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Two-body Dirac equation for semirelativistic quarks

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A semirelativistic quark model for mesons is proposed, based on a comprehensive treatment of the two-body Dirac equation introduced by Breit in 1929. Sixteen-component eigenstates of J^2, J_3, and parity are constructed. They are used to obtain radial equations for the most general local, nonderivative interaction Hamiltonian of order v^2/c^2 containing both scalar and four-vector terms. The Breit interaction is a special case of this interaction Hamiltonian, and the radial equations are considered first for that case. It has been known since 1930 that there is an ambiguity associated with the Breit equation. It arises because there are two different ways to reduce the Breit equation to four-component form. In one method the Breit interaction is regarded as a perturbation and in the other it is not. Only the first method gives correct results. This ambiguity is resolved by a consideration of the radial equations for the Breit equation. In addition, explicit solutions of the radial equations to order α^4 are given for hydrogen and positronium. The conclusion is that the two-body Dirac equation is unambiguous and correct to order α^4 in QED. The radial equations are used next in the context of QCD to determine which of the 24 possible combinations of scalar and vector potentials considered can give rise to quark confinement in a normalized theory. If the higher-order potentials are no larger than the zero-order potentials in magnitude, we show that there are only two possible forms for the confinement interaction: (1) the scalar interaction $S(r)\,(1-\hat{r}\cdot\vec{\alpha}_1\hat{r}\cdot\vec{\alpha}_2)\,\beta_1\beta_2$ and (2) the combined scalar-vector interaction $[V(r)+S(r)\beta_1\beta_2]\,(1-\hat{r}\cdot\vec{\alpha}_1\hat{r}\cdot\vec{\alpha}_2)$ where $S\geq V$ for large r.

I. INTRODUCTION

It is generally assumed that the internal dynamics of $q\bar{q}$ systems can be described satisfactorily by a static potential and a nonrelativistic wave equation for the case of slowly moving quarks. Calculations of the Υ family of states serve as examples of this assumption.[1] On the other hand, the concept of a static potential may not be applicable to systems of rapidly moving quarks that may require a two-body relativistically covariant wave equation[2] for their description. Between these two limits is the intermediate case of systems composed of semirelativistic quarks which are adequately described by a static interaction but which require something more than a nonrelativistic wave equation.

The simplest relativistic wave equation for fermions is the single-particle Dirac equation. It is useful for $q\bar{Q}$ systems[3] where $M_Q \gg M_q$, but cannot adequately describe two relativistic or semirelativistic quarks of comparable mass. Of course, there is always the possibility of using the single-particle Dirac equation, or even a nonrelativistic wave equation, if an effective potential which takes into account the motion and spin of the other particle is available. However, a general description of the relation between an effective potential and the original interaction on which it is based is lacking at this time. In certain cases effective potentials can be obtained. For example, a nonrelativistic reduction of the Breit interaction yields the effective Fermi-Breit interaction which has been used to obtain the hyperfine splittings of mesons and baryons.[4] However, the Fermi-Breit interaction neglects binding-energy effects, and its use for deeply bound systems is questionable. More accurate effective potentials must be both spin and energy dependent. An example of an effective potential which is "exact" when the interaction is the fourth component of a four-vector, and hence not exact, is given in Sec. VI. Its classical counterpart is obtained in Sec. II.

A noncovariant but approximately relativistic two-body wave equation with a static interaction is needed for systems of semirelativistic particles of comparable mass. In QED there is such an equation. It is the Breit equation[5]

$$(H_{01}+H_{02}+V_C+H_B)\psi=E\psi\ , \tag{1.1}$$

where V_C is the Coulomb potential and H_B is the Breit interaction

$$H_B = -\tfrac{1}{2} V_c(\vec{\alpha}_1 \cdot \vec{\alpha}_2 + \hat{r} \cdot \vec{\alpha}_1 \hat{r} \cdot \vec{\alpha}_2) \ . \tag{1.2}$$

As is well known, the Coulomb potential is the zero-order term in the v/c expansion of the electromagnetic interaction, and the Breit interaction is the v^2/c^2 term. However, the Breit equation, as written, is not correct to order v^2/c^2. This was originally noted by Breit in 1930.[6] The reduction of Eq. (1.1) to four-component form gives rise to a spurious α^2/r^2 term in the effective Hamiltonian as discussed by Baker and Glover[7] and by Feldman, Fulton, and Townsend.[8] Breit[9] suggested that Eq. (1.1) be replaced by the equations

$$(H_{01} + H_{02} + V_C)\psi_0 = E_0 \psi_0 \tag{1.3a}$$

and

$$E = E_0 + \langle \psi_0 H_B \psi_0 \rangle \tag{1.3b}$$

in which H_B is included only at the level of first-order perturbation theory. In that case, the spurious term is not encountered in the reduction to four-component form. A satisfactory explanation of why Eqs. (1.1) and (1.3) give different results has not been given in the literature.

By analogy with QED we propose that the concept of a static potential is valid to order v^2/c^2 for QCD and that an appropriate wave equation for semirelativistic quarks is the two-body Dirac equation. The formalism needed to separate the angular and radial dependence of the wave function is developed in Sec. III and used to derive radial equations in Sec. IV. The general implications of the radial equations are considered in Sec. V for a wide variety of unbounded interaction Hamiltonians, and explicit calculations of the energy levels of hydrogen and positronium to order α^4 are made in Sec. VII.

Two interesting results follow from a consideration of the radial equations. First, the ambiguity regarding the Breit equation is resolved. The central point is that Eq. (1.1) is found to have no normalized solutions due to a singularity at the point $r_0 = \alpha/E$. Although this singularity is not apparent from the form of Eq. (1.1), it is clearly present in the radial equations as shown in Sec. V. It follows that a reduction of Eq. (1.1) to four-component form cannot be expected to yield the correct effective potential, and the spurious term encountered is a reflection of this fact. On the other hand, the two-body Dirac equation for the Coulomb potential, Eq. (1.3a), is not singular at r_0, normalized solutions may be obtained, and the perturbation integral in Eq. (1.3b) is well defined. In this way we can understand the correctness of Breit's original prescription, namely that H_B must be included only as a perturbation. This type of situation is familiar from nonrelativistic quantum mechanics. Consider, for example, the spin-orbit potential $V_{SL} = V_C' \vec{L} \cdot \vec{S}/2m^2 r$. It is too singular at the origin to yield results if included directly into Schrödinger's equation but when treated as a perturbation, it yields correct fine-structure corrections to the zero-order energy levels. The reason for this occurrence is that V_{SL} is an accurate approximation of the energy-dependent effective potential $V_{SL}[1+(\xi - V)/2m]^{-1}$ only for $|r - \alpha/2m| \gg 0$. Similarly, the Breit interaction must be regarded as an accurate approximation only for $|r - r_0| \gg 0$.

Second, the type of scalar and vector static potentials that can confine quarks is determined to order v^2/c^2. It is not sufficient, for example, that the potential increase without bound with increasing separation distance. A distinctive characteristic of relativistic descriptions that include negative-energy states is the ability of particles to move through certain types of potential barriers without the expected damping of their wave function.[10] A strongly repulsive potential can lead to an enhancement of the "small" components of the wave function in regions in which the potential is sufficiently large. The wave function oscillates through these regions and avoids confinement no matter how large the potential. This is the familiar Klein paradox.[11] It occurs for potentials which transform as the fourth component of a four-vector but not for potentials which transform as Lorentz scalars. This fact has been noted for the single-particle Dirac equation[3] and is found to hold for the two-body Dirac equation as well. The conclusion is that in this formalism quark confinement requires a Lorentz scalar interaction. Scalar interactions have been included in many quark-model calculations,[12] and their importance noted by various authors.[13]

In addition to the difficulties associated with the Klein paradox, confinement potentials tend to generate singularities in the wave function which destroy its normalization. In fact, we find it impossible to obtain normalized solutions for any zero-order confinement potential. Since radial equations for higher-order potentials have not been given in the literature, we include such potentials in our considerations.[14] All possible local, nonderivative scalar and four-vector potentials of order v^2/c^2 are considered. Several are found that yield normalized wave functions but there are only two interactions

for which the higher-order potentials do not exceed the zero-order potentials in magnitude. They are the scalar interaction

$$H_S = S(r)(1-\hat{r}\cdot\vec{\alpha}_1\hat{r}\cdot\vec{\alpha}_2)\beta_1\beta_2 \tag{1.4}$$

and the scalar-vector combination

$$H_{SV} = [V(r)+S(r)\beta_1\beta_2](1-\hat{r}\cdot\vec{\alpha}_1\hat{r}\cdot\vec{\alpha}_2) , \tag{1.5}$$

where $S(r) \geq V(r)$ for large r. Numerical solutions of the radial equations obtained in Sec. IV for the above confinement interactions will be presented in a later publication.

