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Scattering problem with a complex potential

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A semiclassical method has been generalized and used to study the scattering of a particle from a complex potential. The accuracy of the method is tested by considering a simple potential consisting of a repulsive Coulomb part and an attractive imaginary part of the form $-i\beta/r^2$. The complex phase shifts for the problem can be determined exactly by solving the relevant radial equation. The semiclassical results, including terms of order $\hbar^2$, have been found to give good agreement with the exact results.

I. INTRODUCTION

The extensive use of semiclassical methods in the study of scattering phenomena and, in particular, heavy-ion collisions has brought into focus some limitations of the JWKB method. Though versatile in nature, the JWKB method has not been recognized so far as a practical method because of its lack of accuracy. To overcome the well-known defects of the method, Miller and Good[1] proposed a modification, which has been further generalized by Rosen and Yennie,[2] Lu and Measure,[3] Wald and Lu,[4] and Berry and Mount.[5] The accuracy of this generalized method has also been studied by Wald *et al.*[6] The method has already been used in the study of some problems including the scattering of electrons from a two-center potential.[7,8] The method, however, cannot be applied in a straightforward manner to heavy-ion collisions. The turning points in this case are, in general, complex and also too many in number. With a Woods-Saxon potential, which is commonly used in the optical-model treatment of heavy-ion collisions, the situation is even worse, there being an infinite number of turning points due to the complex poles of the Woods-Saxon function.

The first attempt to study the complex potential was rather casual. Instead of writing for the wave vector

$$k(r)=\left[k^2-\frac{L(L+1)}{r^2}-\frac{2\mu}{\hbar^2}(V_R+iV_I)\right]^{1/2}, \quad (1.1)$$

one expands and keeps terms up to first order in V_I. The resulting JWKB expression for the phase shifts involves an integration over the real path from the classical turning point, determined entirely by the real potential, the imaginary part contributing only a damping factor to each of the partial waves. The approximation can be reliable for a small absorption. However, as the energy increases, more and more inelastic channels open up, making this treatment completely unsuitable. It was, therefore, necessary to look for an alternative way of studying complex potentials in the semiclassical approach.

Koeling and Malfliet[9] studied this problem and suggested a generalization of the semiclassical method which includes contributions from all possible complex trajectories. This prescription, however, is neither a working proposition nor a correct one. A correct solution was given by Knoll and Schaeffer[10,11] who studied the problem analytically and showed that it was not necessary to consider all the complex trajectories. The relative importance of the contributions for single reflections from different turning points and of possible multiple reflections can be estimated following their analysis. It has been found that in realistic cases only a few of the trajectories make dominant contributions. This makes it worthwhile to consider a generalization of the semiclassical method for complex trajectories as a practical method of calculation. The inclusion of higher-order terms in $\hbar^2$ will be another useful step. The semiclassical method mentioned earlier is fairly accurate and is a suitable candidate for adaptation for a complex potential. To illustrate the procedure, we have considered in this paper a simple complex potential which is a combination of a repulsive Coulomb term and an attractive imaginary potential of the type $i\beta/r^2$. We can calculate the complex phase shifts for this problem exactly by solving the relevant radial equations. It is then possible to

study the variation of the phase shifts, in particular, its real parts as the strength of the imaginary potential is varied. The semiclassical method, adapted for complex trajectories, have now been applied to this problem. The first-order term in $\hbar^2$ has also been calculated. A comparison of the semiclassical results with the exact results gives an estimate of the accuracy of the method. The agreement has been found to be very good. The case of a realistic potential, consisting of a Coulomb and a Woods-Saxon–type optical potential, is currently under consideration and will be reported elsewhere.

The presentation of the paper is as follows. In Sec. II, the problem has been studied exactly. Section III gives the semiclassical treatment for the same problem. In Sec. IV, the semiclassical results have been presented and compared with the exact results. Our conclusions are also summarized there.

