Gauge theories on the body-centered hypercubic lattice

William Celmaster
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 8 September 1982)

The four-dimensional body-centered hypercubic lattice has a point symmetry group which is three times as large as that of the simple hypercubic lattice. This enlarged symmetry is implemented by introducing an action consisting of a sum over triangular plaquettes. Here, the theory is presented and some of its properties are described.

Most investigations[1–4] of four-dimensional lattice gauge theory have been conducted on the simple hypercubic (SH) lattice. That lattice is invariant[5,6] under a 384-element subgroup, T, which consists of reflections and multiples of 90° rotations about any site. However, it turns out that there is a lattice, to be called the BCT lattice for reasons to be divulged shortly, whose point symmetry group has 1152 elements. This BCT group contains T as a subgroup. Whereas the SH lattice has 4 symmetry axes, the BCT lattice has 12.

Gauge actions which can be formulated[1] on the BCT lattice will be more rotationally invariant than those formulated on the SH lattice and so might be expected to have a strong-coupling limit which is closer to the continuum. There has been much speculation and study[7–10] on the relationship of the restoration of rotational symmetry to the nature of the crossover region seen in Monte Carlo analyses. In particular, if conjectures about roughening[8] are correct, we would expect, in the BCT theory, a much smoother transition from strong to weak coupling. At the very least, it will be possible to study these speculations in a novel way; at best there may be some significant improvement in the reliability of Monte Carlo simulations and, perhaps, of strong-coupling expansions.

In this paper I will define the BCT theory, show that it has the correct continuum limit, and then briefly describe some of its properties at large g. An especially intriguing possibility to be discussed is that rectangular Wilson loops (yes, there are also triangles, etc.) may be in a rough phase at $g=\infty$. This would be a consequence of the fact that rectangular loops cannot be spanned by planar surfaces of elementary plaquettes.

The BCT lattice is constructed as follows: pack four-space with hypercubes, known also as *tesseracts*,[5] and take the lattice sites to be their corners and centers. The resulting body-centered-tesseract (BCT) lattice can be seen to consist of two staggered ordinary lattices. Body centers form one lattice and corners form the other. The origin of the extra symmetry is simple. Let $\pm e_i$ be the 8 standard unit vectors $(\pm 1,0,0,0)$, etc., which generate the SH lattice. Then the body centers nearest to the origin are the 16 vectors $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$. These can be seen to have unit length—hence the extra symmetry. It is worth noting that in other dimensions body centers are not the same distance from the origin as are the corners. In fact, in three dimensions, no lattice has a higher symmetry than the cubic lattice. For this reason amongst others, it is difficult to formulate a Hamiltonian version of the BCT theory.

Before examining details of the new lattice it is interesting to consider its geometry. The BCT group happens to be the symmetry group of a regular four-dimensional solid known as the 24-cell and whose projection is shown in Fig. 1. Four-space can be solidly packed with these 24-cells.[5] If, after doing such a packing, the centers of these cells are joined to the cell vertices, we obtain a web of equilateral triangles whose vertices are the sites of the BCT lattice and whose edges are the links. All possible nearest neighbors are joined in this way. An advantage of this alternate construction of the lattice is that it provides a geometrical insight into how to implement periodic boundary conditions which respect the BCT group invariance. The method will be to fill, with triangles, a finite region whose boundary is a 24-cell.

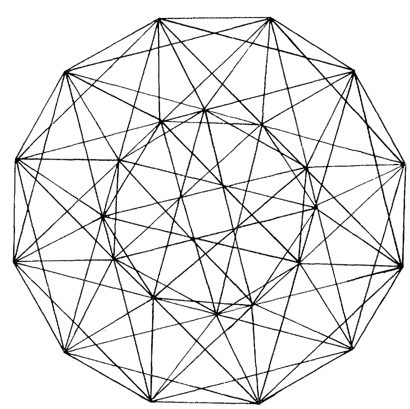

FIG. 1. A two-dimensional projection of the 24-cell.

