# Many-quark problem in two-dimensional gauge theory

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General theorems are given for enumerating and constructing the singlet eigenstates of interaction energy for an arbitrary number of static quarks in SU($N$) gauge theory in $1+1$ dimensions.

## I. MAIN RESULTS

In this paper we discuss diagonalization of the interaction energy for an arbitrary number of fixed quark sources in two-dimensional SU($N$) gauge theory. Here quark is meant to be a carrier of SU($N$) quantum numbers (in any representation in general) and refers to both quark and antiquark unless further specified. We will explicitly restrict ourselves to overall SU($N$) singlets because other states would have infinite energy. The relevant Hamiltonian is

$$H=-\sum_{\substack{k,l=1\\k>l}}^{n}\frac{\alpha}{4}\lambda_i^{(k)}\lambda_i^{(l)}\,|x_k-x_l|\,,\quad \alpha=g^2/2 \tag{1}$$

in the Coulomb gauge, and this expression is exact. [The $\lambda_i$'s are the SU($N$) matrices appropriate to the quark or antiquark representation.] Although the problem looks almost trivial, it nevertheless leads to some interesting consequences. For one thing, we shall find a simple interpretation of the elegant algorithm given by Bralić and others for evaluating Wilson loops. In the following we will first state the main results in a sequence of theorems.

Without losing generality, one may choose the $n$ quarks to be well ordered in position, so that $x_1\le x_2\le\cdots\le x_n$. Other configurations can be obtained by relabeling the quarks, but our considerations will be restricted to one such sector $\Gamma$ at a time.

*Theorem 1.* In $\Gamma$, one may remove the absolute sign in the potential function in Eq. (1). $H$ becomes a sum of separable linear potentials which one can write as

$$\begin{aligned}-H=&\frac{\alpha}{4}\lambda^{(1)}\cdot(\lambda^{(2)}+\lambda^{(3)}+\cdots+\lambda^{(n)})r_{21}\\&+\frac{\alpha}{4}(\lambda^{(1)}+\lambda^{(2)})\cdot(\lambda^{(3)}+\cdots+\lambda^{(n)})r_{32}\\&+\cdots\\&+\frac{\alpha}{4}(\lambda^{(1)}+\cdots+\lambda^{(n-1)})\cdot\lambda^{(n)}r_{n,n-1}\,,\end{aligned} \tag{2}$$

$$r_{k+1,k}=x_{k+1}-x_k\ge 0\,.$$

This follows from writing

$$x_k-x_l=r_{k,k-1}+r_{k-1,k-2}+\cdots+r_{l+1,l}\qquad(k>l)$$

and regrouping the terms in Eq. (1). Each line in Eq. (2) represents a partition of $n$ quarks into left and right subsets $L$ and $R$, which then couple through their respective SU($N$) generators,

$$\lambda^{L,k}=\lambda^{(1)}+\lambda^{(2)}+\cdots+\lambda^{(k)}\,,$$
$$\lambda^{R,n-k}=\lambda^{(k+1)}+\lambda^{(k+2)}+\cdots+\lambda^{(n)}\,.$$

So Eq. (2) may be further rewritten as

$$\begin{aligned}H&=\sum_{k=1}^{n-1}H_k\,,\\H_k&=-\frac{\alpha}{4}\lambda^{L,k}\cdot\lambda^{R,n-k}r_{k+1,k}\\&=-\frac{\alpha}{2}r_{k+1,k}[\,C(L+R,n)-C(L,k)\\&\qquad-C(R,n-k)]\,,\end{aligned} \tag{3}$$

where $C(L,k)$ is the quadratic Casimir operator for $L$:

$$C(L,k)=\lambda^{L,k}\cdot\lambda^{L,k}/4\,,$$

and so on. But because of the restriction to overall singlets, the eigenvalues of the Casimir operators are restricted to

$$C(L+R)=0,\quad C(L)=C(R)\,,$$

 

and $(L,k)$ and $(R,n-k)$ are conjugate representations of each other. This is not always possible for an arbitrary composition of the system, but when it is, knowledge of a left subsystem $(L,k)$ uniquely determines the right subsystem $(R,n-k)$, and vice versa, as far as the representation is concerned. But the Casimir operators of all left subsystems $(L,k)$ commute. One thus has theorem 2.