II. EXACT COMPLEX PHASE SHIFTS

Let us introduce a potential consisting of a repulsive Coulomb part and an attractive imaginary part of the form $-i\beta/r^2$. The radial Schrödinger equation is then given by

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dR_L}{dr}\right]+\left[k^2-\frac{2nk}{r}-\frac{L(L+1)-i\beta}{r^2}\right]\times R_L(r)=0\,, \tag{2.1}$$

where

$$n=\frac{\mu ZZ'e^2}{\hbar^2 k},\quad k=(2\mu E/\hbar^2)^{1/2}\,.$$

Let $l=p+iq$, $p>0$, be a solution of the equation

$$l(l+1)=L(L+1)-i\beta\,. \tag{2.2}$$

Let us substitute

$$R_L(r)=r^l e^{ikr}f_l(r)$$

in Eq. (2.1), which gives the equation

$$rf_l''(r)+(2ikr+2l+2)f_l'(r)+[2ik(l+1)-2nk]f_l(r)=0\,. \tag{2.3}$$

The solution of this equation can be written as

$$f_l(r)=C_l\,{}_1F_1(l+1+in,\ 2l+2,-2ikr)\,, \tag{2.4}$$

where C_l is the normalization constant. We will have to impose the appropriate boundary condition on (2.4). In particular, one has to ensure that there is no attenuation of the incoming wave. The asymptotic form of $R_l(r)$ is then given by

$$R_l(r)\to C_l\frac{\Gamma(2l+2)}{(2k)^l kr}\frac{e^{n\pi/2}e^{q\pi/2}}{\Gamma(p+1+iq-in)}\frac{1}{2i}[e^{is}-(A+iB)e^{-is}]\,, \tag{2.5}$$

where

$$A+iB=\frac{\Gamma(p+1+iq-in)}{\Gamma(p+1+iq+in)}e^{-q\pi} \tag{2.6}$$

and

$$s=kr-\tfrac{1}{2}p\pi-n\ln 2kr\,. \tag{2.7}$$

The case of a real Coulomb potential is well known. The corresponding solution has the asymptotic behavior

$$R_L(r)\to C_l\frac{e^{n\pi/2+i\sigma_L^c}\Gamma(2L+2)}{(2k)^L kr\Gamma(L+1+in)}\sin\left[kr-\frac{L\pi}{2}-n\ln 2kr+\sigma_L^c\right]\,, \tag{2.8}$$

where

$$\sigma_L^c=\arg\Gamma(L+1+in)\,. \tag{2.9}$$

Since the imaginary part of the potential vanishes for large r, it should be possible to rewrite (2.5) in the form (2.8) with the inclusion of an additional phase shift. We, therefore, define the complex phase shifts $\eta_L=\mu+i\lambda$ through the relation

$$\sin\left[kr-\frac{L\pi}{2}+n\ln 2kr+\mu+i\lambda\right]=\frac{1}{2i}[e^{is}-(A+iB)e^{-is}]\,. \tag{2.10}$$

The complex phase shifts η_L are then given by

$$\eta_L=(L-p)\frac{\pi}{2}-\frac{1}{2}\tan^{-1}(B/A)$$
$$+\frac{i}{2}\ln(A^2+B^2)^{1/2}. \quad (2.11)$$

The phase shifts η_L have been determined for $\eta=0.5$, 2.0, and 10.0 and different values of L and β and are given in Tables I–III.

It may be interesting to note at this stage the results that one obtains by the perturbative treatment of the complex potential by the semiclassical method. The real part of the phase shift is obtained by a straightforward application of the JWKB method to the problem with the real part of the potential only. Obviously, the results cannot be reliable, since the real part of the phase shift is independent of β, whereas the exact phase shifts show a fairly good variation as β changes. The imaginary part also shows a very poor agreement, becoming worse as β increases.

III. THE SEMICLASSICAL METHOD

It will be useful to recall the essence of the semiclassical method[1–4] mentioned earlier. Let

$$\frac{d^2\psi(y)}{dy^2}+\frac{t_1(y)}{\hbar^2}\psi(y)=0 \quad (3.1)$$

be the relevant radial equation for the given prob-

TABLE I. The exact and the semiclassical phase shifts for $n=0.5$.

