu^2)} \quad (3.1)$$

is satisfied, with $n_f=4$, by the values of α_s for $c\bar{c}$ and $b\bar{b}$ states.

Our values for the energy levels below the charm and bottom thresholds, together with the values of the parameters, are given in Tables I and II. It is interesting that not only the values of α_s for $c\bar{c}$ and $b\bar{b}$ satisfy (3.1), but also we find that the values of μ are related as

$$\mu_c^2/m_c \approx \mu_b^2/m_b \ . \tag{3.2}$$

For the splitting of observed energy levels below thresholds, we obtain for $c\bar{c}$

$$\begin{aligned} M(\psi')-M(\psi)&=588 \text{ MeV},\\ M(\psi)-M(\eta_c)&=116 \text{ MeV},\\ M(\psi')-M(\eta_c')&=85 \text{ MeV},\\ M(\chi_0)-M(\psi)&=319 \text{ MeV},\\ M(\chi_1)-M(\chi_0)&=99 \text{ MeV},\\ M(\chi_2)-M(\chi_1)&=46 \text{ MeV}, \end{aligned} \tag{3.3}$$

and for $b\bar{b}$

$$\begin{aligned} M(\Upsilon')-M(\Upsilon)&=551 \text{ MeV},\\ M(\Upsilon'')-M(\Upsilon)&=893 \text{ MeV} \ , \end{aligned} \tag{3.4}$$

which all agree with the experimental results[9] within a few MeV.

Furthermore, from the leptonic-width formula[14]

$$\Gamma_{ee}=\frac{16\pi\alpha^2 e_Q^2}{M^2(Q\bar{Q})}|\psi(0)|^2 \ , \tag{3.5}$$

we find for $c\bar{c}$

$$\Gamma_{ee}(\psi)=5.07 \text{ keV}, \ \ \Gamma_{ee}(\psi')=2.55 \text{ keV} \ , \tag{3.6}$$

and for $b\bar{b}$

$$\begin{aligned} &\Gamma_{ee}(\Upsilon)=1.29 \text{ keV}, \ \ \Gamma_{ee}(\Upsilon')=0.62 \text{ keV},\\ &\Gamma_{ee}(\Upsilon'')=0.46 \text{ keV} \ , \end{aligned} \tag{3.7}$$

which are in reasonable agreement with experiments. Note that our theoretical value of $\Gamma_{ee}(\psi')$ actually represents $\Gamma_{ee}(2\,^3S_1)$ and should be compared with the experimental results by keeping in mind that

$$\Gamma_{ee}(2\,^3S_1)\approx\Gamma_{ee}(\psi')+\Gamma_{ee}(\psi'') \ ,$$

as a result of the mixing of the $2\,^3S_1$ and $1\,^3D_1$ states of charmonium.

It is straightforward to extend our treatment to $t\bar{t}$ states, and we have given the results for some energy levels in Table III by assuming the values 20 and 30 GeV for the top-quark mass. For the $1\,^3S_1$, $2\,^3S_1$, and $3\,^3S_1$ levels, the leptonic widths in keV are 5.71, 1.84, and 1.20 for $m_t=20$ GeV, and 6.43, 1.84, and 1.12 for $m_t=30$ GeV.

(iv) Conclusion. We conclude with some remarks on the strong coupling constant α_s and the QCD

TABLE I. Charmonium spectrum with $m_c=1.20$ GeV, $\mu=1.88$ GeV, $\alpha_s(\mu)=0.392$, and $A=0.177$ GeV^2.

State	Mass (GeV)	State	Mass (GeV)
$1\,^3S_1(\psi)$	3.097	$1\,^3P_2(\chi_2)$	3.561
$1\,^1S_0(\eta_c)$	2.981	$1\,^3P_1(\chi_1)$	3.515
		$1\,^3P_0(\chi_0)$	3.416
$2\,^3S_1(\psi')$	3.685	$1\,^1P_1$	3.531
$2\,^1S_0(\eta_c')$	3.600		

TABLE II. $b\bar{b}$ spectrum with $m_b=4.78$ GeV, $\mu=3.75$ GeV, $\alpha_s(\mu)=0.288$, and $A=0.177$ GeV^2.

