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# Variational analysis of lattice gauge theories

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A variational approach to the study of four-dimensional lattice gauge theories is initiated. We illustrate the technique by first analyzing the simpler SU(2) chiral model. A systematic treatment of the SU(2) lattice gauge theory with a mixed fundamental-adjoint action is presented next. This leads to an improvement over the standard mean-field (MF) results; in particular, a "second-order deconfining transition line" in the negative SO(3) coupling region previously identified by a MF technique now disappears. All transitions found are first order. The procedure is then generalized to the large-$N$ limit. In particular, a first-order transition line corresponding to the breaking of a (local) U(1) symmetry (at vanishing fundamental coupling) is again identified.

## I. INTRODUCTION

Much has been learned about various lattice gauge theories by the use of the Monte Carlo (MC) method. However, because of its numerical nature, it is often difficult to extract physical insights from those quantitative results. It is thus desirable to have on hand analytic methods, even if they are only approximate, for exploring various aspects of lattice gauge theories. One promising approach has been the large-$N$ expansion where a semiclassical treatment can be applied.

For directly analyzing theories at finite $N$, the mean-field (MF) approach can also be used provided that a proper interpretation is supplied to ensure local gauge invariance.[1] Although the general reliability of a MF analysis is problematical, it has recently been used to yield impressive results for various lattice gauge theories[2–4] when compared with the MC data. The method has also been applied to study the large-$N$ lattice QCD described by a mixed "fundamental-adjoint" (FA) plaquette action.[5] Furthermore, the corresponding mixed SU(2)-SO(3) model was analyzed using the MF method by Drouffe[3,4]; surprisingly, a "second-order deconfining transition line" was found in the region of negative SO(3) couplings.[4]

In this paper, we begin exploring the use of variational methods for studying lattice gauge theories. The use of a variational approach in a path-integral formalism has recently been advocated by Sakita,[6] who has verified that the method can provide an accurate description for the matrix models for the whole range of coupling strength.[7] We now generalize the method to lattice gauge theories. For clarity, we concentrate on providing a systematic treatment for the SU(2) case. In particular, we demonstrate that the second-order transition line found by Drouffe[4] is actually absent. The analysis is then generalized to the large-$N$ limit.

The essence of the variational method is the inequality for statistical averages, $\langle e^{Q}\rangle \geq e^{\langle Q\rangle}$. For definiteness, consider the SU($N$) lattice theory in $d$ dimensions described by the standard Wilson action,

$$S=\left[\frac{\beta N}{2}\right]\sum_{P}[\mathrm{tr}U(\partial P)+\mathrm{H.c.}]\,, \tag{1.1}$$

where the trace is for an SU($N$) matrix in the fundamental representation and the sum is over all plaquettes with $U(\partial P)$ the product of $U$'s along the boundary of $P$. Since the partition function

$$e^{-W}\equiv Z\equiv\int\prod_{l}dU_l e^{S} \tag{1.2}$$

cannot be evaluated analytically, one now replaces $S$ by a trial action, $S_0(U_l;J_\alpha)$, where $\{J_\alpha\}$ are variational (matrix) parameters. Denote a statistical average with respect to the trial action by

$$\langle Q\rangle_0=Z_0{}^{-1}\int\prod_{l}dU_l Q e^{S_0}\,, \tag{1.3}$$

where $Z_0$ is obtained by replacing $S$ by $S_0$ in (1.2). It follows that an upper bound for the free energy is

$$W\leq\overline{W}+\langle S_0-S\rangle_0\equiv W_{\mathrm{eff}}(\beta;J_\alpha)\,, \tag{1.4}$$

where $\overline{W}=-\ln Z_0$. Given a class of trial actions, the best estimate for the free energy is then ob-

tained by minimizing $W_{\text{eff}}$ with respect to $J_\alpha$. By analyzing the $\beta$ dependence of the variational free energy $W_{\text{eff}}(\beta)$, information on the phase structure can be obtained.

The choice of the trial action $S_0$ is partly dictated by the ease with which the averages (1.3) can be performed. On the other hand, it is reasonable to expect that the variational estimate improves when $S_0$ better approximates the true action, (1.1). In this paper, we consider the *simplest* class of trial actions

$$S_0=\sum_l [\mathrm{tr}(U_l J_l^\dagger + U_l^\dagger J_l)] , \tag{1.5}$$

where the sum is over all links. With (1.5) both $Z_0$ and $\langle S\rangle_0$, $\langle S_0\rangle_0$ can be evaluated in terms of an external source integral,[8]

$$f\equiv e^{-E_0(J,J^\dagger)}=\int dU e^{\mathrm{tr}(UJ^\dagger+U^\dagger J)} . \tag{1.6}$$

Clearly, much improved *Ansätze* than (1.5) can easily be introduced.

In (1.5), $J_l$ and $J_l^\dagger$ are arbitrary matrices which can depend on the link position. For SU(2) described by the Wilson action, a further simplifying assumption on $J_l, J_l^\dagger$ then directly leads to the standard MF result. However, for the case of a mixed fundamental-adjacent [SU(2)-SO(3)] action, an improvement over the standard MF analysis[3,4] is obtained.

To illustrate our procedure, we first treat in Sec. II the simpler case of an SU(2) chiral model. A systematic analysis of the SU(2) lattice gauge theory with a mixed FA action is carried out in Sec. III. An extension of the variational procedure to the large-$N$ limit is presented in Sec. IV. We comment on the relation of our result to that of the standard MF approach in Sec. V.

## II. CHIRAL MODEL IN $d$ DIMENSIONS

We apply first the variational method to the SU(2)×SU(2) chiral model, which is of interest in its own right. The analysis also serves as an illustration of our procedure in a simpler setting. With an SU(2) matrix variable $U_\alpha$ on each lattice site, the action is

$$S=\frac{\beta}{2}\sum_{\langle\alpha,\gamma\rangle}(\mathrm{tr}U_\alpha U_\gamma^\dagger+\mathrm{H.c.}) , \tag{2.1}$$

where the sum is over all links $\langle\alpha,\gamma\rangle$ and a periodic boundary condition can also be imposed.