β	Reη_L	Reσ_L	Reσ'_L	Imη_L	Imσ_L and Imσ'_L
			$L=0$		
0.00	−0.2441	−0.2441	−0.2389	0.0000	0.0000
0.25	−0.2683	−0.2694	−0.2642	0.2069	0.2071
0.50	−0.3211	−0.3220	−0.3210	0.3873	0.3880
1.00	−0.4492	−0.4498	−0.4496	0.6835	0.6840
1.50	−0.5802	−0.5811	−0.5808	0.9254	0.9263
2.25	−0.7677	−0.7685	−0.7681	1.2287	1.2290
2.50	−0.8273	−0.8280	−0.8277	1.3188	1.3190
3.00	−0.9425	−0.9439	−0.9428	1.4869	1.4870
3.50	−1.0526	−1.0529	−1.0529	1.6419	1.6419
4.00	−1.1582	−1.1589	−1.1584	1.7863	1.7863
			$L=1$		
0.00	0.2196	0.2196	0.2200	0.0000	0.0000
0.25	0.2173	0.2172	0.2177	0.1047	0.1048
0.50	0.2106	0.2105	0.2110	0.2089	0.2090
1.00	0.1848	0.1846	0.1851	0.4130	0.4132
1.50	0.1449	0.1446	0.1451	0.6090	0.6093
2.25	0.0660	0.0656	0.0660	0.8843	0.8846
2.50	0.0361	0.0357	0.0361	0.9709	0.9712
3.00	−0.0273	−0.0278	−0.0273	1.1368	1.1370
3.50	−0.0941	−0.0946	−0.0942	1.2934	1.2936
4.00	−0.1630	−0.1635	−0.1631	1.4418	1.4419
			$L=2$		
0.00	0.4646	0.4646	0.4646	0.0000	0.0000
0.25	0.4640	0.4639	0.4640	0.0688	0.0688
0.50	0.4622	0.4621	0.4622	0.1375	0.1375
1.00	0.4551	0.4550	0.4551	0.2745	0.2745
1.50	0.4435	0.4434	0.4435	0.4105	0.4105
2.25	0.4179	0.4178	0.4179	0.6116	0.6117
2.50	0.4074	0.4073	0.4073	0.6777	0.6778
3.00	0.3835	0.3834	0.3835	0.8083	0.8084
3.50	0.3563	0.3561	0.3562	0.9365	0.9366
4.00	0.3260	0.3258	0.3259	1.0622	1.0622

TABLE II. The exact and semiclassical phase shifts for $n=2.0$.

β	$\mathrm{Re}\eta_L$	$\mathrm{Re}\sigma_L$	$\mathrm{Re}\sigma'_L$	$\mathrm{Im}\eta_L$	$\mathrm{Im}\sigma_L$ and $\mathrm{Im}\sigma'_L$
			$L=0$		
0.00	0.1296	0.1295	0.1293	0.0000	0.0000
0.25	0.1290	0.1287	0.1285	0.0625	0.0624
0.50	0.1269	0.1267	0.1265	0.1248	0.1247
1.00	0.1188	0.1188	0.1186	0.2487	0.2487
1.50	0.1059	0.1059	0.1058	0.3709	0.3708
2.25	0.0785	0.0785	0.0785	0.5499	0.5498
2.50	0.0675	0.0675	0.0675	0.6082	0.6082
3.00	0.0432	0.0433	0.0432	0.7228	0.7228
3.50	0.0162	0.0162	0.0162	0.8345	0.8345
4.00	−0.0130	−0.0129	−0.0129	0.9433	0.9433
			$L=1$		
0.00	1.2368	1.2367	1.2367	0.0000	0.0000
0.25	1.2364	1.2363	1.2362	0.0542	0.0542
0.50	1.2352	1.2351	1.2351	0.1083	0.1083
1.00	1.2304	1.2304	1.2303	0.2162	0.2161
1.50	1.2226	1.2225	1.2224	0.3235	0.3234
2.25	1.2054	1.2054	1.2053	0.4825	0.4824
2.50	1.1983	1.1983	1.1982	0.5350	0.5350
3.00	1.1823	1.1823	1.1821	0.6387	0.6387
3.50	1.1639	1.1638	1.1638	0.7409	0.7409
4.00	1.1435	1.1435	1.1434	0.8415	0.8415
			$L=2$		
0.00	2.0222	2.0221	2.0222	0.0000	0.0000
0.25	2.0220	2.0220	2.0219	0.0450	0.0449
0.50	2.0214	2.0213	2.0213	0.0900	0.0899
1.00	2.0189	2.0188	2.0187	0.1799	0.1799
1.50	2.0147	2.0146	2.0146	0.2695	0.2695
2.25	2.0055	2.0054	2.0053	0.4034	0.4034
2.50	2.0016	2.0016	2.0015	0.4478	0.4479
3.00	1.9927	1.9926	1.9926	0.5363	0.5363
3.50	1.9824	1.9824	1.9824	0.6242	0.6242
4.00	1.9706	1.9705	1.9705	0.7114	0.7114