State	Mass (GeV)	State	Mass (GeV)
$1\,^3S_1(\Upsilon)$	9.462	$1\,^3D_3$	10.167
$1\,^1S_0(\eta_b)$	9.427	$1\,^3D_2$	10.162
		$1\,^3D_1$	10.155
$2\,^3S_1(\Upsilon')$	10.013	$1\,^1D_2$	10.163
$2\,^1S_0(\eta_b')$	9.994		
		$2\,^3D_3$	10.459
$3\,^3S_1(\Upsilon'')$	10.355	$2\,^3D_2$	10.454
$3\,^1S_0(\eta_b'')$	10.339	$2\,^3D_1$	10.447
		$2\,^1D_2$	10.455
$1\,^3P_2$	9.910		
$1\,^3P_1$	9.893	$1\,^3F_4$	10.365
$1\,^3P_0$	9.868	$1\,^3F_3$	10.364
$1\,^1P_1$	9.900	$1\,^3F_2$	10.361
		$1\,^1F_3$	10.364
$2\,^3P_2$	10.266		
$2\,^3P_1$	10.252		
$2\,^3P_0$	10.232		
$2\,^1P_1$	10.258		

TABLE III. $t\bar{t}$ spectrum for $m_t=20$ GeV and $m_t=30$ GeV. Only some low-lying energy levels are given in view of the uncertainty regarding the top-quark mass.

State	Mass (GeV) $m_t=20$ GeV	Mass (GeV) $m_t=30$ GeV
$1\,^3S_1$	39.308	59.106
$1\,^1S_0$	39.285	59.083
$2\,^3S_1$	39.947	59.831
$2\,^1S_0$	39.939	59.824
$1\,^3P_2$	39.880	59.769
$1\,^3P_1$	39.872	59.761
$1\,^3P_0$	39.862	59.753
$1\,^1P_1$	39.876	59.765

scale parameter

$$\Lambda = \mu \exp\left(-\frac{6\pi}{(33-2n_f)\alpha_s}\right) \quad . \tag{4.1}$$

According to Tables I and II, our coupling constants for $c\bar{c}$ and $b\bar{b}$ are

$$\alpha_{\rm GR}(1.88\ {\rm GeV}) = 0.392 \tag{4.2}$$

and

$$\alpha_{\rm GR}(3.75\ {\rm GeV}) = 0.288 \quad , \tag{4.3}$$

where the subscript GR refers to the renormalization scheme of Ref. 7. From (4.3), we can also obtain $\alpha_{\rm GR}$ for higher values of μ by applying (3.1) with $n_f = 5$, and thus, for instance,

$$\alpha_{\rm GR}(35\ {\rm GeV}) = 0.161 \quad . \tag{4.4}$$

It follows that the values of $\Lambda_{\rm GR}$ for $n_f = 3$, 4, and 5 are

$$\Lambda_{\rm GR}^{(3)} = 317\ {\rm MeV},\quad \Lambda_{\rm GR}^{(4)} = 274\ {\rm MeV},\quad \Lambda_{\rm GR}^{(5)} = 218\ {\rm MeV} \quad . \tag{4.5}$$

In order to compare our results with those in the modified minimal-subtraction ($\overline{\rm MS}$) scheme, we note that[15]

$$\alpha_{\overline{\rm MS}} = \alpha_{\rm GR}\left(1 - \frac{\lambda\alpha_{\rm GR}}{\pi}\right),\quad \lambda \equiv \frac{1}{12}\left(49 - \frac{10}{3}n_f\right) \quad , \tag{4.6}$$

and, consequently,

$$\Lambda_{\overline{\rm MS}} = \Lambda_{\rm GR} \exp\left(-\frac{6\lambda}{33-2n_f}\right) \quad . \tag{4.7}$$

Thus our α_s and Λ in the $\overline{\rm MS}$ scheme are

$$\alpha_{\overline{\rm MS}}(35\,{\rm GeV}) = 0.14,\quad \Lambda_{\overline{\rm MS}}^{(5)} = 108\ {\rm MeV} \quad , \tag{4.8}$$

which are consistent with the values extracted from high-energy positron-electron annihilation.[10,11]

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[15]The $\overline{\rm MS}$ scheme currently in use represents a *modification* of the modified minimal-subtraction scheme in which not only $-\frac{1}{2}\gamma_E + \frac{1}{2}\ln(4\pi)$ is subtracted along with the $1/(4-n)$ poles in dimensionally regularized integrals, but also the decoupling-theorem-violating terms are dropped.