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Renormalization of interacting scalar field theories in curved space-time

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The renormalization of interacting scalar field theories in general curved space-times is discussed. The background-field method is used to calculate the effective action. Divergences are analyzed using heat-kernel techniques and dimensional regularization. Renormalization of a scalar field theory with cubic and quartic self-interactions is shown at the two-loop level in a four-dimensional space-time. Counterterms, including the gravitational ones, are computed to this order. Renormalization of the one-loop effective action is examined for a scalar field with a cubic self-interaction in a general six-dimensional space-time. As a result of the asymptotic freedom of this theory, the coupling constant appearing in the $R\phi^2$ term is shown using the renormalization group to have an ultraviolet fixed point given by its conformal value of $\frac{1}{5}$.

I. INTRODUCTION

The study of quantum field theory in curved space-time has been the subject of much current research. Most of the work has been concerned with free (i.e., noninteracting) quantum fields, although the effects of interactions are now being studied. The first problem encountered with interacting fields is that of renormalizability.

Renormalization of interacting quantum field theory in curved space-time has been considered by several authors[1–18] (see the review of Birrell[5] for an introduction). In the case where the external gravitational field is weak so that the metric may be expanded about the Minkowski metric, standard momentum-space techniques familiar from flat-space-time calculations may be used.[1–4] This approach cannot be applied to general space-times or topologically nontrivial space-times. Studies in more general space-times are discussed in Refs. 5–18. With the exceptions of Birrell and Ford[8] who considered a spatially flat Robertson-Walker universe with topology $S^1\times R^3$, and Drummond and Shore[13–15,18] who worked in S^4, these calculations have been restricted to space-times which are topologically trivial (i.e., diffeomorphic to space-times with an R^4 topology). This restriction is particularly clear in the momentum-space technique developed by Bunch and Parker[11] which uses Riemannian normal coordinates and thus can probe only what happens in a neighborhood of the origin of the coordinate system. The importance of this restriction is evident from Refs. 8, 19–21, where it is shown that the presence of a nontrivial topology, even in flat space-time, complicates the renormalization procedure. This might seem to be a bit peculiar, since naively (on the basis of the equivalence principle) it would be expected that renormalization might be affected only by *local* curvature terms and not by the global topology. However, there is nothing to prevent this argument from being applied to graphs which contribute separately to the Green's functions where it is known to be incorrect.[8,19,20] This is an indication of how the equivalence principle may not apply when quantum effects are considered (see also Drummond and Hathrell[22]).

The present paper is concerned with the renormalization of interacting scalar field theories in general curved space-times with arbitrary topologies. The method consists of an evaluation of the effective action using the background-field method.[23,24] Dimensional regularization[25] and heat-kernel techniques[26–28] are used to handle the divergences which are present. Because coordinate-space methods are used, no assumptions need to be made about the space-time topology. This means that the results apply equally well to the twisted scalar fields which exist if the space-time is not simply connected.[29]

One advantage of using the background-field method is that it cuts down considerably the number of diagrams which need to be calculated. All contributions at a given order in the loop expansion can be obtained by an evaluation of only vacuum bubbles without the necessity of evaluating the n-point functions separately for each $n=0,1,2,\ldots$. (There are only two bubbles, plus one counterterm diagram for ϕ^4 theory at the two-loop level. See Fig. 1.) Furthermore, recent formulations[30,31] have

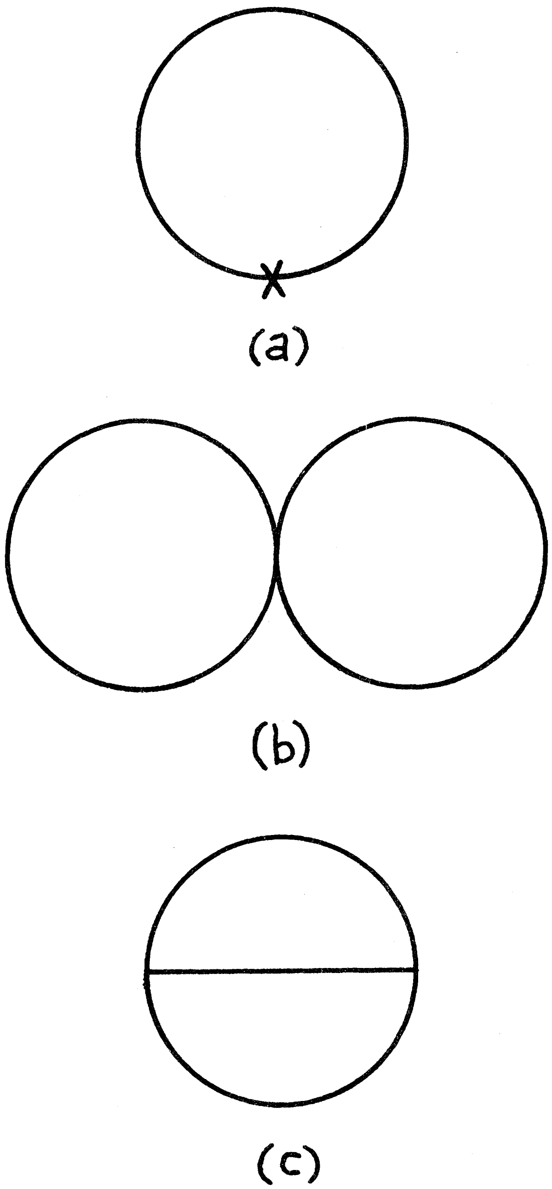

FIG. 1. The vacuum bubbles contributing to the two-loop effective action. The cross in (a) indicates the insertion of the quadratic counterterm vertex defined in Eqs. (2.18) and (4.19).