Consider a trial action

$$S_0=\sum_\alpha \mathrm{tr}(U_\alpha J_\alpha^\dagger + U_\alpha^\dagger J_\alpha) . \tag{2.2}$$

We distinguish two types of variational *Ansätze*.

*Type A: Global Ansatz where $J_\alpha=J$ for all sites.* The (trial) partition function $Z_0$ now factorizes so that it is given by a product of identical external source integrals, (1.6), one for each site. Similarly, the averages $\langle S\rangle_0$ and $\langle S_0\rangle_0$ can also be evaluated since they are now given as sums of integrals involving at most neighboring sites, e.g., $\langle S\rangle_0$ is a sum of integrals of the type

$$\int [dU_\alpha dU_\gamma]\mathrm{tr}(U_\alpha U_\gamma^\dagger+\mathrm{H.c.})e^{\mathrm{tr}[(U_\alpha+U_\gamma)J^\dagger+\mathrm{H.c.}]} . \tag{2.3}$$

These single- and double-link integrals can be obtained from (1.6) by differentiations so that

$$\langle S\rangle_0=Vd\beta\frac{\partial E_0}{\partial J_{ij}}\frac{\partial E_0}{\partial J_{ji}^\dagger} , \tag{2.4}$$

$$\langle S_0\rangle_0=-V\left[\frac{\partial E_0}{\partial J_{ij}}J_{ij}+\frac{\partial E_0}{\partial J_{ij}^\dagger}J_{ij}^\dagger\right] , \tag{2.5}$$

where $V$ is the total volume (lattice spacing $a=1$), and $(Vd)$ is the total number of links. Therefore, once $E_0(J,J^\dagger)$ is known, an upper bound for the free energy per site is obtained:

$$E(\beta;J,J^\dagger)=\frac{W_{\text{eff}}}{V}=E_0-(\langle S\rangle_0-\langle S_0\rangle_0)/V . \tag{2.6}$$

*Type B: Local Ansatz.* We again consider (2.2) but choose $J_\alpha=V_\alpha J$, where $V_\alpha$ is an arbitrary SU(2) matrix for each site $\alpha$. We now treat both $\{J,J^\dagger\}$ and $\{V_\alpha,V_\alpha^\dagger\}$ as variational parameters.

Because of the invariance of the group integration measure, we can rotate $J_\alpha$ to $J$ so that the partition function $Z_0$ is again given by a product of (1.6), the same as that for the type-A global *Ansatz*. Similarly, since $S_0$ is a sum of single-site terms, $\langle S_0\rangle_0$ remains given by (2.5). However, the average $\langle S\rangle_0$ involves neighboring sites; the effect of rotating $J_\alpha$ to $J$ modifies (2.3) by replacing the factor $\mathrm{tr}(U_\alpha U_\gamma^\dagger+\mathrm{H.c.})$ by $\mathrm{tr}(U_\alpha U_\gamma^\dagger V_{\alpha\gamma}^\dagger+\mathrm{H.c.})$ where $V_{\alpha\gamma}\equiv V_\alpha^\dagger V_\gamma$. Instead of $V_\alpha$ for each site, we can introduce an SU(2) variational matrix $V_{\alpha\gamma}$ for each link, $\langle\alpha,\gamma\rangle$, and the average $\langle S\rangle_0$ can again be written as

$$\langle S\rangle_0=\frac{\beta}{2}\sum_l\left[\frac{\partial E_0}{\partial J_{ij}}\frac{\partial E_0}{\partial J_{jk}^\dagger}V_{ik}(l)+\mathrm{H.c.}\right] , \tag{2.7}$$

where the sum is over all links. Since setting $V(l)=I$ reduces (2.7) to (2.4), the type-A *Ansatz* is a special limit of our more general type-B *Ansatz*, which we consider in what follows.

The single-link integral (1.6) has been evaluated previously[8] for general SU($N$) and U($N$); in the case of SU(2), it gives

$$E_0(J^\dagger,J)=\ln[z/I_1(2z)] , \tag{2.8}$$

where $I_1$ is the modified Bessel function. Note that the dependence on $J$ and $J^\dagger$ is only through a single parameter $z$, where $z^2=\mathrm{tr}(JJ^\dagger)+\det J+\det J^\dagger$. Defining an auxiliary function

$$W_0(z)\equiv -\frac{1}{2}\frac{d}{dz}E_0(z) , \tag{2.9}$$

we obtain from (2.5) and (2.7)

$$\langle S\rangle_0=\beta W_0(z)^2\sum_l \mathrm{tr}V_l , \tag{2.10}$$

$$\langle S_0\rangle_0=2VzW_0(z) . \tag{2.11}$$

Since $V_l$ is an element of SU(2), $\mathrm{tr}V_l=2\cos(\theta_l/2)$, i.e., in addition to $z$ we have an angular variational parameter $\theta_l$ for each link.

To further simplify the problem, we restrict ourselves to the case where $\theta_l=\theta$ for all links so that an upper bound for the free energy per site is

$$E(\beta;z,\theta)=E_0-2\beta dW_0(z)^2\cos\frac{\theta}{2}+2zW_0(z) . \tag{2.12}$$

The stationary conditions $\delta E/\delta\theta=0$ and $\delta E/\delta z=0$ yield

$$\beta dW_0(z)^2\sin\frac{\theta}{2}=0 , \tag{2.13}$$

$$-2\beta dW_0(z)\cos\frac{\theta}{2}+z=0 . \tag{2.14}$$

From (2.13), we see, in general, $\theta=0$, reducing the problem to the type-A case. However, this turns out not to be the case if the action is mixed fundamental-adjoint; thus our effort here will not be in vain.