Periodicity is then imposed by identifying opposite faces of that boundary. In doing Monte Carlo calculations the above considerations will be important.[11]

Next come the details. Points of the lattice can be specified by a 4-tuplet of integers (n_1,n_2,n_3,n_4) corresponding to lattice sites $n_1e_1+n_2e_2+n_3e_3+n_4f$, where e_i are the usual unit vectors (1,0,0,0,) etc., and $f=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Each of the 24 unit vectors described earlier lies on this lattice. As an example, $e_4=-e_1-e_2-e_3+2f$. Those unit vectors form three sets of orthogonal axes: $O_1=\{\pm e_i\}$, $O_2=\{\pm f_i\}$, and $O_3=\{\pm f, \pm f_{1i}\}$. The subscripts on the f's denote the position of "$-\frac{1}{2}$" so, for instance, $f_{12}=(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Elementary triangles are formed by joining nearest neighbors. There are 96 such triangles (unoriented) touching each site. Furthermore, each edge is contiguous to 8 triangles. By contrast, on the SH lattice each site touches 24 elementary squares and each edge is contiguous to 6 squares. An SU(N) action invariant under the BCT group is

$$S_{\rm BCT}=\frac{1}{2g^2}\sum_{\Delta}{\rm Tr}(U_\Delta) \tag{1a}$$

with

$$U_\Delta=U_{n,[v,w]}=U_{n,v}U_{n+v,w}U^\dagger_{n,v+w} \ , \tag{1b}$$

where (n,v) denotes a triangle link pointing from the site n to the site $n+v$, U's are in SU(N) and, as usual, $U_{n,v}=U^\dagger_{n+v,-v}$. This action has the standard classical continuum limit and, at small k, has the standard gauge propagator. No vector-meson doubling occurs. The derivation of these facts is technically rather complicated and details will be presented elsewhere.[12] Here, the only results that I will give are the explicit closed form for the inverse propagator, and the propagator itself near $k=0$.

Begin by defining A^i_n,

$$U_{n,v}=\exp\left[iag\sum_{\mu,b}\tfrac{1}{2}(A^{i,b}_n+A^{i,b}_{n+v})\cdot vT_b\right] \ , \tag{2}$$

where b is the group index, a is the lattice spacing, and i is 1, 2, or 3 according to whether v is a member of the orthogonal axes O_1, O_2, or O_3. The action can be expanded through quadratic order in the A's:

$$S_{\rm BCT}={\rm const}+T(R)\sum_{\substack{i,j,\mu\\ v,n,m}}A^{i,\mu}_nQ^{ij,\mu v}_{n,m}A^{j,v}_m+O(a^3) \ , \tag{3}$$

where, for SU(N), $T(R)=\frac{1}{2}$. Q is the inverse propagator and is diagonal in the group indices, which have been suppressed for the sake of clarity. A closed form for Q is obtained by transforming to momentum space,

$$Q_k\delta(k'-k)=\sum_{n,n'}\exp[-ia(k\cdot n-k'\cdot n')]Q_{n,n'} \ . \tag{4}$$

Q_k is the 12×12 matrix (rows and columns represent the i,j indices):

$$Q^{\mu\nu}_k=\frac{-a^2}{2}\begin{bmatrix}4C_\mu^2 & -\prod C_\omega-C_\mu^2 & -\prod C_\omega-C_\mu^2\\ -\prod C_\omega-C_\mu^2 & 2+2\prod C_\omega-2\prod S_\omega & -\prod C_\omega+C_\mu^2-\frac{1}{2}\sum C_\omega^2\\ -\prod C_\omega-C_\mu^2 & -\prod C_\omega+C_\mu^2-\frac{1}{2}\sum C_\omega^2 & 2+2\prod C_\omega+2\prod S_\omega\end{bmatrix} \quad \text{if } \mu=\nu$$

$$=\frac{-a^2}{2}\begin{bmatrix}0 & -\frac{C_\mu C_\nu}{S_\mu S_\nu}\prod S_\omega & \frac{C_\mu C_\nu}{S_\mu S_\nu}\prod S_\omega\\ -\frac{C_\mu C_\nu}{S_\mu S_\nu}\prod S_\omega & \frac{2C_\mu C_\nu}{S_\mu S_\nu}\prod S_\omega-\frac{2S_\mu S_\nu}{C_\mu C_\nu}\prod C_\omega & \frac{S_\mu S_\nu}{C_\mu C_\nu}\prod C_\omega\\ \frac{C_\mu C_\nu}{S_\mu S_\nu}\prod S_\omega & \frac{S_\mu S_\nu}{C_\mu C_\nu}\prod C_\omega & -\frac{2C_\mu C_\nu}{S_\mu S_\nu}\prod S_\omega-\frac{2S_\mu S_\nu}{C_\mu C_\nu}\prod C_\omega\end{bmatrix} \quad \text{if } \mu\neq\nu \ . \tag{5}$$