given a procedure which respects the invariances (e.g., gauge and general coordinate invariance) which may be present (see also the review of Abbott[32]). This has advantages when calculating the Yang-Mills β function.[33] Finally, once the effective action has been renormalized, expressions such as the expectation value of the stress-energy tensor which is obtained by functional differentiation with respect to the background metric, or the n-point functions which are obtained by functional differentiation with respect to the background fields, will be automatically finite.

The outline of this paper is the following. In Sec. II, the method of the background field is reviewed, showing how higher-loop contributions to the effective action are obtained. The heat-kernel method is discussed in Sec. III. In Sec. IV, the renormalizability of a scalar field theory with no more than a quartic self-interaction is proved at the two-loop level on a general (compact or noncompact) four-dimensional manifold without boundary. The renormalization of a scalar field theory with a cubic self-interaction is discussed in Sec. V at the one-loop level on a general six-dimensional space-time without boundary. This is of interest because the theory is the simplest one which is asymptotically free. Some of the results of this section are in disagreement with those of Gass,[16] who studied ϕ^3 theory in a spatially flat Robertson-Walker model. Sec. VI contains the conclusions and discussion.

II. THE BACKGROUND-FIELD METHOD AND THE EFFECTIVE ACTION

In this section, the background-field approach to computing the effective action is considered. In addition to Refs. 23, 24, 30—32, which were mentioned in the Introduction, Refs. 34, 35 are also useful.

Let $I[\phi]$ denote the classical action functional, and let $\hat{\phi}_i(x)$ denote the arbitrary background fields. The disconnected generating functional, or partition function, may be defined by the following functional integral:

$$Z[J;\hat{\phi}]=\int [d\phi]\exp\left\{-\frac{1}{\hbar}I[\phi]+\frac{1}{\hbar}J_i(\phi-\hat{\phi})_i\right\} . \qquad (2.1)$$

It is assumed here that the space-time has a Riemannian rather than a Lorentzian metric. In (2.1) the index i denotes both internal and space-time indices with a repeated index both summed and integrated over; that is,

$$J_i(\phi-\hat{\phi})_i=\sum_i \int dv_x J_i(x)[\phi_i(x)-\hat{\phi}_i(x)] , \qquad (2.2)$$

where

$$dv_x=[g(x)]^{1/2}d^nx$$

is the invariant volume element on the n-dimensional manifold. If the decomposition

$$\phi_i(x)=\hat{\phi}_i(x)+\phi_i^q(x) \qquad (2.3)$$

is made, where $\phi_i^q(x)$ represents a quantum deviation from the background field, then the sources in (2.1) are seen to be coupled only to the quantum part of the field.

In order to obtain the effective action, the by now well-known procedure described below must be followed. First, form the generating functional which gives rise to the connected Green's functions and is defined in terms of (2.1) by

$$W[J;\hat{\phi}]=-\hbar\ln Z[J;\hat{\phi}] . \qquad (2.4)$$

Next, perform the functional Legendre transformation[36]

$$\Gamma[\Psi;\hat{\phi}]=W[J;\hat{\phi}]-J_i\Psi_i , \qquad (2.5)$$

where

$$\Psi_i(x)=\frac{\delta W[J;\hat{\phi}]}{\delta J_i(x)} . \qquad (2.6)$$

The generating functional $\Gamma[\Psi;\hat{\phi}]$ gives rise only to the one-particle-irreducible Green's functions. Finally, the effective action, which depends only on the background field, is defined by

$$\Gamma[\hat{\phi}]=\Gamma[\Psi=0;\hat{\phi}] . \qquad (2.7)$$

In order to calculate the effective action, some perturbative approach must be adopted. The usual one is the loop expansion, in which $\hbar$ is used as a loop-counting parameter.[37] Expand the classical action in a functional Taylor series about the background field:

$$I[\phi]=I[\hat{\phi}]+(\phi-\hat{\phi})_iI_{1i}+\tfrac{1}{2}(\phi-\hat{\phi})_i(\phi-\hat{\phi})_jI_{2ij}+\frac{1}{3!}(\phi-\hat{\phi})_i(\phi-\hat{\phi})_j(\phi-\hat{\phi})_kI_{3ijk}$$
$$+\frac{1}{4!}(\phi-\hat{\phi})_i(\phi-\hat{\phi})_j(\phi-\hat{\phi})_k(\phi-\hat{\phi})_lI_{4ijkl} , \qquad (2.8)$$

where $I_{1i}, \dots, I_{4ijkl}$ denote the first, . . . , fourth functional derivatives of the classical action evaluated at the background field. Substitution of this expansion into (2.1) and working through (2.4)–(2.7) results in the loop expansion of the effective action in powers of $\hbar$:

$$\Gamma[\hat{\phi}]=\sum_{n=0}^{\infty}\hbar^n\Gamma^{(n)}[\hat{\phi}] . \qquad (2.9)$$