II. THE CLASSICAL EFFECTIVE POTENTIAL

We begin with the simplest problem of all, that of constructing the classical effective potential for two relativistic point masses m_1 and m_2 interacting through a central potential $V(r)$. The total energy in the c.m. system is

$$E = V+(p^2+m_1{}^2)^{1/2}+(p^2+m_2{}^2)^{1/2} . \tag{2.1}$$

To reduce this two-body problem to that of a single particle m_1 of energy $E_1 = E-m_2$ moving in an effective potential $U_{\text{eff}}(r)$, set $U_{\text{eff}} = V-m_2+(p^2+m_2{}^2)^{1/2}$ and use Eq. (2.1) to eliminate the p^2 dependence of U_{eff}. This can be done by solving Eq. (2.1) for $E-V$ and squaring both sides of the resulting equation. This yields the expression

$$(E-V)^2-(2p^2+m_1{}^2+m_2{}^2) = 2[(p^2+m_1{}^2)(p^2+m_2{}^2)]^{1/2} . \tag{2.2}$$

When Eq. (2.2) is squared, the p^4 terms cancel, and we obtain an expression for p^2. When it is added to $m_2{}^2$, a perfect square is obtained. In particular, we find that

$$(p^2+m_2{}^2)^{1/2} = \tfrac{1}{2}[E-V+(m_2{}^2-m_1{}^2)/(E-V)] . \tag{2.3}$$

This gives for the effective potential

$$U_{\text{eff}}(r) = V_{\text{eff}}(r)-m_2 , \tag{2.4a}$$

where

$$V_{\text{eff}}(r) = \tfrac{1}{2}[E+V+(m_2{}^2-m_1{}^2)/(E-V)] . \tag{2.4b}$$

This is the classical limit to which the quantum effective potential in Sec. VI must reduce. It allows Eq. (2.1) to be replaced by the equation

$$E_1 = U_{\text{eff}}(r)+(p^2+m_1{}^2)^{1/2} . \tag{2.5}$$

The tendency of confinement potentials to generate singularities in the wave function can be understood in part from a consideration of the effective potential. For confinement interactions there is necessarily a point r_0 at which the potential equals the total energy. At that point both the classical and quantum effective potentials are unbounded. Although r_0 is in a classically forbidden region, it will be reached by the wave function as it tunnels through the forbidden region. Equation (2.3) is valid only up to the turning point but it is of interest to consider it beyond that point in the spirit of the correspondence principle. For $m_1 \neq m_2$ it shows that p^2 is infinite at r_0 corresponding to a particle moving at the speed of light. Such a state is physically unattainable, and it is not surprising that the wave equation has no physically significant solutions for a potential which is the fourth component of a four-vector, as is shown in Sec. V.

III. TWO-BODY EIGENSTATES OF J^2, J_3, P

The total angular momentum for two spin-$\frac{1}{2}$ particles can be written as $\vec{J} = \vec{Q}+\vec{S}_2$ or $\vec{J} = \vec{L}+\vec{S}$ where $\vec{Q} = \vec{L}+\vec{S}_1$ is the total angular momentum of particle 1 and $\vec{S} = \vec{S}_1+\vec{S}_2$ is the total spin. The wave function ψ is chosen to be an eigenstate of J^2, J_3, and the parity P. Each component of ψ is the product of an angular function and a radial amplitude. The radial amplitudes are labeled by the quantum numbers j,η,s,l or by j,η,q,l where η is the eigenvalue of P. The number of linearly independent radial amplitudes is easily determined. Consider the first set of quantum numbers. Clearly $s = 0,1$. For $s = 1$ three values of l are possible, namely $l = j$, $l = j\pm 1$; for $s = 0$ there is only one possible value, $l = j$. For each of these four amplitudes there is a corresponding "small" amplitude resulting in a total of eight radial amplitudes. For $j = 0$ this number reduces to four because there is only one way to add $\vec{L}$ and $\vec{S}$ with $s = 1$ to get $j = 0$. Using the other set of labels j,η,q,l naturally gives the same result.

Sixteen-component, two-body eigenstates can be constructed as follows. Consider a spin-$\frac{1}{2}$ particle of mass m_1 moving under the influence of an interaction which is invariant under spatial rotations and inversions. It will be described by a four-component spinor ϕ of total angular momentum $\vec{Q}$. For a given value of q there are two possible values

of l, namely $l=q\pm\frac{1}{2}$. The normalized two-component eigenfunctions of Q^2 and Q_3 are the well-known functions[11]

$$\chi_{ql}^{q_3}=a_+\binom{1}{0}Y_l^{q_3-1/2}+a_-\binom{0}{1}Y_l^{q_3+1/2} \tag{3.1a}$$

for $l=q-\frac{1}{2}$ and

$$\chi_{ql}^{q_3}=-a_-\binom{1}{0}Y_l^{q_3-1/2}+a_+\binom{0}{1}Y_l^{q_3+1/2} \tag{3.1b}$$

for $l=q+\frac{1}{2}$ where $a_\pm=[(l\pm q_3+\frac{1}{2})/(2l+1)]^{1/2}$ and Y_l^m are spherical harmonics. In both cases, the parity of $\chi_{ql}^{q_3}$ is $(-1)^l$. Since the orbital angular momentum does not commute with even the single-particle Dirac Hamiltonian, the principle of superposition implies that both the "large" and "small" components of ϕ can be expanded in terms of the χ functions, i.e.,

$$\phi=\begin{bmatrix} C_{LA}\chi_{ql_A}^{q_3}+C_{LB}\chi_{ql_B}^{q_3} \\ C_{SA}\chi_{ql_A}^{q_3}+C_{SB}\chi_{ql_B}^{q_3} \end{bmatrix}, \tag{3.2}$$

where l_A and l_B are the two possible values of l for fixed q. However, ϕ must be a state of definite parity. Let the parity of ϕ be $(-1)^{l_A}$ and recall that the space-inverted spinor is $\beta\phi(-\vec{r})$. Then $C_{LB}=C_{SA}=0$, and we arrive at the usual expression for ϕ,

$$\phi(q,q_3,l_A)=\begin{bmatrix} g_{ql}(r)\chi_{ql_A}^{q_3}(\theta,\phi) \\ if_{gl_A}(r)\chi_{ql_B}^{q_3}(\theta,\phi) \end{bmatrix}, \tag{3.3}$$

where q_{ql_A} and f_{ql_A} are radial amplitudes, l_A is the leading orbital angular momentum, and l_B is the secondary orbital angular momentum. The dependence of the radial amplitudes on j,η is suppressed. To include the second particle in our description we must construct a 16-component eigenfunction ψ_{jj_3} of J^2,J_3 where $\vec{J}=\vec{S}_2+\vec{Q}$. Notice how both the "large" and "small" components of ϕ were built up from a combination of the Pauli spin matrices and the eigenfunctions of L^2,L_3. The next step is completely analogous. We have eigenfunctions of Q^2,Q_3. To form both the "large" and "small" (relative to particle 2) components of ψ_{jj_3} we must combine the Pauli spin matrices for particle 2 with the eigenfunctions of Q^2,Q_3. The "large" and "small" components of ψ will be eight-component spinors which we designate as $\Omega(jj_3ql)$. For a given value of j there are two possible values of q, namely $q_1=j-\frac{1}{2}$ and $q_2=j+\frac{1}{2}$. The corresponding Ω spinors of parity $(-1)^{l_A}$ are

$$\Omega(j,j_3,q,l_A)=b_+\binom{1}{0}\otimes\phi(q,j_3-\tfrac{1}{2},l_A)+b_-\binom{0}{1}\otimes\phi(q,j_3+\tfrac{1}{2},l_A) \tag{3.4a}$$

for $q=q_1$ and

$$\Omega(j,j_3,q,l_A)=-b_-\binom{1}{0}\otimes\phi(q,j_3-\tfrac{1}{2},l_A)+b_+\binom{0}{1}\otimes\phi(q,j_3+\tfrac{1}{2},l_A) \tag{3.4b}$$

for $q=q_2$ where $b_\pm=[(q\pm j_3+\frac{1}{2})/(2q+1)]^{1/2}$. The Ω spinors are analogous to the χ functions but differ in that the parity of the Ω spinors is independent of the angular momentum being added to the spin (q for the Ω spinors and l for the χ functions). To obtain Ω spinors of parity $(-1)^{l_B}$ merely interchange l_A and l_B in Eqs. (3.3) and (3.4). The reasoning used in expressing a 16-component spinor of definite q, $\psi_{jj_3}^q$, in terms of the Ω spinors is the same as was used in going from Eq. (3.2) to Eq. (3.3). Write the upper eight components of $\psi_{jj_3}^q$ as

$$b_{LA}\Omega(jj_3ql_A)+b_{LB}\Omega(jj_3ql_B)$$

and the lower eight components as

$$b_{SA}\Omega(jj_3ql_A)+b_{SB}\Omega(jj_3ql_B)\ .$$

If the parity of $\psi_{jj_3}^q$ is $(-1)^{l_A}$, then $b_{LB}=b_{SA}=0$ and

$$\psi_{jj_3}^q=\begin{bmatrix} \Omega(jj_3ql_A) \\ i\Omega(jj_3ql_B) \end{bmatrix}. \tag{3.5}$$