*Theorem 2.* Let $(L,1)\subset(L,2)\subset\cdots\subset(L,n)$ be a hierarchy of subsystems of quarks as defined above. One can construct an irreducible representation (IR) for the whole system by first specifying an IR$(L,1)$ for $(L,1)$, then adding a quark to form an IR$(L,2)$ for $(L,2)$, and so on, ending finally in a singlet IR$(L,n)$. The resulting state is an eigenstate of $H$ with eigenvalue

$$E=\alpha\sum_{k=1}^{n-1}C(L,k)r_{k+1,k}\ . \tag{4}$$

Enumerating all such hierarchies of IR's (and taking due account of their multiplicities), one exhausts all the singlets of the system. The last point is obvious if the above steps are followed backward starting from $(L,n)$. Physical interpretation of the procedure is also straightforward: The energy of the field flux residing in the interval $r_{k+1,k}$ is determined by the SU$(N)$ quantum numbers of the subsystems to the left and right of it. Note that one does not need any additional magnetic quantum numbers since one is dealing with singlets. Also note that the same IR$(L,k)$ for a given $k$ may come in a multiplet, in which case one must count them as distinct choices.

The above procedure is clearly quite general, but we will for the moment specialize to SU(2) and SU(3), with the quarks in the fundamental representation. An IR of SU(2) is labeled by isospin $I$, or $p\equiv 2I=0,1,\ldots$, whereas an IR of SU(3) is labeled by two integer parameters $p,q\geq 0$. [Essentially, an IR$(p,q)$ behaves like a tensor product of $p$ quarks and $q$ antiquarks, each in a totally symmetric configuration, and zero under contraction of a pair of quark and antiquark indices.] The respective Casimir eigenvalues are

$$\text{SU(2):}\quad C(p)=p^2/4+p/2\ , \tag{5}$$

$$\text{SU(3):}\quad C(p,q)=\tfrac{1}{3}(p^2+pq+q^2)+p+q\ . \tag{6}$$

The IR's in theorem 2 form a correlated sequence because adding a quark to an IR for $(L,k)$ leads to a branching of IR's for $(L,k+1)$. The branching rules are (subject to positivity restrictions)

$$\begin{aligned}&\text{SU(2):}\quad p\to p+1,p-1\ ,\\ &\text{SU(3):}\quad (p,q)\to(p+1,q),(p,q-1),(p-1,q+1)\ \text{for adding a quark } (1,0)\ ,\\ &\qquad\qquad (p,q)\to(p-1,q),(p,q+1),(p+1,q-1)\ \text{for adding an antiquark } (0,1)\ .\end{aligned} \tag{7}$$

Each of the final IR's has a multiplicity of one. An interesting question that arises at this point is the number of singlets, or the dimensionality of physical states, for an $n$-quark system. Obviously it can be derived from the above branching rules. We give the answer below, leaving the proof to the Appendix.

*Theorem 3.* The number of linearly independent singlet states $A_n$ formed by $n$ quarks in SU(2) (Ref. 1) is

$$\begin{aligned}A_n&=\frac{n!}{(n/2)!(n/2+1)!}\quad (n=\text{even})\\ &\sim\frac{2^n}{\sqrt{\pi}(n/2)^{3/2}}\ .\end{aligned} \tag{8}$$

In SU(3), the number $A_n$ for a system consisting only of quarks $(p,q)=(1,0)$ or of antiquarks $(0,1)$ is

$$\begin{aligned}A_n&=\frac{2(n!)}{(n/3)!(n/3+1)!(n/3+2)!}\\ &\sim\frac{1}{\pi}\frac{3^{n+1/2}}{(n/3)^4}\quad (n=0\ \text{mod}3)\ .\end{aligned} \tag{9}$$

The asymptotic forms given above are subject to a simple interpretation. The building up of tensor products of quark states may be regarded as a random-walk process in the vector space of group generators. This space is $N^2-1$ dimensional for SU$(N)$. The number $A_n$ is equal to the number of ways one can come back to the starting point after $n$ steps. Each step has $N$ possible choices of direction, and after $n$ steps the end point will on the

average lie within a radius of $\sqrt{n}$. The number of possibilities leading to the origin is thus

$$A_n \sim N^n/(\sqrt{n})^{N^2-1} .$$

One also expects that in this limit there will be no difference between quarklike steps and antiquarklike steps. Their $N$ allowed directions are equal and opposite, but in the continuum approximation either set may be considered isotropic and equivalent for the present purposes. Thus theorem 4.