The convention for functional derivatives which enter into (2.8) is that of Ref. 38:

$$\frac{\delta\phi_i(x)}{\delta\phi_j(x')}=\delta_{ij}\delta(x,x') , \qquad (2.10)$$

where $\delta(x,x')$ is the biscalar Dirac distribution which satisfies

$$\int dv_x\delta(x,x')\phi_i(x)=\phi_i(x') . \qquad (2.11)$$

It is necessary to write the bare quantities which appear in the action in terms of renormalized ones and counterterms. In the loop expansion, the counterterms are then expanded in powers of $\hbar$ leading to

$$I[\hat{\phi}]=I^R[\hat{\phi}]+\sum_{n=1}^{\infty}\hbar^nI^{(n)}[\hat{\phi}] , \qquad (2.12)$$

where $\hat{\phi}$ is understood to be the renormalized background field, and a superscript R denotes the fact that only renormalized parameters are present. Note that because the quantum parts of the fields get integrated over in the functional integral (2.1), they may be rescaled at the irrelevant cost of changing only the normalization of (2.1). This means that any field renormalization factors may be scaled away in this manner, except for those coming from the background fields. Thus, only the background fields need to be renormalized. By writing the bare quantities appearing in I_{2ij} in terms of renormalized ones and counterterms the following split may be made:

$$I_{2ij}=I_{2ij}^R+I_{2ij}^{CT} . \qquad (2.13)$$

I_{2ij}^{CT} involves counterterms and so is at least of order $\hbar$ in the loop expansion. Define analogously to (2.12)

$$I_{2ij}^{CT}=\sum_{n=1}^{\infty}\hbar^nI_{2ij}^{(n)} . \qquad (2.14)$$

This latter term will be treated as an additional vertex.

From the references given at the beginning of this section, the first two terms in the expansion (2.9) are found to be

$$\Gamma^{(0)}[\hat{\phi}]=I^R[\hat{\phi}] , \qquad (2.15)$$

$$\Gamma^{(1)}[\hat{\phi}]=I^{(1)}[\hat{\phi}]+\tfrac{1}{2}\ln \mathrm{Det} I_{2ij}^{R} \ . \tag{2.16}$$

The $\ln \mathrm{Det} I_{2ij}^{R}$ arises from performing a Gaussian functional integration. The inverse of the operator I_{2ij}^{R} is the propagator in the presence of the background field which is denoted by Δ_{ij}; that is,

$$I_{2ij}^{R}\Delta_{jk}=\delta_{ik} \ . \tag{2.17}$$

Higher-order contributions to (2.9) are obtained by evaluating only one-particle-irreducible vacuum bubbles at a given order in the loop expansion with the vertices obtained from the effective interaction

$$I_{\mathrm{INT}}=\tfrac{1}{2}\phi_i\phi_j I_{2ij}^{\mathrm{CT}}+\frac{\hbar^{1/2}}{3!}\phi_i\phi_j\phi_k I_{3ijk} +\frac{\hbar}{4!}\phi_i\phi_j\phi_k\phi_l I_{4ijkl} \ . \tag{2.18}$$

The exact propagator in the presence of the background field is used on internal lines. The factors of $\hbar^{1/2}$ and $\hbar$ in front of the last two terms in (2.18) have come from a rescaling $\phi\to\hbar^{1/2}\phi$ in the functional integral.[34,35] Note that because the background field is taken to be arbitrary, there will be an effective-mass term in Eq. (2.17) for Δ_{ij} which is not a constant. This means that momentum-space techniques cannot be used to obtain the propagator in the presence of the background field, even in Minkowski space-time. One approach is to expand Δ_{ij} in terms of the free propagator as a power series in the background field as in Ref. 38. A different approach is followed here. In addition, because $I_{2ij}^{\mathrm{CT}}, I_{3ijk}$, and I_{4ijkl} are functional derivatives of the action evaluated at the background field, the rules for the vertices arising from (2.18) will involve the background field and differ from the usual ones.

An $\hbar$ expansion for I_{3ijk} and I_{4ijkl} results from expressing the bare quantities in terms of renormalized ones and counterterms:

$$I_{3ijk}=\sum_{n=0}^{\infty}\hbar^n I_{3ijk}^{(n)} \ , \tag{2.19}$$

$$I_{4ijkl}=\sum_{n=0}^{\infty}\hbar^n I_{4ijkl}^{(n)} \ . \tag{2.20}$$

There is an additional factor of $(-\hbar)$ in front of the two-loop and higher contributions to $\Gamma[\hat{\phi}]$ coming from the factor in (2.4). The two-loop vacuum bubbles contributing to $\Gamma^{(2)}$ are shown in Fig. 1 and lead to

$$\Gamma^{(2)}[\hat{\phi}]=I^{(2)}[\hat{\phi}]+\tfrac{1}{2}\Delta_{ij}I_{2ij}^{(1)}+\tfrac{1}{8}\Delta_{ij}\Delta_{kl}I_{4ijkl}^{(0)} -\tfrac{1}{12}\Delta_{il}\Delta_{jm}\Delta_{kn}I_{3ijk}^{(0)}I_{3lmn}^{(0)} \ . \tag{2.21}$$

It has been assumed that I_{3ijk} and I_{4ijkl} are symmetric under a permutation of indices. If this is not the case (for instance, if the index i refers to different types of fields) the modification of (2.21) is easy enough to write down. Higher-loop contributions to the effective action are found in a similar manner.