With $\theta=0$, (2.14) leads to a variational free energy,

$$E=E_0(z)+\frac{z^2}{2\beta d}=\ln[z/I_1(2z)]+\frac{z^2}{2\beta d} . \tag{2.15}$$

The solution to (2.14) has two branches: (1) $z=0$, corresponding to $E=0$ for all $\beta$; (2) $z\neq 0$, which is a solution to the equation $\beta d=F_2(z)$, where

$$F_2(z)=\frac{z^2}{2}\left[\frac{zI_0(2z)}{I_1(2z)}-1\right]^{-1} . \tag{2.16}$$

We plot $F_2(z)$ in Fig. 1(a); it is clear that this second solution exists only if $\beta d\geq 1$. When it does, the free energy $E$ is negative [Fig. 1(b)] so that $\beta_c=1/d$ corresponds to a phase-transition point where the system changes from $E=0$ to $E\leq 0$. In fact, for the chiral model described by the action (2.1), our variational result coincides exactly with the MF result of Ref. 9.

## III. SU(2) LATTICE GAUGE THEORY WITH FIXED FUNDAMENTAL-ADJOINT ACTION

We adopt the normalization where the action for the SU(2)-SO(3) lattice gauge theory is

$$S=\frac{\beta}{2}\sum_P \mathrm{tr}[(U(\partial P)+\mathrm{H.c.})] +\frac{\beta_a}{3}\sum_P \mathrm{tr}_A U(\partial P) , \tag{3.1}$$

where $U(\partial P)=U_1U_2U_3^\dagger U_4^\dagger$ is the product of SU(2) matrices around a plaquette, $\mathrm{tr}_A$ is the SU(2) trace in the adjoint representation, and the sum is over all plaquettes. Much has been learned about this

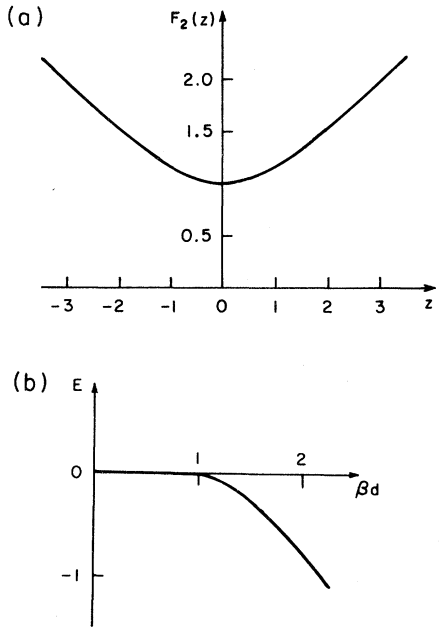


FIG. 1. (a) $F_2(z)$ of Eq. (2.16). (b) Variational free energy for the chiral model.

system either by analytic and MC analyses[10,11] or by various MF methods,[3,4] and a rich phase structure has been identified.

We present here a systematic variational treatment of this system. In order to compare and contrast our approach with the standard MF method, we shall carry out the analysis in three stages with increasing richness in structure.

### A. Minimum scheme

Let us consider the trial action

$$S_0=\sum_l \mathrm{tr}(U_l J^\dagger + U_l^\dagger J) , \tag{3.2}$$

where $J$ is an arbitrary $2\times 2$ matrix. The upper bound $W_{\mathrm{eff}}$, (1.4), can be written as

$$W_{\mathrm{eff}}=\overline{W}-\langle S_f\rangle_0-\langle S_a\rangle_0+\langle S_0\rangle_0 , \tag{3.3}$$

where $\overline{W}=VE_0$, $V$ is the total volume, and $E_0$ is given by (1.6). In (3.3), we have also separated the action (3.3) into the "fundamental" and "adjoint" pieces. The evaluation of the averages in (3.3) can be carried out just as in the case of the chiral model. In analogy with (2.4), we find

$$\langle S_f\rangle_0=\tfrac{1}{4}\beta d(d-1)V\left[\mathrm{tr}\left[\frac{\partial E_0}{\partial J}\frac{\partial E_0}{\partial J}\frac{\partial E_0}{\partial J^\dagger}\frac{\partial E_0}{\partial J^\dagger}\right]+\mathrm{H.c.}\right]$$

$$=\beta d(d-1)VW_0(z)^4 , \tag{3.4}$$

where $W_0(z)=-\frac{1}{2}dE_0/dz$, and $E_0$ given by (2.8).

The evaluation of the adjoint piece $\langle S_a\rangle_0$, is more involved. Using identities such as $4(\partial \det J/\partial J_{k\beta})J_{kl}=(\det J)\delta_{\beta l}$, we find, using (1.6),

$$\langle S_a\rangle_0=\left[\frac{\beta_a}{6}\right]d(d-1)V\left[f^{-4}\frac{\partial^2 f}{\partial J_{ji}^\dagger \partial J_{\alpha\delta}}\frac{\partial^2 f}{\partial J_{kj}^\dagger \partial J_{\delta\gamma}}\frac{\partial^2 f}{\partial J_{lk}\partial J_{\gamma\beta}^\dagger}\frac{\partial^2 f}{\partial J_{il}\partial J_{\beta\alpha}^\dagger}-1\right]$$

$$=\left[\frac{\beta_a}{6}\right]d(d-1)V\left\{3\left[K\left[\frac{dE_0}{dK}\right]^2-K\frac{d^2E_0}{dK^2}\right]^4+\left[K\left[\frac{dE_0}{dK}\right]^2-K\frac{d^2E_0}{dK^2}-2\frac{dE_0}{dK}\right]^4-1\right\},$$

where $K\equiv z^2$ and $E_0=-\ln f$. However, from $UU^\dagger=1$, we have an identity,

$$K\left[\left[\frac{dE_0}{dK}\right]^2-\frac{d^2E_0}{dK^2}\right]-2\frac{dE_0}{dK}=1 ,$$

so that $\langle S_a\rangle_0$ simplifies to

$$\langle S_a\rangle_0=\tfrac{1}{2}\beta_a d(d-1)V\left[W_0(z)^2+\frac{z}{2}\frac{d}{dz}(W_0/z)\right]^4 . \tag{3.5}$$