For $j\neq 0$ $\psi_{jj_3}^q$ cannot be a solution to the two-body Dirac equation because Q^2 does not commute with the Hamiltonian. This means that the wave function can be expanded in terms of the eigenstates of Q^2, as long as they are chosen to have the same parity, i.e.,

$$\psi_{jj_3}=\psi_{jj_3}^{q_1}+\psi_{jj_3}^{q_2}\ , \tag{3.6}$$

where the leading orbital angular momentum of $\psi_{jj_3}^{q_1}$ is $l_{A1}=q_1\pm\frac{1}{2}$, and that of $\psi_{jj_3}^{q_2}$ is $l_{A2}=q_2\pm\frac{1}{2}$. The possible values of l_{A1} and l_{A2} are $j-1$, j and $j,j+1$, respectively. If l_{A1} is chosen to be $q_1+\frac{1}{2}$, then the secondary orbital angular momentum l_{B1} must be $q_1-\frac{1}{2}$, etc. The requirement that $\psi_{jj_3}^{q_1}$ and $\psi_{jj_3}^{q_2}$ have the same parity implies that $l_{A1}=l_{A2}=j$ or that $l_{A1}=j-1$ and $l_{A2}=j+1$. For the fermion-fermion problem the parity of ψ_{jj_3} as given by Eqs. (3.5) and (3.6) is $(-1)^{l_{A1}}$ and for the fermion-antifermion

problem it is $(-1)^{l_{A1}+1}$ due to the opposite intrinsic parity of the antiparticle. For the $q\bar{q}$ problem the wave function with $l_{A1}=l_{A2}=j$ and $\eta=(-1)^{j+1}$ describes the states on the exchange-degenerate pseudoscalar-meson trajectory, and the wave function with $l_{A1}=j-1$, $l_{A2}=j+1$, and $\eta=(-1)^{j}$ describes the states on the exchange-degenerate vector-meson trajectory. The former will be referred to as the PMT solutions and the latter as the VMT solutions (even in fermion-fermion problems). Each of these solutions contains eight radial amplitudes for $j\neq 0$, namely, g_{Ai}, f_{Ai}, g_{Bi}, and f_{Bi} for $i=1,2$ where we have simplified the notation by letting $g_{Ai}=g_{qi,l_{Ai}}$, etc. It is also convenient to represent the ϕ spinor of Eq. (3.3) as $\phi_A(g_A,if_A)$ or just ϕ_A. More generally $\phi_{An}(g,if)$ is the ϕ spinor with leading orbital angular momentum l_{An}, secondary orbital angular momentum l_{Bn}, parity $(-1)^{l_{An}}$, and radial functions g and f. Likewise $\Omega_{An}(g,if)$ is the Ω spinor with leading orbital angular momentum l_{An}, parity $(-1)^{l_{An}}$, and radial functions g and f. In this notation, for example,

$$\Omega_{A1}(g,if)=\begin{bmatrix} b_+\phi_{A1}^{j_3-1/2}(g,if) \\ b_-\phi_{A1}^{j_3+1/2}(g,if) \end{bmatrix} . \tag{3.7}$$

This is identical to Eq. (3.4a) when $g=g_{A1}$ and $f=f_{A1}$. Similar expressions hold for $\Omega_{A2}(g,if)$ and $\Omega_{Bn}(g,if)$. For $\Omega_{An}(g_{An},if_{An})$ one merely writes Ω_{An}, etc. Then Eq. (3.6) becomes

$$\psi_{jj_3}=\begin{bmatrix} \Omega_{A1} \\ i\Omega_{B1} \end{bmatrix}+\begin{bmatrix} \Omega_{A2} \\ i\Omega_{B2} \end{bmatrix} . \tag{3.8}$$

Notice that there is only one value of q for $j=0$, namely, $q_2=\frac{1}{2}$. In that case the first term in Eq. (3.8) is not present.

IV. THE RADIAL EQUATIONS

The two-body Dirac equation consists of two Dirac operators $H_{0i}=\vec{\alpha}_i\cdot\vec{p}_i+m_i\beta_i$ plus an interaction Hamiltonian H_I written in the direct-product space arising from the four-components required for the description of each spin-$\frac{1}{2}$ particle. The wave function may be written as 16-component column matrix, and 16×16 two-body Dirac matrices may be defined for each particle such that $\vec{\alpha}_1=I\otimes\vec{\alpha}$, $\beta_1=I\otimes\beta$, $\vec{\alpha}_2=\vec{\alpha}\otimes I$, and $\beta_2=\beta\otimes I$ where $\vec{\alpha},\beta$ are the usual 4×4 Dirac matrices and I is the 4×4 unit matrix. The two-body Dirac equation in the c.m. system is

$$(\vec{\alpha}_1\cdot\vec{p}+\beta_1 m_1-\vec{\alpha}_2\cdot\vec{p}+\beta_2 m_2+H_I)\psi=E\psi , \tag{4.1}$$

where $\vec{p}=-i\vec{\nabla}_r$ and $\vec{r}=\vec{r}_1-\vec{r}_2$. The wave function in Eq. (4.1) has components $\psi_{\mu\nu}$ where the first (second) subscript refers to particle 2 (particle 1); the components are ordered in the column matrix ψ such that the second subscript ν varies most rapidly. Breit[5] introduced this equation for the special case $H_I=V_c+H_B$ and applied it to various systems. However, he did not obtain the corresponding radial equations. In fact, radial equations for the Breit equation have not appeared in the literature.

In this section we obtain radial equations for a very general interaction Hamiltonian which contains the Breit interaction as a special case. The greater generality is required since we are interested in $q\bar{q}$ states, and the Klein paradox suggests the presence of a scalar confinement interaction which, to zero order in v/c, is of the form $S(r)\beta_1\beta_2$. The development of singularities in the radial equations necessitates the consideration of higher-order terms in H_I. As corrections to the zero-order scalar interaction we include the v^2/c^2 terms $S_1(r)\hat{r}\cdot\vec{\alpha}_1\hat{r}\cdot\vec{\alpha}_2\beta_1\beta_2$ and $S_2(r)\vec{\alpha}_1\cdot\vec{\alpha}_2\beta_1\beta_2$; as corrections to $V(r)$ we include the terms $V_1(r)\hat{r}\cdot\vec{\alpha}_1\hat{r}_1\vec{\alpha}_2$ and $V_2(r)\vec{\alpha}_1\cdot\vec{\alpha}_2$. This results in a six-term interaction Hamiltonian

$$H_I=V+V_1\hat{r}\cdot\vec{\alpha}_1\hat{r}\cdot\vec{\alpha}_2+V_2\vec{\alpha}_1\cdot\vec{\alpha}_2$$
$$+(S+S_1\hat{r}\cdot\vec{\alpha}_1\hat{r}_1\cdot\vec{\alpha}_2+S_2\vec{\alpha}_1\cdot\vec{\alpha}_2)\beta_1\beta_2 . \tag{4.2}$$

The above is the most general form for H_I to order v^2/c^2 containing both scalar and four-vector potentials that may be constructed from the vectors $\hat{r}$, $\vec{\alpha}_1$, and $\vec{\alpha}_2$.

The derivation of the radial equations is complicated by the necessity of introducing three different sets of radial amplitudes: (1) the eight linearly independent lower-case amplitudes defined in Sec. III, g_{Ai}, f_{Ai}, g_{Bi}, and f_{Bi} for $i=1,2$ (hereafter referred to as the LC amplitudes), (2) an intermediate set of eight linearly dependent upper-case amplitudes (hereafter referred to as the UC amplitude), and (3) a set of four linearly independent F amplitudes. The LC amplitudes have the advantage of being the amplitudes most directly related to the radial amplitudes of the single-particle Dirac equation and the disadvantage of satisfying a relatively complicated set of radial equations. The radial equations are much simpler when expressed in terms of the F amplitudes. The UC amplitudes are defined in terms of the LC amplitudes as follows:

$$\begin{aligned} G_{An} &= g_{An} - [(-1)^n f_{Bn} - p f_{Bk}]/\lambda , \\ F_{An} &= f_{An} + [(-1)^n g_{Bn} - p g_{Bk}]/\lambda , \\ G_{Bn} &= g_{Bn} + [(-1)^n f_{An} - p f_{Ak}]/\lambda , \\ F_{Bn} &= f_{Bn} - [(-1)^n g_{An} - p g_{Ak}]/\lambda \end{aligned} \tag{4.3}$$

for $n=1,2$ where $p=2[j(j+1)]^{1/2}$, $\lambda=2j+1$, and $k=3-n$. It is easily verified that for $n=1,2$ and $k=3-n$

$$\begin{aligned} G_{An} &= -[(-1)^n F_{Bn} - p F_{Bk}]/\lambda , \\ F_{An} &= [(-1)^n G_{Bn} - p G_{Bk}]/\lambda , \\ G_{Bn} &= [(-1)^n F_{An} - p F_{Ak}]/\lambda , \\ F_{Bn} &= -[(-1)^n G_{An} - p G_{Ak}]/\lambda . \end{aligned} \tag{4.4}$$