III. THE HEAT-KERNEL EXPANSION

In this section, the heat-kernel expansion and its application to regularizing the effective action are discussed. This method has been used by many authors[24,26–28,39–43] to regularize the one-loop effective action or expectation value of the stress tensor for free scalar fields in curved space-time. Boulware[44] has also used it to discuss the one-loop effective action for a scalar field in combined classical gravitational and electromagnetic backgrounds. The first three coefficients in the asymptotic expansion (3.13) below were first given by DeWitt[24] and later extended to one more term by Sakai.[45] Gilkey[46,47] has derived them rigorously using more powerful methods.

From (2.16), one contribution to the one-loop effective action is seen to be (ignoring the term $I^{(1)}$)

$$\Gamma^{(1)}[\hat{\phi}]=\tfrac{1}{2}\ln \mathrm{Det} I_2 \tag{3.1}$$

$$=\tfrac{1}{2}\mathrm{Tr}\ln I_2 \ . \tag{3.2}$$

The indices and superscript R will be dropped in this section. The determinant in (3.1) and trace in (3.2) are functional as well as over any indices which I_2 might have.

Variation of (3.2) with respect to the background field gives

$$\delta\Gamma^{(1)}=\tfrac{1}{2}\mathrm{Tr}[I_2{}^{-1}\delta I_2] \ . \tag{3.3}$$

This may be written as

$$\delta\Gamma^{(1)}=-\tfrac{1}{2}\delta\left\{\mathrm{Tr}\left[\int_0^\infty \frac{dt}{t}e^{-tI_2}\right]\right\} , \tag{3.4}$$

which leads to the definition

$$\Gamma^{(1)}[\hat{\phi}]=-\tfrac{1}{2}\mathrm{Tr}\left[\int_0^\infty \frac{dt}{t}e^{-tI_2}\right] . \tag{3.5}$$

(The arbitrary integration constant which arises may be absorbed into the bare gravitational action.) Regularization of this quantity is still required.

Let $K(t,x,y,I_2)$ be the kernel associated with the operator e^{-tI_2}; that is,

$$(e^{-tI_2}\phi)(x)=\int dv_y(K(t,x,y,I_2)\phi)(y)\ , \qquad (3.6)$$

where ϕ is any cross section of the bundle over space-time relevant to the type of field of interest. Let $\{\lambda_i,\psi_i\}_{i=1}^{\infty}$ be a spectral resolution of I_2 which is assumed to be self-adjoint with respect to the fiber inner product. The cross section may be written as

$$\phi=\sum_{i=1}^{\infty} a_i\psi_i\ . \qquad (3.7)$$

Hence,

$$(e^{-tI_2}\phi)(x)=\sum_{i=1}^{\infty} a_i(e^{-tI_2}\psi_i)(x)$$
$$=\sum_{i=1}^{\infty} a_i e^{-t\lambda_i}\psi_i(x)\ . \qquad (3.8)$$

(This is seen properly by using the functional calculus of Seeley.[48]) The kernel function may, therefore, be written as

$$K(t,x,y,I_2)=\sum_{i=1}^{\infty} e^{-\lambda_i t}\psi_i(x)\otimes\bar{\psi}_i(y)\ . \qquad (3.9)$$

[Substitution of this expression into (3.6) is seen to lead to (3.8).] It then follows from (3.9) that $K(t,x,y,I_2)$ satisfies the "heat equation"

$$\frac{\partial}{\partial t}K(t,x,y,I_2)+I_2K(t,x,y,I_2)=0\ . \qquad (3.10)$$

The boundary condition

$$K(0,x,y,I_2)=\delta(y,x) \qquad (3.11)$$

follows from the fact that as $t\to 0^+$, $(e^{-tI_2}\phi)(x)\to\phi(x)$. This is most easily seen from (3.8).

The ζ function used by Dowker and Critchley[26,27] is the Mellin transform of the heat kernel (with zero modes of the operator I_2 projected out). The ζ function used by Hawking[28] is the Mellin transform of the traced heat kernel.