*Theorem 4.* For large $n$, the entropy $S=(\ln A_n)/n$ per quark in SU($N$) is

$$S \sim \ln N - \frac{(N^2-1)}{2n}\ln n + 0\left[\frac{1}{n}\right], \tag{10}$$

provided that $n \gg N^2$. In the strict thermodynamic limit, the second term is zero, implying that the restriction to singlets may be ignored.

## II. WILSON LOOP AVERAGES

A Wilson loop describes the gauge field action induced by a quark source following a closed trajectory. A general algorithm was derived by Bralić[2] and by Kazakov and Kostov[3] for evaluating the action for an arbitrary number of Wilson loops. The present Hamiltonian approach gives a simple interpretation of their results.

If a configuration of Wilson loops is viewed in its time sequence, various pairs of quark and antiquark are created, recombined, and eventually annihilated. At creation and annihilation, each pair is in the singlet state. At any time slice, the quarks are spatially ordered in a definite way. So one can, for example, start from a singlet state composed of pairs separated from each other. Its energy is the sum of the energies of the individual pairs. At the instant when a crossing of quarks from different pairs occurs, the system goes over to a different space ordering, which defines a new set of singlet eigenstates. The initial state should then be decomposed into new eigenstates and each of them will develop with its own energy eigenvalue. Repeating this process until all quarks are annihilated in pairs, one gets a sum of terms carrying different action factors. It is not difficult to see that each such action can be expressed in terms of weighted areas enclosed by the various quark lines.

## III. COMMENTS

Many of the results obtained here are due to the peculiarities of linear potentials in one dimension, which are additive and separable within a sector characterized by a specific ordering of quarks. In a given sector one may choose a particular singlet state. Since the corresponding eigenpotential is additive, one can plug it into a Schrödinger or Dirac Hamiltonian and solve it for individual quarks separately. Unfortunately, difficulties arise in joining the solutions in different sectors (except for the trivial case of two quarks). So one does not gain much advantage by making use of the present method in tackling the many-quark problem. For Dirac quarks, moreover, the problem of Klein's paradox would persist; it could be dealt with only by treating the quarks as quantized fields.

On the other hand, the insight gained here may prove useful in a statistical treatment of large numbers of quarks. This will be pursued in later communications.

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## APPENDIX: GENERATING FUNCTIONS FOR DECOMPOSITION OF TENSOR PRODUCTS[4]

The counting of the number of singlet states contained in an $n$-quark system can be done for SU(2) or SU(3) by elementary methods, but here we follow the more general method of generating functions. Let a quark correspond to the fundamental representation of U($N$). The Young tableau with $N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_N$ specifies an IR of U($N$), of which the sequence $n_1-n_2$, $n_2-n_3, \ldots, n_{N-1}-n_N$ stands for an IR of SU($N$). A quark (not an antiquark) represents a block in the tableau. The number of times an IR with $(n_1, \ldots, n_N)$ occurs in a direct product of $n=n_1+n_2+\cdots+n_N$ quarks is equal to the number $A(n_1,n_2, \ldots, n_N)$ of ways the corresponding tableau can be built by adding a quark one by one. A quark may be added to any row of the tableau if the restrictions $n_1 \geq n_2 \geq \cdots$ are not violated. So define a generating function with $N$ variables

$$f(x_i)=\sum_{\{n_i\}} A(n_i){x_1}^{n_1}\cdots{x_N}^{n_N}\ . \tag{A1}$$

An addition of a quark contributes a factor $x_1+x_2+\cdots+x_N$ except for the above restrictions. These restrictions are effectively met if one can force $f$ to be zero whenever $n_{i+1}=n_i+1$ for some $i$, which in turn can be realized if $f\times{x_1}^{N-1}{x_2}^{N-1}\ldots x_{N-1}$ is extended to a totally antisymmetric function. Now the function