Finally, $\langle S_0\rangle_0$ can be evaluated in analogy with (2.5); we thus arrive at an upper bound for the free energy per link,

$$E\equiv(1/Vd)W_{\mathrm{eff}}$$
$$=E_0+2zW_0(z)-\beta(d-1)W_0(z)^4$$
$$-\tfrac{1}{2}\beta_a(d-1)\left[W_0{}^2-\frac{1}{2z}W_0+\frac{W_0'}{2}\right]^4 . \tag{3.6}$$

The variational free energy can then be found by first enforcing the stationary condition $\delta E/\delta z=0$. Anticipating our subsequent improved treatment, we now only consider the limiting case where $\beta_a=0$.

With $\beta_a=0$, the problem reduces to that for the usual Wilson action; the condition $\delta E/\delta z=0$ leads to

$$z=2\beta(d-1)W_0{}^3(z) \tag{3.7}$$

whose solution again has two branches: (1) $z=0$, corresponding to $E=0$; (2) $z\neq 0$, which is a solution to the equation $\beta(d-1)=F_4(z)$ where

$$F_4(z)=\left[\frac{z^4}{2}\right]\left[\frac{zI_0(2z)}{I_1(2z)}-1\right]^{-3} . \tag{3.8}$$

In Fig. 2(a) we plot $F_4(z)$ exhibiting the fact that $F_4(z)$ is bounded from below. Therefore, this second solution exists only for $\beta\geq\beta_c$ where $\beta_c(d-1)\simeq 3.35$. For $d=4$, this leads to $\beta_c\simeq 1.12$ which is precisely the MF result previously found by Greensite *et al.*[2] However, the free energy on this branch remains positive in the neighborhood of $\beta_c$; the actual transition does not take place until $\beta_c'\simeq 1.41$ where $E$ turns negative. In fact, two

sets of solutions exist [Fig. 2(b)]; the "lower" branch corresponds to a lower free energy.

### B. Local variational Ansatz

Following the analysis of the type-B *Ansatz* for the chiral model, we consider the situation $J_l = V_l J$ where $V_l$ is an SU(2) matrix for each link. To simplify the problem further, we shall set all $V_l$ within a plaquette to be $I$ except one; this single matrix can be assigned the plaquette label $V(P)$. A moment of reflection easily convinces oneself that this configuration can be achieved. Furthermore, while allowing $V(P)$ to vary from plaquette to plaquette, we restrict

$$\begin{aligned} &\mathrm{tr} V(P) = 2\cos\frac{\theta}{2}\,, \\ &\mathrm{tr}_A V(P) = 1 + 2\cos\theta \end{aligned} \tag{3.9}$$

with $0 \le \theta \le 4\pi$.

Under the above variational *Ansatz*, we need only modify (3.4) and (3.5) by multiplying them by $\cos(\theta/2)$ and $\frac{1}{3}(1+2\cos\theta)$, respectively. On the other hand, the upper bound for the free energy per link, $E$, is now a function of $z$ and $\theta$,

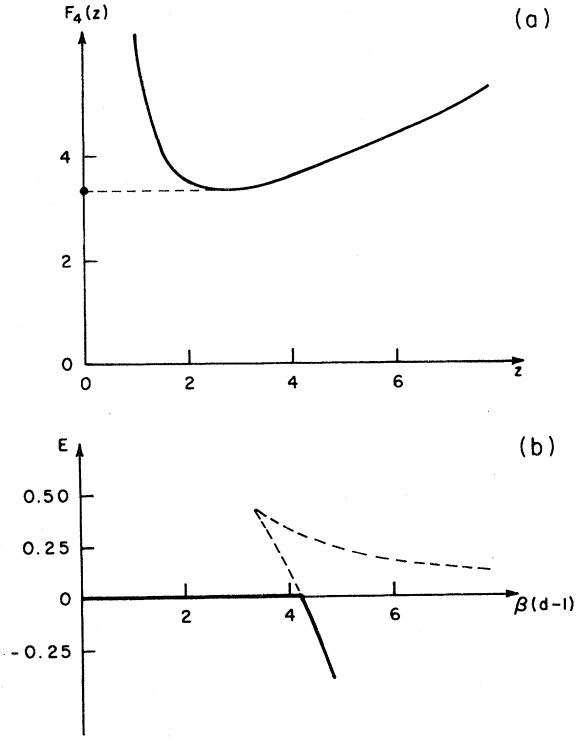


FIG. 2. (a) $F_4(z)$ of Eq. (3.8). (b) Variational free energy for four-dimensional QCD with a Wilson action.

$$E(\beta,\beta_a;z,\theta) = E_0 + 2zW_0 - \beta(d-1)W_0{}^4\cos\frac{\theta}{2} - \frac{\beta_a}{6}(d-1)\left[W_0{}^2 - \frac{1}{2z}W_0 + \tfrac{1}{2}W_0'\right]^4 (1+2\cos\theta)\,. \tag{3.10}$$

The stationary conditions are $\delta E/\delta z = 0$ and

$$\frac{\delta E}{\delta\theta} = \sin\frac{\theta}{2}\left[\left[\frac{\beta}{2}\right]\left[\frac{I_0}{I_1} - \frac{1}{z}\right]^4 + \left[\frac{\beta_a}{3}\right]\left[1 + \frac{2}{z^2} - \frac{2}{z}\frac{I_0}{I_1}\right]^4 \cos\frac{\theta}{2}\right] = 0\,, \tag{3.11}$$

where the arguments of $I_0$ and $I_1$ are both $2z$.