Using (3.6) and (3.5), it follows that (interchanging the order of integration)

$$\Gamma^{(1)}[\hat{\phi}]=-\tfrac{1}{2}\int dv_x\int_0^{\infty}\frac{dt}{t}\mathrm{tr}K(t,x,x,I_2)\ , \qquad (3.12)$$

where the functional trace has been performed leaving only the trace over any indices which K might have. The heat kernel has the following asymptotic expansion as $t\to 0^+$:

$$K(t,x,x,I_2)\sim(4\pi t)^{-n/2}\sum_{m=0}^{\infty} t^m E_m(x,I_2)\ , \qquad (3.13)$$

where the $E_m(x,I_2)$ are endomorphisms of the fiber which are *local* invariants constructed from quantities occurring in I_2. This means that they do not depend upon the space-time topology or whether or not the bundle is trivial. Gilkey[46,47] assumes that the manifold is an n-dimensional compact Riemannian manifold without boundary. The extension to the noncompact case has been discussed by Wald[41] for the Klein-Gordon operator. The results given in the present paper will, therefore, extend to the noncompact case provided that surface terms may be ignored. (See the discussion concerning this point in Sec. VI.) Terms which are space-time integrals of total divergences will be neglected in this paper.

Split the integration in (3.12) into

$$\int_0^{\infty}\frac{dt}{t}K(t,x,x,I_2)=\int_0^{t_0}\frac{dt}{t}K(t,x,x,I_2)+\int_{t_0}^{\infty}\frac{dt}{t}K(t,x,x,I_2)\ , \qquad (3.14)$$

where t_0 is arbitrary. The second integral in (3.14) is convergent. Since t_0 is arbitrary, it may be taken to be as small as desired. The asymptotic expansion (3.13) may then be used in the first term of (3.14). This leads to

$$\int_0^{t_0}\frac{dt}{t}K(t,x,x,I_2)=(4\pi)^{-n/2}\sum_{m=0}^{\infty}\left[\frac{-2}{n-2m}\right]t_0^{\,m-n/2}E_m(x,I_2) \qquad (3.15)$$

provided that $n<2m$. Adopting dimensional regularization[25] where the space-time dimension n is treated as a regularizing parameter, it is seen that (3.12) is analytic for $\mathrm{Re}(n)<0$. If $\Gamma^{(1)}[\hat{\phi}]$ is analytically continued out of this region, simple poles are seen to occur at $n=0,2,4,6,\ldots$. As a result, the one-loop effective action is seen to be finite in space-times of odd dimensions, even in the presence of interactions. Let n_0 be the physical space-time dimension. If n_0 is even, the one-loop effective action has a pole part which is given from (3.15), (3.12) by

$$\mathrm{P.P.}\{\Gamma^{(1)}[\hat{\phi}]\}=(4\pi)^{-n_0/2}(n-n_0)^{-1}\times\int dv_x\mathrm{tr}E_{n_0/2}(x,I_2)\ . \qquad (3.16)$$

It is customary to define

$$a_n(x,I_2)=\mathrm{tr}E_n(x,I_2)\ , \qquad (3.17)$$

$$a_n(I_2)=\int dv_x a_n(x,I_2)\ , \tag{3.18}$$

in which case,

$$\text{P.P.}\{\Gamma^{(1)}[\hat{\phi}]\}=(4\pi)^{-n_0/2}(n-n_0)^{-1}\times\int dv_x a_{n_0/2}(x,I_2) \tag{3.19}$$

$$=(4\pi)^{-n_0/2}(n-n_0)^{-1}a_{n_0/2}(I_2)\ . \tag{3.20}$$

To save writing, the following definition is made:

$$\epsilon=(4\pi)^{n_0/2}(n-n_0)\ . \tag{3.21}$$

Note that in order to obtain (3.16), it is necessary that the extra dimensions which occur in the dimensional regularization procedure be flat (but not necessarily topologically trivial) with the n-dimensional manifold a direct product of these extra dimensions and the physical space-time. This has the consequence that the expansion coefficients occurring in (3.15) are formed using quantities formed from the physical space-time part of the operator I_2. See Refs. 28 and 41 for comments in the case where this is not done.

An expression for the pole part of the coincidence limit of the propagator in the presence of the background field may also be found using these methods. From (2.17),

$$\Delta=I_2{}^{-1}=\int_0^\infty dte^{-tI_2}\ . \tag{3.22}$$

Therefore, from the definition of the heat kernel

$$\Delta(x,x')=\int_0^\infty dtK(t,x,x',I_2)\ . \tag{3.23}$$

Let $x=x'$, split up the integration as in (3.14), and use the asymptotic expansion (3.13). This leads to

$$\text{P.P.}\{\Delta(x,x)\}=-2\epsilon^{-1}E_{n_0/2-1}(x,I_2) \tag{3.24}$$

for even n_0, and zero for odd n_0.