$$\tilde{f}(x_i)=\frac{1}{1-\sum x_i}\prod_{i>j}\left[1-\frac{x_i}{x_j}\right]$$

$$=\sum\tilde{A}(n_i){x_1}^{n_1}\cdots{x_N}^{n_N} \tag{A2}$$

satisfies the required antisymmetry. When the first factor is expanded in powers of $x_i$, the $n$th power clearly generates $n$-quark states, whereas the second factor is homogeneous of degree zero, and does not disturb this power. It follows that $\tilde{f}$ coincides with $f$ in those terms satisfying the power restrictions.

The number $A_n$ of SU($N$) singlets made of $n$ quarks is the coefficient of the term with $n_1=n_2=\cdots=n_N=n/N$. Direct evaluation gives

$$A_n=\frac{1!2!\cdots(N-1)!n!}{(n/N)!(n/N+1)!..(n/N+N-1)!}$$

$$(n=0\ \mathrm{mod}N). \tag{A3}$$

In order to incorporate antiquarks, one notes that addition of an antiquark amounts to removal of a block from the tableau, or multiplication by a factor $\sum(1/x_i)$. To be more precise, the correspondence is $x_i\to 1/x_{N-i+1}$, under which the antisymmetrizing factor is invariant. So Eq. (A2) is generalized to

$$F(x_i,t)=\frac{1}{1-t\sum x_i}\,\frac{1}{1-t\sum(1/x_i)}$$

$$\times\prod_{i>j}\left[1-\frac{x_i}{x_j}\right]$$

$$=\sum\tilde{A}(n_i,m)\prod{x_i}^{n_i}t^m\ . \tag{A4}$$

TABLE I. Selected list of the numbers $A(n',n'')=A(n'',n')$ of SU(3) singlets formed by $n'$ quarks and $n''$ antiquarks.

| $n=n'+n''$ | $n'$ | $n''$ | $A(n',n'')$ |
|---|---|---|---|
| 2 | 1 | 1 | 1 |
| 3 | 3 | 0 | 1 |
| 4 | 2 | 2 | 2 |
| 5 | 4 | 1 | 3 |
| 6 | 6 | 0 | 5 |
| | 3 | 3 | 6 |
| 7 | 5 | 2 | 11 |
| 8 | 7 | 1 | 21 |
| | 4 | 4 | 23 |
| 9 | 9 | 0 | 42 |
| | 6 | 3 | 47 |
| 10 | 8 | 2 | 98 |
| | 5 | 5 | 103 |
| 11 | 10 | 1 | 210 |
| | 7 | 4 | 225 |
| 12 | 12 | 0 | 462 |
| | 9 | 3 | 498 |
| | 6 | 6 | 513 |
| 18 | 18 | 0 | 87 516 |
| | 15 | 3 | 91 091 |
| | 12 | 6 | 93 500 |
| | 9 | 9 | 94 359 |
| 24 | 24 | 0 | 23 371 634 |
| | 21 | 3 | 23 959 494 |
| | 18 | 6 | 24 410 334 |
| | 15 | 9 | 24 695 139 |
| | 12 | 12 | 24 792 705 |

Here $m$ counts the total number of quarks and antiquarks, and $\sum n_i$ counts the number of quarks minus the number of antiquarks, with $n_i$ now running over both positive and negative values. [The Young tableau must obviously be generalized to allow negative numbers of blocks, but still keeping the restrictions $n_1\geq n_2\geq\cdots$.] Numerical values of $\bar{A}$ for SU(3) are computed in Table I. One sees that for a fixed total number $n=n'+n''$ of quarks and antiquarks, $A(n',n'')$ is larger for more even mixture as may be expected, but the variation becomes smaller as the number increases.

[1]See, for example, E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1959), p. 131.
[2]N. Bralić, Phys. Rev. D 22, 3090 (1980).
[3]V. A. Kazakov and I. K. Kostov, Nucl. Phys. B176, 199 (1980); V. A. Kazakov, *ibid.* B179, 283 (1980).
[4]This is a variant of the standard methods, derived in the present context. See M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Massachusetts, 1962).