Note that $E(\beta,\beta_a;z,\theta)$ is invariant under $\beta \to -\beta$, $\theta \to \theta + 2\pi$. We, therefore, restrict ourselves to the region $\beta \ge 0$. From (3.11), one obvious solution corresponds to $\theta = 0$; the situation is then identical to the case of constant $J_l$ described earlier. However, a nontrivial solution to (3.11) in $\theta$ exists for $\beta_a$ sufficiently negative. When this is the case, we must evaluate the variational free energy $E(\beta,\beta_a;z,\theta)$ to decide on the phase structure.

In Fig. 3 we exhibit the phase structure of the SU(2)-SO(3) lattice gauge theory under our present variational *Ansatz*. The parameter space is divided into three phase regions, separated by *first-order* transition lines (solid lines). We exhibit also the MF result of Ref. 4 as dashed-dotted lines. In particular, our solution does not indicate the presence of any *second-order* transition line (dotted line) found by a MF method in Ref. 4. (Further discussion in Sec. V.)

### C. The gauge center

In order to account for the special role of the SU(2) center, we follow the standard procedure[4,10] by writing the first term in (3.1) as

$$S_f = \frac{\beta}{2}\sum_P \sigma_P[\mathrm{tr}U(\partial P) + \mathrm{H.c.}]\,, \tag{3.12}$$

where $\sigma_P = \prod_{l\in P}\sigma_l$, and $\sigma_l$ are elements of $Z_2$ associated with links. One now sums over $\sigma_l$ in defining the partition function.

For our variational trial action, we add to our

previous analysis a term $\sum_l \sigma_l m_l$, where the sum is over all links. Therefore, in addition to $J_l, V_l$, we now also have $m_l$ as variational parameters. In the spirit of simplicity, we choose $m_l$ to be independent of $l$, and together with the previous *Ansatz*, we obtain an improved upper bound:

$$E(\beta,\beta_a;z,\theta,m)=E_0+2zW_0+m\tanh m-\ln\cosh m-\beta(d-1)W_0{}^4(\tanh m)^4\cos\frac{\theta}{2}$$
$$-\frac{\beta_a(d-1)}{6}\left[W_0{}^2-\frac{1}{2z}W_0+\tfrac{1}{2}W_0'\right]^2(1+2\cos\theta)\,.$$

Proceeding as before by finding stationary points of $E$ with respect to $z$, $\theta$, and $m$, the phase structure can be determined. This is exhibited in Fig. 4. As expected, the pocket (phase 4), related to the liberation of $Z_2$ monopoles[10] now appears. Furthermore, a fifth region, characterized by $W_0\neq 0$, $m=0$, $\theta=\pi$, also appears in the lower left corner. For comparison, we again exhibit the MF result of Ref. 4 by dashed and dotted lines. We emphasize that all transitions are first order and no signal for the presence of a second-order line in the $\beta_a<0$ region was detected.

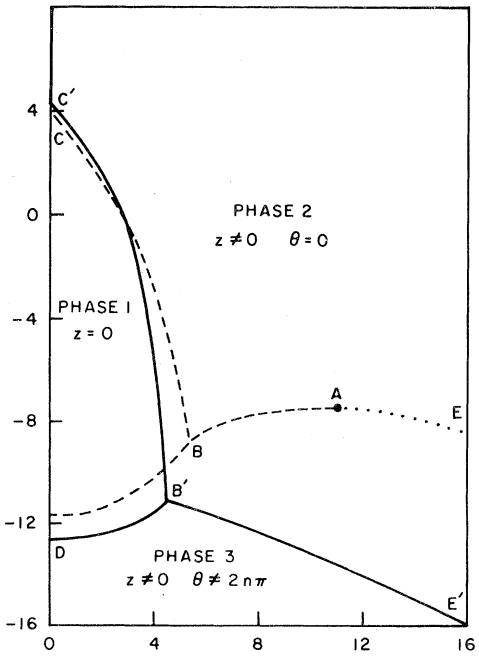


FIG. 3. Phase structure for four-dimensional QCD with a mixed FA action. Solid lines are variational results for first-order transitions. Dashed-dotted lines are MF results (the dotted line is the second-order transition line previously identified in Ref. 4).

## IV. LARGE-$N$ LIMIT

We have previously studied the large-$N$ limit of (3.1) by a MF approach[5,12]; we now repeat the analysis from the variational viewpoint. Conforming to the large-$N$ convention, we normalize the action as[5,13]

$$S=\sum_P\{\beta N[\mathrm{tr}U(\partial P)+\mathrm{H.c.}]$$
$$+\beta_a[\,|\,\mathrm{tr}U(\partial P)\,|^2-1]\}\,. \qquad (4.1)$$

In order to compare with the MF result, we first adopt the minimum *Ansatz* for the variational trial action

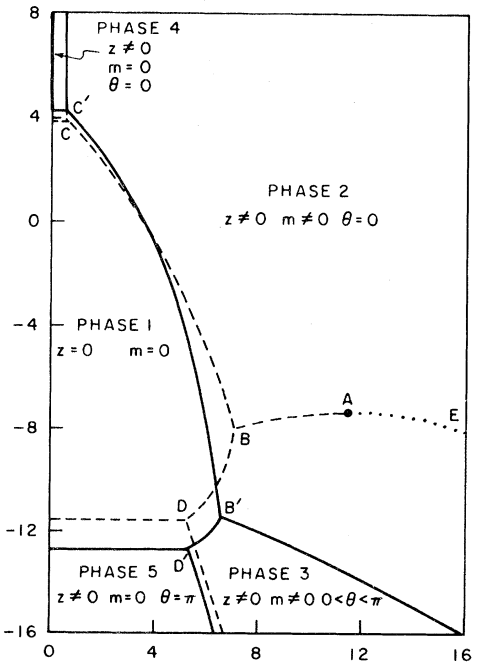


FIG. 4. Improved phase structure when the gauge center is treated properly. Dashed-dotted lines are MF results.

$$S_0 = N \sum_l \mathrm{tr}(U_l J_l^\dagger + U_l^\dagger J_l) , \tag{4.2}$$

where the sum is over all links. We assume $J_l = J$ for all $l$. It follows that (3.3)–(3.6) apply except $E_0$ is now different and we must also watch out for the $N$ factors.