More generally consider

$$F(x,x_1,\ldots,x_k)=\Delta(x,x_1)\Delta(x_1,x_2)\cdots\Delta(x_k,x)\ . \tag{3.25}$$

Then,

$$\int dv_{x_1}\cdots dv_{x_k}\text{tr}F(x,x_1,\ldots,x_k)=\text{Tr}[\Delta^k]=\text{Tr}[I_2{}^{-k}]=\frac{1}{(k-1)!}\int_0^\infty dtt^{k-1}\text{tr}K(t,x,x,I_2)\ . \tag{3.26}$$

This expression contains a pole term only if n_0 is even and $k\le\frac{1}{2}n_0$. Using the asymptotic expansion as before, it is found that

$$\int dv_{x_1}\cdots dv_{x_k}\text{tr}F(x,x_1,\ldots,x_k)=\frac{-2}{(k-1)!}\epsilon^{-1}\text{tr}E_{n_0/2-k}(x,I_2)+\text{ finite terms}\ . \tag{3.27}$$

Therefore, up to terms which are total divergences,

$$\text{P.P.}\{F(x,x_1,..,x_k)\}=\frac{-2}{(k-1)!}\epsilon^{-1}E_{n_0/2-k}(x,I_2)\delta(x_1,x)\cdots\delta(x_k,x)\ , \tag{3.28}$$

if n_0 is even and $k\le\frac{1}{2}n_0$, and vanishes otherwise.

In the case of multicomponent scalar fields, I_2 will always be of the form

$$I_2=-\Box+Q\ , \tag{3.29}$$

where Q is in general a matrix. For this operator[46,47]

$$E_0(x,I_2)=1\ , \tag{3.30}$$

$$E_1(x,I_2)=\tfrac{1}{6}R-Q\ , \tag{3.31}$$

$$E_2(x,I_2)=\tfrac{1}{360}(12\Box R+5R^2-2R^{\mu\nu}R_{\mu\nu}+2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-60RQ+180Q^2-60\Box Q) \tag{3.32}$$

give the first three coefficients in the asymptotic expansion (3.13). (The sign conventions for the curvature used here are those of Ref. 49, which are opposite to those of Gilkey.[46,47]) In Sec. V, the result for E_3 is required. For operators of the form

(3.29), the result is easily found in Refs. 46 and 47. [Note a misprint in Ref. 47 where the last term in the third last line of theorem 4.3(d) should read $-4R_{ijik}\mathscr{E}_{;jk}$ in Gilkey's notation.] It is sufficient for the purposes of the present paper to have an expression for $a_3(I_2)$ in the case of a single scalar field, so that the fiber is the real line and Q in (3.29) is just a function. After using a number of curvature identities, and discarding terms which are integrals of total divergences, it follows that

$$a_3(I_2)=\frac{1}{7!}\int dv_x\Big(\tfrac{142}{9}R\Box R+\tfrac{26}{9}R^{\mu\nu}\Box R_{\mu\nu}+\tfrac{7}{9}R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma}+\tfrac{35}{9}R^3-\tfrac{14}{3}RR^{\mu\nu}R_{\mu\nu}+\tfrac{14}{3}RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$$
$$-4R_{\mu\nu}R^{\mu}{}_{\sigma}R^{\nu\sigma}+\tfrac{20}{9}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}-\tfrac{8}{9}R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^{\nu}{}_{\lambda\rho\sigma}+\tfrac{8}{3}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\tau}R^{\rho\sigma\lambda\tau}\Big)$$
$$+\tfrac{1}{360}\int dv_x(30Q\Box Q-12Q\Box R-60Q^3+30RQ^2-5R^2Q+2R^{\mu\nu}R_{\mu\nu}Q-2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}Q)\,. \tag{3.33}$$

It should be noted that the expressions in (3.30)–(3.33) are universal ones, and in particular the coefficients do not depend upon the dimension of the manifold.

IV. APPLICATION TO ϕ^4 THEORY IN FOUR DIMENSIONS

Although the results of the previous sections are applicable to multicomponent scalar fields, attention shall be restricted to only a single scalar field. This shows all of the relevant features of the calculation and has the advantage of shortening the expressions which are involved.

Take the bare matter action for the scalar fields to be

$$I_M[\phi]=\int dv_x\Big[-\tfrac{1}{2}\phi_B(x)\Box\phi_B(x)+\tfrac{1}{2}M_B{}^2\phi_B{}^2(x)+\tfrac{1}{2}\xi_B R(x)\phi_B{}^2(x)+\frac{1}{3!}g_B\phi_B{}^3(x)+\frac{1}{4!}\lambda_B\phi_B{}^4(x)$$
$$+h_B\phi_B(x)+\eta_B R(x)\phi_B(x)\Big]\,. \tag{4.1}$$

In addition to the linear or tadpole term $h_B\phi_B$ which is required in flat space-time for renormalizability, it is also necessary to include the nonminimal gravitational tadpole term $\eta_B R\phi_B$ if the theory is to be renormalizable in curved space-time. The term $\xi_B R\phi_B{}^2$ is the well-known one which leads to the so-called improvement term in the stress-energy tensor.[50] It is also necessary to include a bare Einstein-Hilbert gravitational action of the form[2]

$$I_G=-\int dv_x(\Lambda_B+\kappa_B R+\alpha_{1B}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\alpha_{2B}R^{\mu\nu}R_{\mu\nu}+\alpha_{3B}R^2)\,. \tag{4.2}$$

Here Λ_B is the bare cosmological constant and κ_B is related to the bare Newtonian gravitational constant.