It is clear from Eq. (4.4) that no more than four of the UC amplitudes can be linearly independent. For example, G_{A1}, F_{A1}, G_{B2}, and F_{B2} are a linearly independent set of amplitudes. The F amplitudes are defined in terms of this set as follows:

$$\begin{aligned} F_1 &= \sqrt{j} F_{A1} + \sqrt{j+1} G_{B2} , \\ F_2 &= -\sqrt{j} G_{A1} + \sqrt{j+1} F_{B2} , \\ F_3 &= \sqrt{j+1} G_{A1} - \sqrt{j} F_{B2} , \\ F_4 &= -\sqrt{j+1} F_{A1} - \sqrt{j} G_{B2} . \end{aligned} \tag{4.5}$$

The derivation of the radial equations has two parts. The first is the application of the separation-of-variables method based on the eigenfunctions of J^2, J_3, and P obtained in Sec. III. This requires the expansions of the various matrix products in Eq. (4.1) in terms of the Ω spinors. These expansions are given in the Appendix. Although ψ has 16 components, this method yields only 8 radial equations because each is obtained twice. The resulting radial equations contain both LC and UC amplitudes and, what is more important, they are not linearly independent. The second part of the derivation is the reduction of the eight linearly dependent radial equations to four linearly independent ones. This can be done by eliminating the LC amplitudes or by eliminating the UC amplitudes. In the former case we obtain four radial equations for the UC amplitudes which we simplify by introducing the F amplitudes. In the latter case we obtain four radial equations for the LC amplitudes.

Equation (3.8) is the basis of the separation-of-variables method. Different results are obtained depending on whether the PMT solution or the VMT solution is used. A sign convention is used to distinguish between the two cases, the upper sign corresponding to the PMT solution, and the lower sign to the VMT solution. Using Eq. (3.8) and the expansions in the Appendix, one can show that Eq. (4.1) can be rewritten as

$$\begin{bmatrix} \Omega_{A1}(h_1, ih_2) \\ i\Omega_{B1}(h_3, ih_4) \end{bmatrix} + \begin{bmatrix} \Omega_{A2}(h_5, ih_6) \\ i\Omega_{B2}(h_7, ih_8) \end{bmatrix} = 0 \tag{4.6}$$

for the interaction Hamiltonian of Eq. (4.2) where h_i, $i=1,8$, are linear combinations of the radial amplitudes and their first derivatives. Since the angular functions defined in Eq. (3.1) are orthogonal, Eq. (4.6) can be satisfied only if each h_i vanishes. This yields eight rather complicated radial equations. The details of these equations are not important for our considerations but a knowledge of their general form is needed. It can be shown that for $n=1,2$

$$\begin{aligned} G'_{An} + (1+\kappa_{AN}) G_{An}/r &= c_{An} , \\ F'_{An} + (1-\kappa_{An}) F_{An}/r &= d_{An} , \\ G'_{Bn} + (1+\kappa_{Bn}) G_{Bn}/r &= c_{Bn} , \end{aligned} \tag{4.7}$$

etc., where c_{An}, d_{An}, c_{Bn}, etc., are functions of r, the LC amplitudes, and the potentials. The four κ's are defined in accordance with their definition in the single-particle Dirac equation. In particular, $\kappa_{A1}=\pm j$, $\kappa_{B1}=\mp j$, $\kappa_{A2}=\mp(j+1)$, and $\kappa_{B2}=\pm(j+1)$, the upper (lower) sign corresponding to the PMT (VMT) solution as indicated previously. This sign convention is used throughout the remainder of the paper. Equation (4.4) implies that the eight radial equations of the form indicated in Eq. (4.7) are not linearly independent. If we differentiate Eq. (4.4) and replace the derivatives of the UC amplitudes by their values from Eq. (4.7), we obtain a set of eight algebraic relations between the UC and LC amplitudes. Consider, for example, the derivative of the first equation in (4.4), $\lambda G'_{A1} = F'_{B1} + p F'_{B2}$. Using the values of the κ's and Eqs. (4.7) and (4.4), we find that

$$c_{A1} - (d_{B1} + p d_{B2})/\lambda = \pm p F_{B2}/r . \tag{4.8}$$

This is the first of eight algebraic relations that can be obtained from Eqs. (4.4) and (4.7). These algebraic relations may be regarded as a system of eight equations for the eight linearly independent LC amplitudes. When this system is solved for the LC amplitudes, one finds that for $n=1,2$ with $k=3-n$

$$
\begin{aligned}
4g_{An} &= 2G_{An} + m_s\left[\frac{G_{An}\pm F_{Bn}}{D_1} + \frac{G_{An}\mp F_{Bn}}{D_2}\right] + (-1)^n\frac{\epsilon p}{r}\left[\frac{F_{Ak}-G_{Bk}}{D_1} - \frac{F_{Ak}+G_{Bk}}{D_2}\right], \\
4f_{Bn} &= 2F_{Bn} \mp m_s\left[\frac{G_{An}\pm F_{Bn}}{D_1} - \frac{G_{An}\mp F_{Bn}}{D_2}\right] + (-1)^n\frac{\epsilon p}{r}\left[\frac{F_{Ak}-G_{Bk}}{D_1} + \frac{F_{Ak}+G_{Bk}}{D_2}\right], \\
4g_{Bn} &= 2G_{Bn} \pm m_d\left[\frac{F_{An}\pm G_{Bn}}{D_3} - \frac{F_{An}\mp G_{Bn}}{D_4}\right] + (-1)^n(1-\epsilon)\frac{p}{r}\left[\frac{G_{Ak}+F_{Bk}}{D_3} + \frac{G_{Ak}-F_{Bk}}{D_4}\right], \\
4f_{An} &= 2F_{An} - m_d\left[\frac{F_{An}\pm G_{Bn}}{D_3} + \frac{F_{An}\mp G_{Bn}}{D_4}\right] - (-1)^n(1-\epsilon)\frac{p}{r}\left[\frac{G_{Ak}+F_{Bk}}{D_3} - \frac{G_{Ak}-F_{Bk}}{D_4}\right],
\end{aligned}
\tag{4.9}
$$

where $\epsilon=0$ (1) for the PMT (VMT) solution, $m_s=m_1+m_2$, and $m_d=m_1-m_2$. The denominators in (4.9) are defined as follows: $D_1=(x_+ + A_+ - B_+)\mp B_+$, $D_2=D_1-2B_+$, $D_3=(x_- + A_- - B_-)\pm B_-$, and $D_4=D_3-2B_-$ where $x_\pm = E-(V\pm S)$, $A_\pm = V_2 - V_1 \pm (S_2 - S_1)$, and $B_\pm = V_2 \pm S_2$. Radial equations for the four linearly independent UC amplitudes, G_{A1}, F_{A1}, G_{B2}, and F_{B2}, are obtained by using Eq. (4.9) to eliminate the LC amplitudes from the right side of Eq. (4.7). Then the UC amplitudes are written in terms of the F amplitudes by inverting Eq. (4.5). In this way we obtain the following radial equations for the F amplitudes:

$$
\begin{aligned}
F_1' + (2/r)F_1 &= \tfrac{1}{2}(x_\pm - A_\pm + 4B_\pm)F_2 - \tfrac{1}{2}(m_R{}^2/U_4 + p^2/r^2U_2)F_2 \mp (m_0 p/2rU_2)F_4\,, \\
F_2' &= -\tfrac{1}{2}(x_\mp - A_\mp)F_1 + (m_0{}^2/2U_1)F_1 \pm (m_0 p/2rU_1)F_3\,, \\
F_3' + (1/r)F_3 &= -\tfrac{1}{2}(x_\mp - A_\mp + 2B_\mp)F_4 + (m_0{}^2/2U_2)F_4 \pm (m_0 p/2rU_2)F_2\,, \\
F_4' + (1/r)F_4 &= \tfrac{1}{2}(x_\pm - A_\pm + 2B_\pm)F_3 - \tfrac{1}{2}(m_R{}^2/U_3 + p^2/r^2U_1)F_3 \mp (m_0 p/2rU_1)F_1\,,
\end{aligned}
\tag{4.10}
$$

where $m_0=m_1\mp m_2$, $m_R=m_1\pm m_1$, $U_1=x_\mp + A_\mp$, $U_2=x_\mp + A_\mp - 2B_\mp$, $U_3=x_\pm + A_\pm - 2B_\pm$, and $U_4=x_\pm + A_\pm - 4B_\pm$. Note that for $j=0$ all the LC and UC amplitudes corresponding to $q=q_1$ vanish. It follows from Eq. (4.5) that $F_3=F_4=0$, and Eq. (4.10) reduces to a set of two radial equations for F_1 and F_2.