In Eq. (9), $C_\omega=\cos(\frac{1}{2}ak_\omega)$, $S_\omega=\sin(\frac{1}{2}ak_\omega)$, and sums or products range over indices 1 to 4. Gauge fixing can be done by imposing the Lorentz condition

$$\sum_\mu(A^{1,\mu}_{n+e_\mu}-A^{1,\mu}_{n-e_\mu})=C_n \ . \tag{6}$$

This has the effect[13] of adding to $Q^{ij,\mu\nu}_k$ a term $\propto C_\mu S_\mu C_\nu S_\nu\delta_{1i}\delta_{1j}$.

The result, Q^α_k, when inverted near $ak=0$ is, in the Feynman gauge,

$$(Q^{\alpha F}_k)^{-1\mu\nu}=-\frac{4}{k^2a^4}\left[\delta^{\mu\nu}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}+O(a)\right] \ . \tag{7}$$

This shows that at small momenta all A^i's can be treated as degenerate and the usual perturbation theory follows.[14] The propagator at large values of ak is used only for computing renormalization constants.

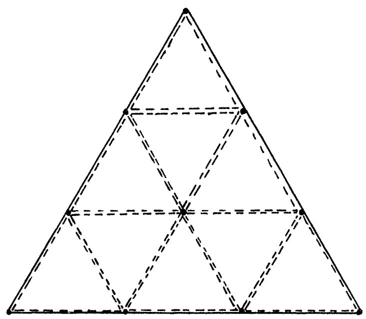

FIG. 2. A triangle of length 3, tiled with elementary triangles (dashed lines).

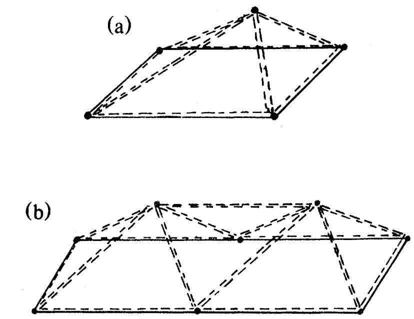

FIG. 3. (a) A 1×1 loop spanned by plaquettes. (b) The "Toblerone graph": a 1×2 loop spanned by plaquettes.

If such a calculation is necessary, Q^α can easily be inverted numerically. However, I have not succeeded in analytically inverting Q^α.

The above summarizes the present study of the quantum continuum limit. Clearly, the BCT lattice is not an elegant way to regularize QCD perturbation theory and its virtues will lie in the strong and medium coupling regime. A strong-coupling expansion can be done in the usual way[1,15,16] but using triangular, rather than square, plaquettes. We can compute expectation values not only of rectangular Wilson loops but also of triangles and various other shapes. For triangles, an area law can be proven: Let

$$W(\Delta)=\left\langle\frac{1}{N}\prod_T U_{ij}\right\rangle \ , \tag{8}$$

where U_{ij} are the path-ordered links of the boundary of an equilateral triangle T, whose edges have length n. T is similar to one of the elementary triangles and can, in fact, be tiled with n^2 of these triangles (see Fig. 2). The leading $1/g^2$ contribution to $W(\Delta)$ is found to be

$$W(\Delta)=\frac{1}{(g^2N)^{n^2}}=\exp\left[-\frac{4}{\sqrt{3}}\ln(g^2N)\right]A \ , \tag{9}$$

where A is the area of T.

It is less easy to prove an area law for square loops. These cannot be spanned by a planar surface of plaquettes. For instance, the smallest surface of plaquettes whose edges bound a 1×1 square is a pyramid as shown in Fig. 3(a). There are four such pyramids each contributing $1/(g^2N)^4$ to the strong-coupling expansion of that loop. Larger rectangles are spanned by 1×1 loops and their minimal surfaces can be constructed by attaching pyramids to one another. If those were the only possible surfaces then an area law would be obtained as in Eq. (9). However, there are other minimal surfaces such as the "Toblerone" graph of Fig. 3(b), and grand pyramids, which are magnified versions of Fig. 3(a). These will modify the above law in a way reminiscent[17] of the roughening behavior seen in expansions of off-axis loops on the SH lattice.[8,10] We are thus led to the conjecture that rectangular Wilson loops are rough at $\beta=0$ ($g=\infty$). A proof will require a systematic classification of all possible minimal surfaces. In the meantime, various of these ideas are being tested via Monte Carlo simulations.[11]