All of the bare quantities appearing in (4.1) and (4.2) are now expressed in terms of renormalized ones plus counterterms, where the counterterms are given as a sum of pole terms at $n=4$.

$$M_B{}^2=M^2+\delta M^2\,, \tag{4.3a}$$
$$\xi_B=\xi+\delta\xi\,, \tag{4.3b}$$
$$\mu^{n/2-2}g_B=g+\delta g\,, \tag{4.3c}$$
$$\mu^{n-4}\lambda_B=\lambda+\delta\lambda\,, \tag{4.3d}$$
$$\mu^{2-n/2}h_B=h+\delta h\,, \tag{4.3e}$$
$$\mu^{2-n/2}\eta_B=\eta+\delta\eta\,, \tag{4.3f}$$
$$\mu^{4-n}\Lambda_B=\Lambda+\delta\Lambda\,, \tag{4.3g}$$
$$\mu^{4-n}\kappa_B=\kappa+\delta\kappa\,, \tag{4.3h}$$
$$\mu^{4-n}\alpha_{iB}=\alpha_i+\delta\alpha_i\quad(i=1,2,3)\,. \tag{4.3i}$$

The unit of mass μ is introduced[51,52] so that all of the renormalized couplings appearing in (4.3) have the same dimensions for all n as they do for $n=4$. In addition to (4.3), there is a renormalization of the background field

$$\mu^{2-n/2}\hat{\phi}_B(x)=Z^{1/2}\hat{\phi}(x)\ , \tag{4.4a}$$

where

$$Z=1+\delta Z\ . \tag{4.4b}$$

All of the counterterms appearing in (4.3) and (4.4) are given in the loop expansion as series in $\hbar$ of the form

$$\delta C=\sum_{n=1}^{\infty}\hbar^n\delta C^{(n)}\ , \tag{4.5}$$

where C stands for any of the couplings occurring in (4.3) or Z in (4.4a).

A. The one-loop effective action

The required expression for the one-loop effective action has been given in (2.16). Since I_{2ij} is defined to be the second functional derivative of the action evaluated at the background field, it follows from (4.1) that

$$I_2(x,x')=\mu^{n-4}[-\Box_x+M_B{}^2+\xi_BR(x)+g_B\hat{\phi}_B(x)+\tfrac{1}{2}\lambda_B\hat{\phi}_B{}^2(x)]\delta(x,x')\ . \tag{4.6}$$

Substitution of (4.3) and (4.4) into (4.6) and making the split defined in (2.13) leads to

$$I_2(x,x')=[-\Box_x+M^2+\xi R(x)+g\hat{\phi}(x)+\tfrac{1}{2}\lambda\hat{\phi}^2(x)]\delta(x,x')\ , \tag{4.7}$$

where the factor of μ^{4-n} may be dropped. Only the first term in the $\hbar$ expansion (2.14) of $I_2^{\rm CT}$ is required. This is

$$I_2^{(1)}(x,x')=\mu^{4-n}[\delta M^{2(1)}+\delta\xi^{(1)}R(x)+(\delta g^{(1)}+\tfrac{1}{2}\delta Z^{(1)}g)\hat{\phi}(x)+\tfrac{1}{2}(\delta\lambda^{(1)}+\delta Z^{(1)}\lambda)\hat{\phi}^2(x)]\delta(x,x')\ . \tag{4.8}$$

Higher-order terms are easily found.

The pole part of $\frac{1}{2}\ln {\rm Det} I_2^R$ is, from (3.16) and (3.21),

$$\text{P.P.}\{\tfrac{1}{2}\ln {\rm Det}I_2^R\}=\epsilon^{-1}\mu^{n-4}\int dv_x {\rm tr}E_2(x,I_2^R)\ . \tag{4.9}$$

The factor of μ^{n-4} ensures the correct dimensionality of $\Gamma^{(1)}[\hat{\phi}]$. I_2^R is seen from (4.7) to be of the form (3.29) where

$$Q=M^2+\xi R(x)+g\hat{\phi}(x)+\tfrac{1}{2}\lambda\hat{\phi}^2(x)\ . \tag{4.10}$$

From (3.32), noting that since the fiber is one dimensional the trace is trivial, it is found that

$$\begin{aligned}\text{P.P.}\{\tfrac{1}{2}\ln {\rm Det}I_2^R\}=\epsilon^{-1}\mu^{n-4}\int dv_x\big(&\tfrac{1}{180}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-\tfrac{1}{180}R^{\mu\nu}R_{\mu\nu}+\tfrac{1}{2}(\xi-\tfrac{1}{6})^2R^2+M^2(\xi-\tfrac{1}{6})R+\tfrac{1}{2}M^4\\&+[M^2+(\xi-\tfrac{1}{6})R]g\hat{\phi}(x)+\tfrac{1}{2}\{[M^2+(\xi-\tfrac{1}{6})R]\lambda+g^2\}\hat{\phi}^2(x)\\&+\tfrac{1}{2}g\lambda\hat{\phi}^3(x)+\tfrac{1}{8}\lambda^2\hat{\phi}^4(x)\big)\ .\end{aligned} \tag{4.11}$$

(Total divergences have been discarded.) The pole term which is linear in the background field necessitates the two tadpole terms occurring in (4.1) for renormalizability.