A principal advantage of the radial equations for the F amplitudes is the fact that for the equal mass case the first two equations for the PMT solution decouple from the last two (since $m_0=0$). In that case, the first two equations are called the radial equations for the exchange-degenerate π trajectory, and the last two the radial equations for the exchange-degenerate A_1 trajectory. The reason for this identification stems from the work of Suura[15] who has constructed $q\bar{q}$ eigenstates of J^2, J_3, P and charge conjugation for the equal-mass case. Although the qq wave equation introduced in Ref. 15 is not of the same form as Eq. (4.1), the qq wave function χ of Ref. 15 and ψ also differ. When the relationship $\psi=\gamma_{2y}\chi^T$ is used, the wave equation of Ref. 15 can be shown to be the same as Eq. (4.1) with $H_I=V(r)$. If we set $m_d=V_1=V_2=S=S_1=S_2=0$ in Eq. (4.10) and derive the second-order equations satisfied by F_2 and F_3, we find that

$$
\begin{aligned}
&F_2'' + [2/r + V'/(E-V)]F_1' \\
&\qquad + [\tfrac{1}{4}(E-V)^2 - m^2 - p^2/4r^2]F_2 = 0\,, \\
&F_3'' + [2/r + V'/(E-V)]F_3' \\
&\qquad + [\tfrac{1}{4}(E-V)^2 - m^2 - p^2/4r^2 \\
&\qquad\quad + V'/r(E-V)]F_3 = 0\,,
\end{aligned}
\tag{4.11}
$$

which are identical to Eqs. (2.24) and (2.38) of Ref. 15. It follows that F_1, F_2 refer to the π trajectory and F_3, F_4 to the A_1 trajectory. It is also convenient to refer to the VMT radial equations as the equations for the ρ trajectory when $m_1=m_2$. Notice that the equations do not decouple in this case. Hence four F amplitudes are needed for the ρ trajectory.

Equation (4.10) can also be compared to the radial equations obtained by Krolikowski and Rzewuski[16] for the interaction $H_I=V(r)$. Two sets of radial equations are given in Ref. 16, one set for the amplitudes

$$f_1^+, f_2^+, f_3^-, f_4^-, g_1^+, g_2^+, g_3^-, g_4^-$$

and one set for

$$f_1^-, f_2^-, f_3^+, f_4^+, g_1^-, g_2^-, g_3^+, g_4^+ .$$

Presumably the first set corresponds to our PMT solution and the second to our VMT solution, no mention being made in Ref. 16 of the parity of a given set alternating with j. At any rate, if we make the following identifications for the PMT (VMT) amplitudes, $F_3 = if_3^-(-f_2^-)$, $F_2 = if_2^+(f_3^+)$, $F_3 = -g_2^+(ig_3^+)$, and $F_4 = g_3^-(-ig_2^-)$, Eq. (4.10) can be shown to be equivalent to the radial equations of Ref. 16 for $V_1 = V_2 = S = S_1 = S_2 = 0$.

Radial equations can also be derived for the LC amplitudes by differentiating the algebraic relations like Eq. (4.8) and eliminating the UC amplitudes and their derivatives from Eq. (4.7). However, they are very complicated for the general interaction considered in this paper and are only needed for the case $H_I = V(r)$. For that case we find that

$$\begin{aligned}
&g'_{A1} + (1+\kappa_{A1}) g_{A1}/r = (E - V_{\text{eff}} + m_1) f_{A1} + U_+ g_{A1} - \epsilon U_0 (p f_{A1} - f_{A2}) , \\
&f'_{A1} + (1-\kappa_{A1}) f_{A1}/r = -(E - V_{\text{eff}} - m_1) g_{A1} + U_- f_{A1} + (1-\epsilon) U_0 (p g_{A1} - g_{A2}) , \\
&g'_{A2} + (1+\kappa_{A2}) g_{A2}/r = (E - V_{\text{eff}} + m_1) f_{A2} + U_+ g_{A2} + \epsilon U_0 (f_{A1} + p f_{A2}) , \\
&f'_{A2} + (1-\kappa_{A2}) f_{A2}/r = -(E - V_{\text{eff}} - m_1) g_{A2} + U_- f_{A2} - (1-\epsilon) U_0 (g_{A1} + p g_{A2}) ,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
U_0 &= pV'/(E - V + m_s)(E - V - m_d)\lambda r , \\
U_+ &= m_s V'/(E - V)(E - V + m_s) , \\
U_- &= -m_d V'/(E - V)(E - V - m_d) ,
\end{aligned} \tag{4.13}$$

and V_{eff} is defined by Eq. (2.4b).

V. THE INCLUSION OF UNBOUNDED POTENTIALS

The singular points of the differential equations satisfied by the F amplitudes are at $r = 0$, $r = \infty$, and the points at which the denominators U_i in Eq. (4.10) vanish. If the latter occur at positive, real values of r, the wave function is too singular to be normalized. Therefore, the presence of the pole terms in Eq. (4.10) poses a problem for potentials the maximum value of which equals or exceeds the total energy. Problems can also arise for potentials whose minimum value equals or is smaller than $-E$.

In this section we are interested in the implications of the pole terms and in the type of potential that can produce permanently confined $q\bar{q}$ bound states. The criterion for the inclusion of an unbounded potential in Eq. (4.10) is that it not lead to singular points at $r > 0$. If this criterion is not satisfied, the potential may still be included but only as a perturbation term. The criterion for permanent confinement is simply that all solutions of Eq. (4.10) vanish exponentially as $r \to \infty$.

It is convenient to rewite Eq. (4.10) in matrix form. If we let F be a four-component column matrix with elements F_i $(i = 1,4)$, we have

$$F' + (J/r)F = TF + (\text{pole terms})F + (\text{constant terms})F , \tag{5.1}$$

where

$$T = \frac{1}{2} \begin{bmatrix} 0 & 4B_\pm - C_\pm & 0 & 0 \\ C_\mp & 0 & 0 & 0 \\ 0 & 0 & 0 & C_\mp - 2B_\mp \\ 0 & 0 & 2B_\pm - C_\pm & 0 \end{bmatrix} \tag{5.2}$$

and $C_\pm = (V \pm S) + A_\pm$ and J is a constant 4×4 matrix. The pole terms are associated with the vanishing of the denominators U_i in Eq. (4.10). These denominators are of the form $U_i = E + W_i$ where W_i is a combination of potentials. Since E may be chosen to be arbitrarily large, U_i will necessarily vanish for some $r > 0$ if $W_i \to -\infty$ as $r \to \infty$. Therefore, each W_i must be non-negative for large r. This results in a set of inequalities that the interaction amplitudes must satisfy. They may be written most succinctly in the form

$$V - A_- \le S \le A_+ - V , \tag{5.3a}$$

$$V - A_- + 4B_- \le S \le A_+ - V - 4B_+ . \tag{5.3b}$$

Note that the inequality

$$V - A_- + 2B_- \le S \le A_+ - V - 2B_+$$

is automatically satisfied if Eq. (5.3) is satisfied.

Consider the case of the Coulomb plus Breit interaction in QED, $H_I = V_c + H_B$. Then $V = V_c$,

$S=S_1=S_2=0$, and $V_1=V_2=-\frac{1}{2}V_c$. Equation (5.3a) implies that $V_c\leq 0$ and (5.3b) that $V_c\geq 0$. Therefore, Eq. (5.3) is not satisfied for $V_c\neq 0$. Using the definitions of U_i, we note that in the case $U_1=E-V_c$, $U_2=E=U_3$, $U_4=E+V_c$. The troublesome factor is the latter since it vanishes at $r=\alpha/E$. We conclude that the Breit interaction is a good approximation of the non-Coulombic part of the electromagnetic interaction only for $|r-\alpha/E|\gg 0$ and that it is incorrect to include it directly in Eq. (4.1). Instead it must be treated in first-order perturbation theory. When this is done, the usual results for hydrogen and positronium are obtained to order α^4 as is shown in Sec. VII.