lem, and let

$$\frac{d^2\phi(s)}{ds^2}+\frac{t_2(s)}{\hbar^2}\phi(s)=0 \tag{3.2}$$

be a model equation, whose exact solution is known. The functions $t_1(y)$ and $t_2(s)$ should be qualitatively similar, i.e., they should have the same number of physical roots (real, non-negative), similar behavior asymptotically and near the singular points, if any. Moreover, it will be convenient if neither of them has any extremum beyond its largest real root. When such conditions are met, we have found a model equation. The function $\psi(y)$ is then obtained through a transformation,

$$\psi(y)=T(y)\phi[s(y)]\ . \tag{3.3}$$

The consistency condition among the relations (3.1), (3.2), and (3.3) gives the difference between phase shifts. In practice, it is possible to satisfy the condition only up to a given order of $\hbar^2$, so that one gets a power-series expansion in $\hbar^2$ of the phase-shift difference. The method has been used by a number of workers to determine phase shifts for scattering from real potentials. It has been shown by Lu and Wald[3,4] and Mukherjee and Chandel[7,8] that the method gives very accurate results even if one considers terms only up to order $\hbar^2$.

We now generalize this method to the case of a complex potential. There are two complex roots of the equation

$$t_1(y)=0$$

in the present problem. One of these, however, does not qualify as a turning point because it has a negative real part. As the model equation, we choose, at the first instance, the radial equation for scattering from a point charge, with the same Sommerfeld parameter n. Thus the turning point, the trajectory, and the phase shifts are all real for the model equation. It is nevertheless possible to obtain the transformation (3.3). Let y_t denote the complex turning point for the present problem. Following the method and notation of Refs. 7 and 8 we get the complex phase shift in the zeroth order as

$$\begin{aligned}\sigma^0_{\text{complex}}&=\sigma^c_L+\lim_{\substack{y\to\infty\\ s\to\infty}}\left(s-y-n\ln\frac{s}{y}\right)\\ &\simeq\sigma^c_L+\int_{y_t}^{\tilde{y}}[y^2-2ny-(L+\tfrac{1}{2})^2+i\beta]^{1/2}\frac{dy}{y}-[\tilde{y}^2-2n\tilde{y}-(L+\tfrac{1}{2})^2]^{1/2}\\ &\quad+n\ln\{\tilde{y}-n+[\tilde{y}^2-2n\tilde{y}-(L+\tfrac{1}{2})^2]^{1/2}\}-\tfrac{1}{2}n\ln[n^2+(L+\tfrac{1}{2})^2]\\ &\quad-(L+\tfrac{1}{2})\left[\sin^{-1}\left[\frac{n\tilde{y}+(L+\frac{1}{2})^2}{\tilde{y}[n^2+(L+\frac{1}{2})^2]^{1/2}}\right]-\frac{\pi}{2}\right]\\ &\quad+\frac{i\beta}{2(L+\frac{1}{2})}\left[\sin^{-1}\left[\frac{n\tilde{y}+(L+\frac{1}{2})^2}{\tilde{y}[n^2+(L+\frac{1}{2})^2]^{1/2}}\right]-\sin^{-1}\left[\frac{n}{[n^2+(L+\frac{1}{2})^2]^{1/2}}\right]\right]\end{aligned}\tag{3.4}$$

with σ^c_L given by (2.9). The second term on the right-hand side is obtained by integrating first along a line parallel to the imaginary axis, and then along the real axis. $\tilde{y}$ is arbitrary but very large and real (because of our choice of the path, any large y is real). Langer's substitution has been made in obtaining (3.4).

The divergence difficulty that one encounters in calculating terms of order $\hbar^2$ can be solved exactly in the same manner as in the case of a real potential. This consists essentially in going over to a contour from $\infty-i\epsilon$ to $\infty+i\epsilon$ around the turning point, avoiding any other complex root and eliminating the troublesome denominator by repeated partial integrations. When the integrand is free from divergence, we may go back to the original contour. Thus, the first-order correction term Δ_L is given by

$$\begin{aligned}\Delta_L=&-\tfrac{1}{12}\int_{s_t}^{\infty}\mathscr{D}[T_2(s)][t_2^L(s)]^{1/2}ds\\ &+\tfrac{1}{12}\int_{y_t}^{\infty}\mathscr{D}[T_1(y)][t_1^L(y)]^{1/2}dy\ ,\end{aligned}\tag{3.5}$$