Let us now normalize the external source integral as

$$e^{-N^2 E_0(J,J^\dagger)} = \int dU\, e^{N\,\mathrm{tr}(UJ^\dagger + U^\dagger J)} . \tag{4.3}$$

In the large-$N$ limit, $E_0$ approaches a finite limit[14]

$$E_0 = -\frac{2}{N}\sum_{i=1}^{N}\sqrt{\lambda_i + c} + \frac{1}{2N^2}\sum_{i,j}\ln(\sqrt{\lambda_i + c} + \sqrt{\lambda_j + c}) + c + \tfrac{3}{4} , \tag{4.4}$$

where $\{\lambda_i\}$ are eigenvalues of $JJ^\dagger$. This formal solution has two branches distinguished by a parameter $s = (1/N)\sum_i \lambda_i^{-1/2}$.

(i) *Strong-coupling region* ($s \geq 2$): The constant $c$ is determined by the condition $\sum_i (\lambda_i + c)^{-1/2} = 2N$.

(ii) *Weak-coupling region* ($0 \leq s \leq 2$): The constant $c$ vanishes. Note that $E_0$ is continuous at $s = 2$.

The evaluations of $\langle S_f \rangle_0$, $\langle S_a \rangle_0$, and $\langle S_0 \rangle_0$ can also be carried out as in the SU(2) case. However, a simplification arises whose origin is the same as that for the large-$N$ factorization property. To illustrate this point, let us discuss the limit where $\lambda_i = z^2$ for all $i$ so that $N^{1/2}z$ plays the role of $z$ in (3.4) and (3.5). However, since $E_0$ has now been replaced by $N^2 E_0$, the term leading to $z(d/dz) \times (1/z) W_0$ in (3.5) is now of lower order, so that

$$\langle S_f \rangle_0 = \beta d(d-1) V N^2 W_0(z)^4 , \tag{4.5}$$

$$\langle S_a \rangle_0 \simeq \frac{\beta_a}{2} d(d-1) V N^2 W_0(z)^8 , \tag{4.6}$$

where $W_0 = -\frac{1}{2}E_0'$ and

$$E_0(z) = -z^2\theta(\tfrac{1}{2} - |z|) + (-2|z| + \tfrac{1}{2}\ln 2|z| + \tfrac{3}{4}) \times \theta(|z| - \tfrac{1}{2}) . \tag{4.7}$$

The upper bound for the free energy per link (normalized by $N^2$) in the large-$N$ limit then follows:

$$E(\beta,\beta_a;z) = E_0(z) - \beta(d-1)W_0{}^4(z) - \frac{\beta_a}{2}(d-1)W_0{}^8(z) + 2|z|\,W_0(z) . \tag{4.8}$$

To obtain the variational free energy, we next enforce the stationary condition $\delta E/\delta z = 0$, leading to the relation

$$z = 2(d-1)[\beta W_0{}^3(z) + \beta_a W_0{}^7(z)] \tag{4.9}$$

which is precisely the MF condition [Eq. (11)] of Ref. 5.

The limit $\beta_a = 0$ for $N = \infty$ has previously been investigated by Greensite *et al.* in Ref. 2 where a strong-weak crossover transition was identified (for our normalization) at $\beta \simeq 0.39$, in apparent agreement with the independent numerical estimate of 0.396.[15] However, as explained in Ref. 5, at $\beta = 0.39$, the nontrivial solution to (4.9) ($W_0 \neq 0$) still leads to a positive free energy whereas the trivial solution ($z = W_0 = 0$) always has $E = 0$. Therefore, the system remains in the strong-coupling phase at $\beta = 0.39$; the actual transition, under our minimum variational *Ansatz* (which is equivalent to the standard MF analysis), takes place at $\beta \simeq 0.535$. In view of the crudeness of the variational *Ansatz*, it is comforting to know that it leads to a higher value so that there is room for maneuvering when a better estimate for the free energy is obtained.

Before completing specifying the phase structure for $\beta_a \neq 0$, we first generalize to a local variational *Ansatz* by allowing $J_l = V_l J$ where $V_l$ is an SU($N$) matrix for each link. For practical reasons, we constrain $V_l$ such that, for $N$ large,

$$\frac{1}{N}\mathrm{tr} V_l = \cos\theta, \quad \frac{1}{N^2}\mathrm{tr}_A V_l \simeq \cos^2\theta , \tag{4.10}$$

where $0 \leq \theta \leq 2\pi$. Furthermore, because the center of SU($N$) approaches U(1) as $N$ increases, special handling, as for SU(2), is unnecessary.

With (4.10), Eqs. (4.5) and (4.6) are modified by having an extra factor $\cos\theta$ and $\cos^2\theta$, respectively; this in turn modifies (4.8) so that it also becomes a function of $\theta$. The stationary condition $\delta E/\delta z = 0$ is now

$$z = 2(d-1)W_0(z)^3\cos\theta[\beta + \beta_a W_0(z)^4\cos\theta] \tag{4.11}$$

and the new condition $\delta E/\delta\theta = 0$ leads to

$$0=\sin\theta[\beta+\beta_a W_0(z)^4\cos\theta] \ . \tag{4.12}$$

Unlike the SU(2) case, the nontrivial solution to (4.12) in $\theta$, when substituted back into (4.11), always leads to a trivial solution $z=0$ so that $W_0(0)=0$. Therefore, "phase 3" of Figs. 3 and 4 for SU(2) does not exist in the large-$N$ limit. Also, since the center no longer plays a special role, phases 4 and 5 of Fig. 4 also disappear in this limit.