I am grateful to Norman Christ for some helpful remarks and to many of my colleagues at Northeastern for their interest in the body-centered tesseract. Special thanks are due to Fred Green, Frank Krausz, Paul Rujan, and Fred Wu, all of whom made valuable suggestions at various stages of this work. Also, I wish to thank the Mathlab group at MIT for the use of MACSYMA which helped me out with the inverse propagator. This work was supported in part by a grant from the National Science Foundation.

[1] K. Wilson, Phys. Rev. D 10, 2445 (1974).
[2] M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rev. Lett. 42, 1390 (1979); Phys. Rev. D 20, 1915 (1979); M. Creutz, Phys. Rev. Lett. 43, 553 (1979); Phys. Rev. D 21, 2308 (1980).
[3] There are many reviews of lattice gauge theory. See, for instance, P. Hasenfratz, CERN Report No. CERN-TH-3157 (1981) and M. Creutz, in *Perturbative Quantum Chromodynamics, 1981,* edited by D. W. Duke and J. F. Owens, AIP Conf. Proc. No. 74 (AIP, New York, 1981).
[4] For one approach *not* on a hypercubic lattice, see N. H. Christ, R. Friedberg, and T. D. Lee, Columbia University Report No. CU-TP 205, 1982 (unpublished).
[5] H. S. M. Coxeter, *Regular Polytopes* (Dover, New York, 1973); *Regular Complex Polytopes* (Cambridge Univ. Press, London, 1974).

[6]M. Baake, B. Gemünden, and R. Oedingen, University of Bonn Report No. Bonn-HE-81 13, 1982 (unpublished). (1981).

[7]A. Hasenfratz, E. Hasenfratz, and P. Hasenfratz, Nucl. Phys. B181, 353 (1981); C. Itzykson, M. Peskin, and J. B. Zuber, Phys. Lett. 95B, 259 (1980); M. Lüscher, G. Münster, and P. Weisz, Nucl. Phys. B180, 1 (1980); G. Münster and P. Weisz, *ibid.* B180, 13 (1981); B180, 330 (1981); J. M. Drouffe and J. B. Zuber, *ibid.* B180 [FS2], 253 (1981); B180 [FS2], 264 (1981).

[8]J. B. Kogut, D. K. Sinclair, R. B. Pearson, J. Richardson, and J. Shigemitsu, and Phys. Rev. D 23, 2945 (1981); J. B. Kogut and D. K. Sinclair, *ibid.* 24, 1610 (1981).

[9]C. B. Lang and C. Rebbi, CERN Report No. CERN-TH 3270, 1982 (unpublished).

[10]F. Green, Northeastern University Report No. NUB-2556, 1982 (unpublished); paper in preparation.

[11]W. Celmaster, work in progress. Preliminary Monte Carlo results on the specific heat indicate a smoother transition than in the SH lattice.

[12]W. Celmaster, paper in preparation.

[13]See, for instance, S. Coleman, in *Laws of Hadronic Matter,* edited by A Zichichi (Academic, London, 1975).

[14]A. Hasenfratz and Peter Hasenfratz, Phys. Lett. 93B, 165 (1978); R. Dashen and D. Gross, Phys. Rev. D 23, 2340 (1981); P. Weisz, Phys. Lett. 100B, 331 (1981); W. Celmaster and D. J. Maloof, Phys. Rev. D 24, 2730 (1981); 24, 3208 (1981); H. Kawai, R. Nakayama, and K. Seo, Nucl. Phys. B189, 40 (1981).

[15]Strong-coupling expansions are described in Wilson, Ref. 1, and in R. Balian, J. Drouffe, and C. Itzykson, Phys. Rev. D 10, 3376 (1974); 11, 2098 (1975); 11, 2104 (1975).

[16]Details of the character expansion are explained in G. Münster, Nucl. Phys. B180, 23 (1981); F. Green and S. Samuel, *ibid.* B190, 113 (1982); B194, 107 (1982).

[17]Because of grand pyramid graphs, infinitely large loops have minimal surfaces stretching arbitrarily far from the plane of the loop.