In the remainder of this section we consider confinement potentials for QCD which increase without bound as $r\rightarrow\infty$. The asymptotic behavior of F is controlled by the matrix T. For simplicity we assume that the magnitude of the confinement potentials increase linearly with r. In that case $F\rightarrow F_0\exp(\frac{1}{4}\beta r)$ for large r where $\beta=\beta_0 r$, β_0 is a constant, and F_0 is a constant matrix. Substituting this expression into Eq. (5.1) yields an eigenvalue equation for β, namely $TF_0=\frac{1}{2}\beta F_0$. The vanishing of the determinant of the coefficients yields two values of β^2 and hence four values of β. The former are easily found to be

$$\beta^2=(S\pm V\pm A_\pm\mp 4B_\pm)(S\mp V\mp A_\mp) \tag{5.4a}$$

and

$$\beta^2=(S+V+A_+-2B_+)(S-V-A_-+2B_-)\,. \tag{5.4b}$$

Bound-state solutions correspond to positive β^2 and negative β. Equation (5.4) shows that β^2 is the product of two factors. Hence there are two ways *a priori* to make $\beta^2>0$. Both factors must be positive or both must be negative. However, the latter possibility is excluded because it is inconsistent with Eq. (5.3). Consider Eq. (5.4a) for the VMT solution, for example. The condition $S+V<-A_+$ is inconsistent with $S+V\leq A_+$ from Eq. (5.3a) since both S and V are assumed to be positive for large r. We make this assumption because we wish to maintain as close a connection as possible with previous $q\bar{q}$ calculations. Similar arguments can be made for Eq. (5.4a) for the PMT solution and also for Eq. (5.4b). We conclude that both factors in the expressions for β^2 must be positive. In comparing these conditions with those in Eq. (5.3), we find that there are only two new conditions,

$$S>V+A_- \text{ and } S>V+A_--4B_-\,. \tag{5.5}$$

In the following analysis it is understood that the higher-order terms V_1, V_2 and S_1, S_2 are nonzero only if the respective zero-order interaction amplitudes V and S are nonzero.

Our first conclusion is that the interaction must contain a scalar term. If it does not, $S=0$ and the inequalities $V\leq A_-$ and $V<-A_-$ from Eqs. (5.3) and (5.4) cannot be satisfied by a positive V, with or without the higher-order terms V_1 and V_2. Therefore, a four-vector interaction cannot give rise to normalized quark confinement.

The simplest interaction containing a scalar term in $\beta_1\beta_2 S$. It is consistent with Eq. (5.4) and can produce confined, bound states but it does not satisfy Eq. (5.3), i.e., it does not give normalized solutions.

Consider the five two-term interaction Hamiltonians, (SV), (SS_1), (SS_2), (VV_1), and (VV_2) where we designate a particular interaction Hamiltonian H_I by a set of parentheses enclosing the nonzero interaction amplitudes H_I contains, e.g., $(VV_2)=V+\vec{\alpha}_1\cdot\vec{\alpha}_2 V_2$, etc. The interaction (SV) is inadequate since $V\leq S\leq -V$ is not satisfied. For the next interaction (SS_1), $A_\pm=\mp S_1$ and $B_\pm=0$. From Eqs. (5.3) and (5.4), $S\leq -S_1$, $S\geq -S_1$, and $S>S_1$, all of which are satisfied if S_1 is negative and $S=-S_1$. The corresponding interaction Hamiltonian is satisfactory and is given by Eq. (1.4). For (SS_2), $A_\pm=B_\pm=\pm S_2$, and Eq. (5.3) requires that $S=S_2=-3S_2$ which is impossible. The last two interactions are inadequate since they do not contain S.

There are six three-term interaction Hamiltonians, namely, (SS_1S_2), (SS_1V), (SS_2V), (SVV_1), (SVV_2), and (VV_1V_2). We consider each in turn. For (SS_1S_2), $A_\pm=\pm(S_2-S_1)$, $B_\pm=\pm S_2$, and Eq. (5.3) implies that $S=S_2-S_1=-3S_2-S_1$ which is impossible for $S_2\neq 0$. For (SS_1V), $A_\pm=\mp S_1$, $B_\pm=0$, and the condition $V-S_1\leq S\leq -S_1-V$ from (5.3a) is not satisfied. For (SS_2V), $A_\pm=\pm S_2 B_\pm$, and Eq. (5.3) implies that $S=S_2=-3S_2$ which is impossible. For (SVV_1), $A_\pm=-V_1$, $B_\pm=0$, and Eqs. (5.3) and (5.4) imply that $V-V_1<S\leq -V-V_1$ which cannot be satisfied for $V>0$. For (SVV_2), $A_\pm=V_2=B_\pm$, and Eq. (5.3) shows that $-V_2+V\leq S\leq V_2-V$ and $3V_2+V\leq S\leq -3V_2-V$. The former is valid only if $V_2>V$, and the latter, only if $-3V_2>V$. Hence Eq. (5.3) is not satisfied. As indicated earlier (VV_1V_2) cannot satisfy Eq. (5.3) because it does not contain S.

There are six four-term interaction Hamiltonians, (SS_1VV_1), (SS_2VV_1), (SS_1VV_2), (SS_2VV_2), (SS_1S_2V), and (SVV_1V_2). It is easily shown that

the first two obey Eqs. (5.3) and (5.4) whereas the others do not.

There are four five-term interactions, $(SS_1S_2VV_1)$, $(SS_1S_2VV_2)$, $(SS_1VV_1V_2)$, and $(SS_2VV_1V_2)$. Of these only the first is consistent with Eqs. (5.3) and (5.4) as is readily verified.

Finally the six-term Hamiltonian of Eq. (4.2) can also be shown to be consistent with Eqs. (5.3) and (5.4). However, even in this general case the four-vector terms cannot enter in the form $V+H_B$ as is the case in QED.

This makes a total of five acceptable confinement Hamiltonians, namely, (SS_1) with $S_1=-S$, (SS_1VV_1), (SS_2VV_1), $(SS_1S_2VV_1)$, and $(SS_1S_2VV_1V_2)$. Since V_1 and V_2 are the potentials for higher-order corrections to the zero-order potential V, it is not unreasonable to assume that V_1 and V_2 should not be larger in magnitude than V. This assumption reduces the number and arbitrariness of the acceptable interactions. It is easily verified on this basis that only the first two interactions are acceptable and that V_1 in the second one is no longer arbitrary, i.e., we have (SS_1) with $S_1=-S$ and (SS_1VV_1) with $S_1=-S$ and $V_1=-V$ where $S<V$. The corresponding interaction Hamiltonians are given by Eqs. (1.4) and (1.5).

VI. LIMITING CASES, AN EFFECTIVE POTENTIAL, AND THE COULOMB-LAMB SHIFT

In this section we consider only the single-term interaction Hamiltonian $H_I=V(r)$. For $j=0$ the radial equations for the LC amplitudes, Eq. (4.12) reduce to

$$g'_{A2}+[(1+\kappa_{A2})/r]g_{A2} =(E-V_{\text{eff}}+m_1)f_{A2}+U_+g_{A2}\,, \quad (6.1)$$

$$f'_{A2}+[(1-\kappa_{A2})/r]f_{A2} =-(E-V_{\text{eff}}-m_1)g_{A2}+U_-f_{A2}\,.$$

These equations are reminiscent of the radial equations obtained from the single-particle Dirac equation. In fact, they imply that the four-component ϕ spinor defined by Eq. (3.3) satisfies the effective single-particle Dirac equation

$$[C\vec{\alpha}\cdot(\vec{p}+i\vec{U}/c)+m_1c^2\beta+V_{\text{eff}}]\phi_{A2}=E\phi_{A2}\,, \quad (6.2a)$$

where

$$\vec{U}=\hbar c\begin{bmatrix}0 & U_-\\ U_+ & 0\end{bmatrix}\hat{r} \quad (6.2b)$$

and appropriate factors of h,c have been inserted to facilitate evaluating the effective potential in the classical limit. Note, in particular, that $\vec{U}\to 0$ as $h\to 0$ whereas V_{eff} is independent of h. Since the effective potential in Eq. (6.2) consists of the three-vector $\vec{U}$ and $V_{\text{eff}}-m_2$ [replacing E in Eq. (6.2) by E_1+m_2], the quantum effective potential is seen to reduce to the classical value, Eq. (2.4a), as $h\to 0$.

It is instructive to consider two other limiting cases, first, the single-particle limit and second, the nonrelativistic limit. In the former, we let $m_2\to\infty$ in which case $\vec{U}\to 0$ and $E-V_{\text{eff}}\to E_1-V$. See Eq. (6.3) below. Thus Eq. (6.2) reduces to the single-particle Dirac equation as expected. In the latter, we assume the masses m_1 and m_2 are much larger than $|\xi-V|$ and $|V'|$ where ξ is the binding energy. It is easily seen that $\vec{U}$ is of the order V'/m_2 or smaller and that

$$E-V_{\text{eff}}=m_1+(\mu/m_1)(\xi-V) -\tfrac{1}{2}(m_1-m_2)\sum_{k=2}^{\infty}(\xi-V)^k/(m_1+m_2)^k\,, \quad (6.3)$$

where μ is the reduced mass. If we keep only the first two terms in Eq. (6.3), Eq. (6.2) simplifies to

$$\left[\vec{\alpha}\cdot\vec{p}+m_1\beta+\frac{\mu}{m_1}V\right]\phi_{A2}=\left[m_1+\frac{\mu}{m_1}\xi\right]\phi_{A2}\,. \quad (6.4)$$

This equation can be reduced to Schrödinger form by eliminating the "small" components of ϕ_{A2} in the usual manner. Neglecting the smaller spin-dependent terms for simplicity, we obtain the reduced mass equation $(p^2/2\mu+V)\phi_L=E\phi_L$ where ϕ_L is the "large" component of ϕ_{A2}.