The trivial solutions to (4.12) are $\theta=0$ and $\pi$. For $\beta>0$, the solution to (4.11) corresponding to a lower free energy has $\theta=0$ and $z\geq 0$. On the other hand, for $\beta<0$, the solution $\theta=\pi$, $z<0$ dominates. Indeed, as pointed out earlier for SU(2), the system is invariant under $\beta\rightarrow-\beta$, $\theta\rightarrow\theta+\pi$ simultaneously.

Under our variational *Ansatz*, the one-plaquette expectation is

$$W_1\equiv\frac{1}{N}\langle \mathrm{tr}U(P)\rangle = W_0(z)^4\cos\theta \ .$$

For $\theta=\pi$, it leads to $W_1\leq 0$; when $W_0\neq 0$, it corresponds to the "frustrated" configuration.[5,16] Schematically, the phase diagram is depicted in Fig. 5, which was obtained previously in Ref. 5. Note that the line of first-order transition for $\beta=0$ and $\beta_a>\beta_a^*\simeq 1.398$, which corresponds to the breaking of a (local) U(1) symmetry for the adjoint action. For $\beta_a$ sufficiently small, all loop expectation values in the fundamental representations vanish; the symmetry is broken when $\beta_a$ is increased. For two-dimensional QCD, the exact transition point is $\beta_a^*=1$.[5]

For $\beta>0$, the parameter space is separated into the strong- and weak-coupling regions. The transition, under our minimum variational *Ansatz*, is first order. However, the general large-$N$ argument suggests that the transition in the neighborhood of $\beta_a=0$ should be again third order.[5,17] It is interesting to ask if improved variational *Ansätze* can alter the situation. In Figs. 6(a) and 6(b) we exhibit the numerical solution to (4.1). In Fig. 6(b), we have also indicated the existence of phase II′ (Refs. 12 and 17) where $E<0$, $z\neq 0$. However, in this phase, $0<W_1<\frac{1}{2}$, which is characteristic of the strong-coupling region. Therefore, we interpret II and II′ to both be strong-coupling phases and expect the transition line between them to disappear when an improved variational scheme is used.

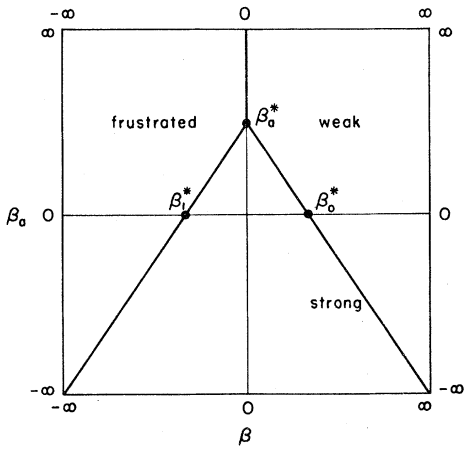


FIG. 5. Schematic representation for the large-$N$ phase diagram for a mixed FA action.

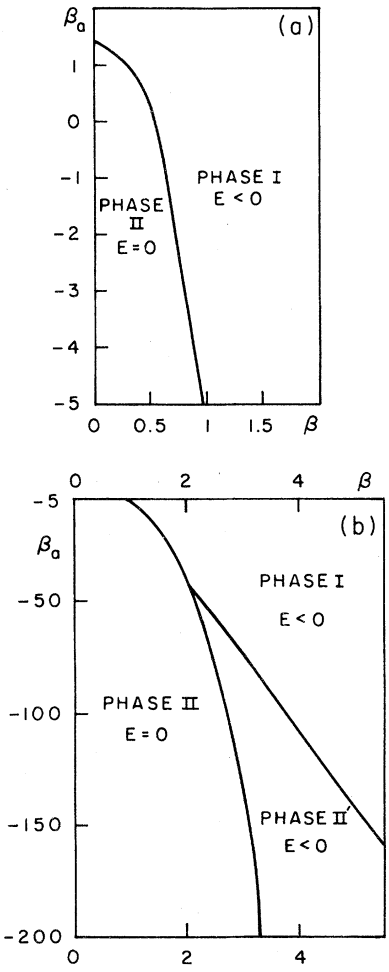


FIG. 6. Numerical results for the large-$N$ phase structure.

## V. REMARKS

In this paper we have begun a variational analysis of lattice gauge theories. The general variational approach is flexible as well as powerful; much remains unexplored. For instance, instead of our simple trial action (1.5) we can introduce one with correlation among neighboring plaquettes. Furthermore, one can also use a trial action where the gauge invariance of the procedure is manifest. Preliminary investigation indicates that these improvements are indeed feasible without dramatically increasing analytic difficulties. These analyses will be reported in subsequent studies.

Since our trial action (1.5) involves each link variable separately, it also leads to nonvanishing gauge-variant expectation values. However, because of local gauge invariance, our stationary solutions in the variational parameters, $\{J_l\}$, are in fact degenerate, i.e., all sets of $\{J_l\}$ related by local gauge transformations are equivalent and are to be summed over. As a consequence, gauge-noninvariant quantities vanish after summing over the degeneracy.[1] On the other hand, for gauge-invariant expectation values, the last step is redundant.

The variational *Ansatz* (1.5) has in fact been used as a possible procedure for "deriving" the mean-field approximation for lattice gauge theories with a Wilson-type action.[1] Indeed, for the case $\beta_a=0$, Eq. (3.7) is the standard MF equation where $W_0(z)$ is the MF value for each link and $z$, given by (3.7), is the "mean-field" due to neighboring links seen by each link. However, this equivalence breaks down when a mixed fundamental-adjoint action is involved.