where

$$\mathscr{D}[T_i]=\frac{T_i''''}{T_i'^2}-4\frac{T_i'''T_i''}{T_i'^3}+3\frac{T_i''^3}{T_i'^4}\ ,\tag{3.6}$$

$$T_1=y^2t_1^L(y),\quad T_2=s^2t_2^L(s)\ ,\tag{3.7}$$

and the primes indicate the number of times the function is differentiated. The expression for (3.5) can be simplified to

$$\begin{aligned}\Delta_L=&-\frac{n}{24[n^2+(L+\frac{1}{2})^2]}\\ &+\frac{n}{8}\int_{y_t}^{\infty}\frac{[t_1^L(y)]^{1/2}}{(y-n)^4}y\,dy\end{aligned}\tag{3.8}$$

and easily evaluated along the original contour. In equation (3.7), $t_1^L(y)$ and $t_2^L(s)$ are given by

$$t_1^L(y)=1-\frac{2n}{y}-\frac{(L+\frac{1}{2})^2-i\beta}{y^2}$$

and

$$t_2^L(s)=1-\frac{2n}{s}-\frac{(L+\frac{1}{2})^2}{s^2}\ .\tag{3.9}$$

The semiclassical phase shifts are then obtained as

$$\sigma_L = \sigma^0_{\text{complex}} + \Delta_L \ . \tag{3.10}$$

It is instructive to repeat the calculations with another model equation, which is given by the radial equation for a free particle, viz.,

$$\frac{d^2\phi(s)}{ds^2} + \frac{t_2(s)}{\hbar^2}\phi(s) = 0 \tag{3.11}$$

with

$$t_2(s) = 1 - \frac{L(L+1)}{s^2} \ . \tag{3.12}$$

The phase shifts in this case are given by

$$\begin{aligned}
\sigma'_L &= \lim_{\substack{y\to\infty \\ s\to\infty}} (s - y + n \ln 2y) \\
&\simeq \int_{y_t}^{\tilde{y}} [y^2 - 2ny - (L+\tfrac{1}{2})^2 + i\beta]^{1/2} \frac{dy}{y} - n - [\tilde{y}^2 - 2n\tilde{y} - (L+\tfrac{1}{2})^2]^{1/2} \\
&\quad + n \ln\{\tilde{y} - n + [\tilde{y}^2 - 2n\tilde{y} - (L+\tfrac{1}{2})^2]^{1/2}\} \\
&\quad - (L+\tfrac{1}{2}) \left[\sin^{-1}\left(\frac{n\tilde{y} + (L+\tfrac{1}{2})^2}{\tilde{y}[n^2 + (L+\tfrac{1}{2})^2]^{1/2}} \right) - \sin^{-1}\left(\frac{n}{[n^2 + (L+\tfrac{1}{2})^2]^{1/2}} \right) - \frac{\pi}{2} \right] \\
&\quad + \frac{i\beta}{2(L+\tfrac{1}{2})} \left[\sin^{-1}\left(\frac{n\tilde{y} + (L+\tfrac{1}{2})^2}{\tilde{y}[n^2 + (L+\tfrac{1}{2})^2]^{1/2}} \right) - \sin^{-1}\left(\frac{n}{[n^2 + (L+\tfrac{1}{2})^2]^{1/2}} \right) \right] + \Delta'_L \ ,
\end{aligned} \tag{3.13}$$

where Δ'_L, the first-order correction term, is given by

$$\Delta'_L = \frac{n}{8} \int_{y_t}^{\infty} \frac{[t_1^L(y)]^{1/2}}{(y-n)^4} y \, dy \tag{3.14}$$

and $t_1^L(y)$ is given by Eq. (3.9). The integration in (3.13) and (3.14) are to be performed along the complex path as discussed before.

IV. RESULTS AND DISCUSSIONS

The phase shifts calculated for $n = 0.5$, 2.0, 10.0, and various β for the two model equations are given, respectively, in Tables I, II, and III. It may be pointed out that both the real and the imaginary parts of the phase shifts agree fairly well with the exact results. The results for $L = 1$ are, however, much better than for $L = 0$. The accuracy improves with higher L. From our results we can draw the following conclusions.