The simplicity of Eq. (6.2) suggests its application to some standard problem e.g., the hydrogen atom. A very simple calculation shows that there is a Coulomb-Lamb shift of order $m_1{}^2\alpha^4/m_2$ according to Eq. (6.2). Set $V=-\alpha/r$ and consider the $k=2$ term in Eq. (6.3). We retain only the $V^2/2m_2$ term since the others do not contribute to the Lamb shift. We use first-order perturbation theory with $H_0=\vec{\alpha}\cdot\vec{p}+m_1\beta-\alpha/r$ and $H_1=\alpha^2/2m_2r^2$. Consider, in particular, the $2P_{1/2}$ and $2S_{1/2}$ states. The zero-order wave functions are the $2S_{1/2}$ and $2P_{1/2}$ Dirac wave functions for a Coulomb potential but the corresponding

Schrödinger wave functions are adequate. The expectation value of H_1 in the state ϕ_{nl} is $(m_1^2\alpha^4/m_2n^3)(2l+1)^{-1}$. This produces a Lamb shift of $\alpha^4m_1^2/12m_2$ between the $2P_{1/2}$ and $2S_{1/2}$ states. This is a surprising result, at least at first sight, because there is not supposed to be a Lamb-shift contribution of this order. The agreement between theory and experiment on the value of the Lamb shift is excellent, and $\alpha^4m_1^2/12m_2$ is not one of the terms included in the comparison. In the following section we show that the resolution of this apparent discrepancy lies in the inclusion of the Breit interaction.

VII. APPLICATION TO HYDROGEN AND POSITRONIUM

There are two problems for which the radial equations obtained in Sec. IV can be solved analytically to order v^2/c^2 and compared with well-established results. One is hydrogen, and the other is positronium. In this section we calculate the energy levels of hydrogen to order $m_1^2\alpha^4/m^2$ and of positronium to order α^4, including both the Coulomb and Breit interactions. Both calculations are based on the radial equations for the LC amplitudes given by Eq. (4.12). Notice that these equations have the structure of two coupled single-particle Dirac equations. When $U_0=0$, the first two equations correspond to a Dirac equation with $q=q_1$ and leading orbital angular momentum l_{A1}, and the second two equations to a Dirac equation with $q=q_2$ and leading orbital angular momentum l_{A2}. The size of the coupling between q_1, l_{A1} and q_2, l_{A2} can be estimated from the average value of U_0. The coupling is small for nonrelativistic systems and vanishes in the single-particle limit.

Consider first the calculation for hydrogen. In that case $\langle U_0\rangle$, $\langle U_+\rangle$, and $\langle U_-\rangle$ are of order $m_1^3\alpha^4/m_2^2$ where m_1 (m_2) is the electron (proton) mass. They are, therefore, negligible. Dropping these terms, we are left with two effective single-particle Dirac equations of the form given by Eq. (6.2). They reduce to the form given by Eq. (6.4) when only the first two terms in Eq. (6.3) are kept. This defines the zero-order problem. The zero-order eigenstates are therefore characterized by a definite q and a leading l. The solutions of the Dirac equation for an attractive $1/r$ potential are well known. The zero-order energy ξ_0 is obtained from the equation

$$m_1+\frac{\mu}{m_1}\xi_0=m_1[1+(\mu\alpha/m_1)^2(n'+s)^2]^{1/2}\,, \tag{7.1}$$

where $n'=n-(q+\frac{1}{2})$ and $s=[(q+\frac{1}{2})^2-(\mu\alpha/m_1)^2]^{1/2}$. Since we are only interested in terms of order $m_1^2\alpha^4/m^2$ or smaller, it is convenient to expand ξ_0 in powers of α and m_1/m_2 to obtain

$$\xi_0=-\tfrac{1}{2}\mu(\alpha/n)^2 -\tfrac{1}{2}m_1(\mu/m_1)^3[n/(q+\tfrac{1}{2})-\tfrac{3}{4}](\alpha/n)^4\,. \tag{7.2}$$

There are only two perturbation terms to evaluate. The first is the term encountered in Sec. IV, i.e., the $k=2$ term in Eq. (6.3). The average value of this term is the Coulomb correction ξ_c where

$$\xi_c=(m_1^2/m_2)[n/(2l+1)-\tfrac{3}{8}](\alpha/n)^4\,. \tag{7.3}$$

The second is the Breit interaction. We have explained in Sec. V that the correct way to include H_B is as a first-order perturbation. Therefore, we need the average value of H_B with respect to the full 16-component wave function as defined by Eq. (3.8). Using the results in the Appendix we find that for the PMT solution

$$\lambda\langle H_B\rangle_{\rm PMT}=\int_0^\infty\{g_{A2}[3pf_{B1}-2(j+2)f_{B2}]-f_{A2}(pg_{B1}+2jg_{B2})+g_{A1}[3pf_{B2}-2(j-1)f_{B1}] -f_{A1}[pg_{B2}+2(j+1)g_{B1}]\}V_cr^2dr\,. \tag{7.4}$$

The expression for $\lambda\langle H_B\rangle_{\rm VMT}$ can be obtained from Eq. (7.4) by multiplying Eq. (7.4) by -1 and then interchanging corresponding g and f amplitudes. The zero-order LC amplitudes are solutions of the single-particle Dirac equation (6.4), and it turns out to be sufficient to use the more convenient nonrelativistic forms for g_{Ai}. This makes possible the analytic evaluation of all the integrals in $\langle H_B\rangle$. The result is that for both the PMT and VMT solutions $\langle H_B\rangle$ cancels out the l-dependent term in ξ_c which is the one responsible for the Lamb shift. In particular, we find that for both PMT and VMT

solutions,

$$\langle H_B\rangle = -\xi_c + 5m_1{}^2\alpha^4/8m_2n^4 - (m_1{}^2\alpha^4/m_2n^3)(q+\tfrac{1}{2})^{-1} + \xi_{\rm HF}\ , \tag{7.5}$$

where $\xi_{\rm HF}$ is the well-known term for the hyperfine structure,[17] i.e.,

$$\xi_{\rm HF} = \frac{m_1{}^2\alpha^4}{2m_2n^3}\left[\frac{j(j+1)-\frac{3}{4}-q(q+1)}{q(q+1)(l_A+\frac{1}{2})}\right] . \tag{7.6}$$

Since the two-body Dirac equation does not account for the anomalous magnetic moment, $\xi_{\rm HF}$ must be multiplied by the factor $1+\kappa_p=2.79$ for the proton's magnetic moment. Then the binding energy to order $m_1{}^2\alpha^4$ is given by the sum $\xi=\xi_0+\xi_c+\langle H_B\rangle$. This result can be written in various forms, perhaps the most convenient of which is in terms of the correction factor of Breit and Brown,[18] $(m_1{}^2-W^2)/2m_2$ where W is the total energy of an electron in the single-particle Dirac equation for $V=-\alpha/r$. It is easily shown from Eqs. (7.2), (7.3), and (7.5) that

$$\xi = -\tfrac{1}{2}m_1(\alpha/n)^2 - \tfrac{1}{2}m_1(\alpha/n)^4[n/(q+\tfrac{1}{2})-\tfrac{3}{4}] + \xi_{\rm HF} + (m_1{}^2-W^2)/2m_2\ . \tag{7.7}$$

The first term gives the Bohr energy levels, the second the fine structure, the third the hyperfine structure, and the fourth is the relativistic correction term. This may be compared, for example, with the result obtained by Lepage[19] from the Bethe-Salpeter equations. It is easily shown that his result agrees with Eq. (7.7) to order $m_1{}^2\alpha^4/m_2$.