Under the standard MF analysis, for the mixed action (3.1), the mean field seen by each link would be

$$z=2\beta(d-1)W_0{}^3\cos\frac{\theta}{2}$$
$$+\tfrac{2}{3}\beta_a(d-1)W_0{}^7(1+2\cos\theta)\,, \qquad (5.1)$$

where we have included the angular variable $\theta$ which now characterizes the relative "orientation" of link variables within a plaquette. Comparing (5.1) with $\delta E/\delta z=0$ derived from (3.10), they no longer agree. Had we incorrectly replaced (3.5) by $\frac{1}{2}\beta_a d(d-1)VW_0(z)^8$, the variational result would have coincided with the MF result of Ref. 4.

Since the MF result could conceivably have been arrived at by other "physical arguments," it is meaningful to compare it with our variational analysis. We have thus carried out an analysis for SU(2) with the MF choice for (3.5) and arrived at the result of Ref. 4 as indicated by dashed-dotted lines in Figs. 3 and 4. In the region $\beta_a<0$, our variational free energy is always lower than that of the MF analysis, so that the second-order transition lines found in Ref. 4 should not have been present. For $\beta_a>0$, the MF free energy is actually slightly lower than our variational result. On the other hand, since the MF result as formulated in Ref. 4 does not follow from a stationary condition, its reliability is questionable.

It is interesting to note that the MF-type analysis is correct in the large-$N$ limit due to the large-$N$ factorization property. Since $|\mathrm{tr}U|^2=\mathrm{tr}_A U$ in the large-$N$ limit, together with the fact that (4.6) holds, the large-$N$ stationary condition (4.11) agrees with (5.1) if we replace $(1+2\cos\theta)/3$ by $\cos^2(\theta/2)$, and then $\theta$ by $2\theta$ in (5.1). Because of these facts the phase structure of the large-$N$ lattice gauge theory with a mixed fundamental-adjoint action differs from that of SU(2) by the absence of phase 3 in Fig. 3, and it is correctly depicted in Ref. 5 (see Fig. 5).

Last, it should be mentioned that both the MF and variational results near $\beta_a=0$ are incorrect for SU(2) since no transition should take place there. This problem has been addressed by Blyvbjerg *et al.* in Ref. 2 for the MF analysis at $\beta_a=0$ where it was shown the fluctuations about the MF result would restore analyticity. We therefore expect a similar procedure to work also in our case for small $\beta_a$.

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[1]B. Balian, J. Drouffe, and C. Itzykson, Phys. Rev. D 10, 3376 (1974); E. Brezin and J. Drouffe, Nucl. Phys. B200 [FS4], 93 (1982).
[2]J. Greensite and B. Lautrup, Phys. Lett. 104B, 41 (1981); J. Greensite, T. H. Hanson, N. D. Hari Dass, and P. G. Lauwers, *ibid.* 105B, 201 (1981); D. J. Pritchard, *ibid.* 106B, 193 (1981); P. Cvitanovic, J. Greensite, and B. Lautrup, *ibid.* 105B, 197 (1981); N. D. Hari Dass and P. G. Lauwers, Report No. NBI-HE-81-48 (unpublished); H. Flyvbjerg, B. Lautrup, and J. B. Zuber, Phys. Lett. 110B, 279 (1982).
[3]J. M. Drouffe, Phys. Lett. 105B, 46 (1981).
[4]J. M. Drouffe, Nucl. Phys. B205, 27 (1982).
[5]Chen T.-L., C.-I Tan, and Zheng X.-T., Phys. Lett. 109B, 383 (1982).
[6]B. Sakita, in Nonperturbative Studies in Quantum Chromodynamics, proceedings of Brown Workshop, 1981, edited by A. Jevicki and C.-I Tan (unpublished).
[7]Chen T.-L., P. Suranyi, C.-I Tan, and Zheng X.-T (unpublished); A. Guha, Stony Brook report (unpublished).
[8]R. Brower, P. Rossi, and C.-I Tan, Nucl. Phys. B190, 699 (1981), and references therein.
[9]J. B. Kogut, M. Snow, and M. Stone, Nucl. Phys. B200 [FS4], 211 (1982).
[10]I. G. Halliday and A. Schwimmer, Phys. Lett. 101B, 327 (1981); 102B, 337 (1981); L. Caneschi, I. G. Halliday, and A. Schwimmer, Nucl. Phys. B200 [FS4], 409 (1982); R. Brower, D. Kessler, and H. Levine, *ibid.* B205 [FS5], 77 (1982); Phys. Rev. Lett. 47, 621 (1981); J. Greensite and B. Lautrup, *ibid.* 47, 9 (1981).
[11]G. Bhanot and M. Creutz, Phys. Rev. D 24, 3212 (1981); G. Bhanot, IAS report, 1981 (unpublished).
[12]M. Ogilvie, private communication.
[13]S. B. Khokhlacher and Yu. M. Makeenko, Phys. Lett. 101B, 403 (1981); Yu. M. Makeenko and M. I. Polikarpov, Moscow Report No. ITEP-152, 1981 (unpublished); and more recently, S. Samuel, Phys. Lett. 112B, 237 (1982).
[14]R. Brower and M. Nauenberg, Nucl. Phys. B180, 221 (1981); E. Brezin and D. Gross, Phys. Lett. 97B, 120 (1980); R. Brower, P. Rossi, and C.-I Tan, Phys. Rev. D 23, 942 (1981); 23, 953 (1981).
[15]F. Green and S. Samuel, Phys. Lett. 103B, 48 (1980).
[16]Chen T.-L. and C.-I. Tan, Phys. Lett. 108B, 127 (1982); Phys. Rev. D 26, 2018 (1982).
[17]João P. Rodrigues, this issue, Phys. Rev. D 26, 2940 (1982).