(a) The real part of the phase shift shows a significant dependence on the imaginary part of the potential. For $L = 1$ the real part even changes sign as the imaginary potential becomes stronger. The perturbative method, on the other hand, gives a real part which does not depend on the imaginary potential.

(b) The imaginary part of the phase shift shows a monotonic increase as β increases, though not as fast as is given by the perturbative method. With an increase in L, $\text{Im}\eta_L$ decreases. Physically this means that partial waves with higher L are less absorbed because of the centrifugal barrier.

(c) The correction term of order $\hbar^2$ makes a small contribution to the phase shift in the present problem in the case of the first model equation. But for the second model equation this contribution is significant. This is easy to understand. In the case of the first model equation, there is a cancellation between the correction terms that does not happen in the other case. Terms of higher order in $\hbar^2$ depend on the higher derivatives of the function $t_1(y)$ which are anyway small for the present problem. However, one may consider a potential which changes rapidly in the vicinity of the turning point. The correction terms may be quite large in that case. For higher L, the correction term will become smaller.

(d) As is evident from the tables, both model equations give almost equally good results.

(e) Although the calculations were done by a particular semiclassical method, there are other semiclassical methods which may be equally appli-

TABLE III. The exact and semiclassical phase-shifts for $n = 10.0$.

β	$\mathrm{Re}\eta_L$	$\mathrm{Re}\sigma_L$	$\mathrm{Re}\sigma'_L$	$\mathrm{Im}\eta_L$	$\mathrm{Im}\sigma_L$ and $\mathrm{Im}\sigma'_L$
			$L=0$		
0.00	13.8029	13.8030	13.8030	0.0000	0.0000
0.25	13.8029	13.8029	13.8029	0.0125	0.0124
0.50	13.8029	13.8028	13.8028	0.0250	0.0249
1.00	13.8028	13.8028	13.8027	0.0500	0.0500
1.50	13.8027	13.8026	13.8025	0.0750	0.0750
2.25	13.8025	13.8024	13.8023	0.1125	0.1125
2.50	13.8024	13.8024	13.8023	0.1250	0.1250
3.00	13.8022	13.8022	13.8021	0.1500	0.1500
3.50	13.8019	13.8019	13.8018	0.1750	0.1750
4.00	13.8016	13.8016	13.8015	0.1999	0.1999
			$L=1$		
0.00	15.2740	15.2741	15.2741	0.0000	0.0000
0.25	15.2740	15.2740	15.2740	0.0124	0.0123
0.50	15.2740	15.2739	15.2739	0.0248	0.0247
1.00	15.2740	15.2739	15.2739	0.0497	0.0496
1.50	15.2739	15.2738	15.2738	0.0745	0.0745
2.25	15.2736	15.2735	15.2735	0.1118	0.1118
2.50	15.2735	15.2735	15.2734	0.1242	0.1242
3.00	15.2733	15.2732	15.2732	0.1490	0.1490
3.50	15.2730	15.2729	15.2729	0.1738	0.1738
4.00	15.2727	15.2727	15.2727	0.1987	0.1987
			$L=2$		
0.00	16.6474	16.6475	16.6475	0.0000	0.0000
0.25	16.6474	16.6474	16.6474	0.0123	0.0121
0.50	16.6474	16.6473	16.6473	0.0245	0.0244
1.00	16.6474	16.6474	16.6473	0.0490	0.0489
1.50	16.6473	16.6473	16.6471	0.0735	0.0735
2.25	16.6470	16.6469	16.6469	0.1103	0.1103
2.50	16.6470	16.6469	16.6468	0.1226	0.1226
3.00	16.6467	16.6467	16.6466	0.1471	0.1471
3.50	16.6465	16.6464	16.6464	0.1716	0.1716
4.00	16.6462	16.6462	16.6461	0.1961	0.1961

cable to this simple problem. The phase-integral method of Fröman and Fröman,[12,13] in particular, could be an alternative method.

(f) The accuracy of the semiclassical results in the present problem makes us feel confident to take up a more complicated problem, viz., that occurring in heavy-ion collisions. This is currently under our consideration. Knoll and Schaeffer[10,11] have shown that in a typical problem with a Coulomb and Woods-Saxon potential, a good approximation is obtained, for most of the partial waves, by considering a single reflection from one turning point only. Even the exceptional cases were not too difficult. Although in the semiclassical method each problem has to be studied separately for its special features, this simple situation is likely to survive in most of the cases of physical interest.

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