The next application is to positronium. It is interesting to determine how the singlet and triplet states of positronium emerge from radial equations written in a representation which emphasizes the total angular momentum $\vec{Q}$ of particle 1 rather than the total spin of the two particles. The critical difference in the hydrogen and positronium problems is that $\langle U_0\rangle$ is of order α^4 for the PMT state of positronium. This causes a mixing between q_1, l_{A1} and q_2, l_{A2} in zero order and has a bearing on the choice of the zero-order Hamiltonian. The zero-order Hamiltonian cannot be Eq. (6.4) as it was for hydrogen because the eigenstates of Eq. (6.4) for different q_i are nondegenerate with an energy difference of order α^4. See Eq. (7.2). Second-order perturbation terms of the form $\langle n\,|\,H'\,|\,k\rangle^2/(E_n-E_k)$ would be of order α^4, rather than $(m_1\alpha/m_2)^4$ as for hydrogen, and this is the same order of magnitude as the first-order perturbation term. Therefore, Eq. (6.4) must be modified so the energy difference between different zero-order states is of order α^2. This is done by subtracting out the q-dependent terms in Eq. (7.2) and choosing the zero-order Hamiltonian

$$H_0 = \vec{\alpha}\cdot p + m\beta + \tfrac{1}{2}V + (m\alpha^4/32n^3)(q+\tfrac{1}{2})^{-1}\ , \tag{7.8}$$

where

$$H_0\phi_0 = \tfrac{1}{2}E_0\phi_0$$

and

$$E_0 = 2m - m\alpha^2/4n^2 + 3m\alpha^4/64n^4\ .$$

This has the effect of making states of different q_i degenerate to order α^4. The perturbation consists of the negative of the term added to Eq. (6.4), namely $H_0' = -(m\alpha^4/32n^3)(q+\frac{1}{2})^{-1}$, the Breit interaction, and the terms in Eq. (4.12) containing U_0, U_+, and U_-. We designate the latter effects collectively as H_U. For the ρ trajectory $\langle U_0\rangle \cong 0(\alpha^6)$ and $q_1 l_{A1}$ and $q_2 l_{A2}$ are not mixed.

First-order degenerate perturbation theory couples the q_1 and q_2 PMT states to give the zero-order eigenvectors

$$\begin{aligned}\lambda\psi_s^0 &= \sqrt{j}\,(^1_0)\otimes\phi^0_{A1} - \sqrt{j+1}\,(^0_1)\otimes\phi^0_{A2}\ ,\\ \lambda\psi_t^0 &= \sqrt{j+1}\,(^1_0)\otimes\phi^0_{A1} + \sqrt{j}\,(^0_1)\otimes\phi^0_{A2}\ ,\end{aligned} \tag{7.9}$$

where ψ_s is the singlet state and ψ_t is the $j=l$ triplet state. The contributions of the various perturbation terms are given in Table I. Combining these contributions with the zero-order energy, we find that

$$E_s = E_t + m\alpha^4/\lambda p^2 n^3 = \epsilon - m\alpha^4/2\lambda n^3\ , \tag{7.10}$$

$$E_{p,k} = \epsilon - \frac{m\alpha^4}{2n^3(2l+1)}\times\left[1+\tfrac{1}{2}(-1)^k\frac{(3l-5k+9)}{(l+2-k)(2l-4k+7)}\right]$$

TABLE I. The contributions of the perturbations H_B and $H_0'+H_U'$ in units of $m\alpha^4/8n^4$ for the singlet state (s), the triplet state (t), the ρ-trajectory solution with $q_1, l_{A1}(\rho_1)$, and the ρ-trajectory solution with $q_2, l_{A2}(\rho_2)$.

	H_B	$H_0'+H_U$
s	$1-3n/\lambda$	$-n/\lambda$
t	$(1-4\lambda^2/p^2)n/\lambda+1$	$-(1+4/p^2)n/\lambda$
ρ_1	$1-(3j+1)n/j\lambda$	$-n(j-1)/j(2j-1)$
ρ_2	$1-(3j+2)n/(j+1)\lambda$	$-n(j+2)/(j+1)(2j+3)$

for $k=1,2$ where $\epsilon=2m-m\alpha^2/4n^2+11m\alpha^4/64n^4$ and E_{pk} is the energy on the ρ trajectory with $q=q_k$, $l=l_{Ak}$. The Coulomb and Breit interactions correspond to instantaneous, transverse photon exchange. Therefore, Eq. (7.10) does not include the annihilation energy corresponding to the annihilation diagram. When this contribution is included, Eq. (7.10) is in complete agreement with the results given by Ferrell[20] and by DeBenedetti and Corben.[21]

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APPENDIX

The derivation of many of the results given in this paper are too lengthy and cumbersome to reproduce in their entirety. However, it is possible to give the operator and matrix expansions that are needed in the derivations. These identities have not been given elsewhere and are offered without proof. In the following, ψ is defined by Eq. (3.18) and $\vec{S}_2=\frac{1}{2}\vec{\Sigma}\otimes I$ where $\vec{\Sigma}$ is the usual 4×4 generalization of the Pauli matrices. For PMT and VMT solutions,

$$\vec{\alpha}_2\cdot\vec{p}\psi=-i\hat{r}\cdot\vec{\alpha}_2\left[\frac{\partial\psi}{\partial r}+\frac{2}{r}\psi\right]+\frac{i}{r}(\vec{L}\cdot\vec{S}_2)(\hat{r}\cdot\vec{\alpha}_2)\psi\ , \tag{A1}$$

$$\hat{r}\cdot\vec{\alpha}_2\psi=\frac{-1}{\lambda}\begin{bmatrix}i\Omega_{A1}(g_{B1}+pg_{B2},if_{B1}+pif_{B2})\\ \Omega_{B1}(g_{A1}+pg_{A2},if_{A1}+pif_{A2})\end{bmatrix}+\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A2}(g_{B2}-pg_{B1},if_{B2}-pif_{B1})\\ \Omega_{B2}(g_{A2}-pg_{A1},if_{A2}-pif_{A1})\end{bmatrix}, \tag{A2}$$

$$(\hat{r}\cdot\vec{\alpha}_1)(\hat{r}\cdot\vec{\alpha}_2)\psi=\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A1}(if_{B1}+pif_{B2},g_{B1}+pg_{B2})\\ \Omega_{B1}(if_{A1}+pif_{A2},g_{A1}+pg_{A2})\end{bmatrix}+\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A2}(-if_{B2}+pif_{B1},-g_{B2}+pg_{B1})\\ \Omega_{B2}(-if_{A2}+pif_{A1},-g_{A2}+pg_{A1})\end{bmatrix}. \tag{A3}$$

For VMT solutions,

$$\vec{\alpha}_1\cdot\vec{\alpha}_2\psi=\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A1}(\lambda if_{B1},2pg_{B2}+(2-\lambda)g_{B1})\\ \Omega_{B1}(2pif_{A2}+(2-\lambda)if_{A1},\lambda g_{A1})\end{bmatrix}+\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A2}(\lambda if_{B2},2pg_{B1}-(\lambda+2)g_{B2})\\ \Omega_{B2}(2pif_{A1}-(2+\lambda)if_{A2},\lambda g_{A2})\end{bmatrix}, \tag{A4}$$

$$\vec{L}\cdot\vec{S}_2\begin{bmatrix}i\Omega_{A1}(g_{Bn},if_{Bn})\\ \Omega_{B1}(g_{An},if_{An})\end{bmatrix}=\frac{(j+1)(2j-1)}{\lambda}\begin{bmatrix}i\Omega_{A1}(0,if_{Bn})\\ \Omega_{B1}(g_{An},0)\end{bmatrix}-\frac{p}{\lambda}\begin{bmatrix}i\Omega_{A2}(0,if_{Bn})\\ \Omega_{B2}(g_{An},0)\end{bmatrix}+(j-1)\begin{bmatrix}i\Omega_{A1}(g_{Bn},0)\\ \Omega_{B1}(0,if_{An})\end{bmatrix}, \tag{A5}$$

$$\vec{L}\cdot\vec{S}_2\begin{bmatrix}i\Omega_{A2}(g_{Bn},if_{Bn})\\ \Omega_{B2}(g_{An},if_{An})\end{bmatrix}=-\frac{j(2j+3)}{\lambda}\begin{bmatrix}i\Omega_{A2}(0,if_{Bn})\\ \Omega_{B2}(g_{An},0)\end{bmatrix}-\frac{p}{\lambda}\begin{bmatrix}i\Omega_{A1}(0,if_{Bn})\\ \Omega_{B1}(g_{An},0)\end{bmatrix}-(j+2)\begin{bmatrix}i\Omega_{A2}(g_{Bn},0)\\ \Omega_{B2}(0,if_{An})\end{bmatrix}. \tag{A6}$$

For the PMT solutions,

$$\vec{\alpha}_1\cdot\vec{\alpha}_2\psi=\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A1}(2pif_{B2}+(2-\lambda)if_{B1},\lambda g_{B1})\\ \Omega_{B1}(\lambda if_{A1},2pg_{A2}+(2-\lambda)g_{A1}\end{bmatrix}+\frac{1}{\lambda}\begin{bmatrix}i\Omega_{A2}(2pif_{B1}-(\lambda+2)if_{B2},\lambda g_{B2})\\ \Omega_{B2}(\lambda if_{A2},2pg_{A1}-(2+\lambda)g_{A2})\end{bmatrix}. \tag{A7}$$

The identities for the PMT solution analogous to Eqs. (A5) and (A6) for the VMT solution can be obtained from Eqs. (A5) and (A6) by means of the following prescription: (a) interchange the first and second arguments of all the Ω spinors on the right side of Eqs. (A5) and (A6) and (b) make the following replacements $g_{An}\leftrightarrow if_{An}$ and $g_{Bn}\leftrightarrow if_{Bn}$.

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