The complete expression for the pole part of the one-loop effective action is from (2.16)

$$\text{P.P.}\{\Gamma^{(1)}[\hat{\phi}]\}=\text{P.P.}\{I^{(1)}[\hat{\phi}]\}+\text{P.P.}\{\tfrac{1}{2}\ln{\rm Det}I_2^R\}\ , \tag{4.12}$$

where $I^{(1)}$ is the term of order $\hbar$ which occurs when (4.3) and (4.4) are substituted into (4.1) and (4.2); thus,

$$\begin{aligned}\text{P.P.}\{I^{(1)}[\hat{\phi}]\}=\mu^{n-4}\int dv_x\Big[&-\tfrac{1}{2}\delta Z^{(1)}\hat{\phi}(x)\Box\hat{\phi}(x)+\tfrac{1}{2}(\delta M^{2(1)}+M^2\delta Z^{(1)})\hat{\phi}^2(x)+\tfrac{1}{2}(\delta\xi^{(1)}+\xi\delta Z^{(1)})R(x)\hat{\phi}^2(x)\\&+\frac{1}{3!}(\delta g^{(1)}+\tfrac{3}{2}g\delta Z^{(1)})\hat{\phi}^3(x)+\frac{1}{4!}(\delta\lambda^{(1)}+2\lambda\delta Z^{(1)})\hat{\phi}^4(x)+(\delta h^{(1)}+\tfrac{1}{2}h\delta Z^{(1)})\hat{\phi}(x)\\&+(\delta\eta^{(1)}+\tfrac{1}{2}\eta\delta Z^{(1)})R(x)\hat{\phi}(x)-\delta\Lambda^{(1)}-\delta\kappa^{(1)}R-\delta\alpha_1^{(1)}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\\&-\delta\alpha_2^{(1)}R^{\mu\nu}R_{\mu\nu}-\delta\alpha_3^{(1)}R^2\Big]\ .\end{aligned} \tag{4.13}$$

Demanding that $\lim_{\epsilon\to 0}\Gamma^{(1)}[\hat{\phi}]$ be finite, fixes the counterterms in (4.13) to cancel off the poles in (4.11). This gives

$$\delta\Lambda^{(1)}=\tfrac{1}{2}M^4\epsilon^{-1}\ , \tag{4.14a}$$

$$\delta\kappa^{(1)}=M^2(\xi-\tfrac{1}{6})\epsilon^{-1}\ , \tag{4.14b}$$

$$\delta\alpha_1^{(1)}=\tfrac{1}{180}\epsilon^{-1}\ , \tag{4.14c}$$

$$\delta\alpha_2^{(1)}=-\tfrac{1}{180}\epsilon^{-1}\ , \tag{4.14d}$$

$$\delta\alpha_3^{(1)}=\tfrac{1}{2}(\xi-\tfrac{1}{6})^2\epsilon^{-1}\ , \tag{4.14e}$$

$$\delta Z^{(1)}=0\ , \tag{4.14f}$$

$$\delta M^{2(1)}=-(M^2\lambda+g^2)\epsilon^{-1}\ , \tag{4.14g}$$

$$\delta\xi^{(1)}=-(\xi-\tfrac{1}{6})\lambda\epsilon^{-1}\ , \tag{4.14h}$$

$$\delta g^{(1)}=-3g\lambda\epsilon^{-1}\ , \tag{4.14i}$$

$$\delta\lambda^{(1)}=-3\lambda^2\epsilon^{-1}\ , \tag{4.14j}$$

$$\delta h^{(1)}=-M^2g\epsilon^{-1}\ , \tag{4.14k}$$

$$\delta\eta^{(1)}=-(\xi-\tfrac{1}{6})g\epsilon^{-1} \tag{4.14l}$$

as the one-loop counterterms.

B. The two-loop effective action

The expression for the two-loop effective action has been given in (2.21). Explicit expressions for $I^{(0)}_{3ijk}$ and $I^{(0)}_{4ijkl}$ are required. Functional differentiation of (4.6) gives

$$I_3(x,x',x'')=\mu^{3n/2-6}[g_B+\lambda_B\hat{\phi}_B(x)]\delta(x,x')\delta(x,x'')\ , \tag{4.15}$$

$$I_4(x,x',x'',x''')=\mu^{2n-8}\lambda_B\delta(x,x')\delta(x,x'')\delta(x,x''')\ . \tag{4.16}$$

Defining the $\hbar$ expansions as in (2.19) and (2.20) leads to

$$I_3^{(0)}(x,x',x'')=\mu^{n-4}[g+\lambda\hat{\phi}(x)]\delta(x,x')\delta(x,x'')\ , \tag{4.17}$$

$$I_4^{(0)}(x,x',x'',x''')=\mu^{n-4}\lambda\delta(x,x')\delta(x,x'')\delta(x,x''')\ . \tag{4.18}$$

From (4.8), using the fact that no one-loop field renormalization is required (4.14f),

$$I_2^{(1)}(x,x')=\mu^{n-4}[\,\delta M^{2(1)}+\delta\xi^{(1)}R(x)+\delta g^{(1)}\hat{\phi}(x)+\tfrac{1}{2}\delta\lambda^{(1)}\hat{\phi}^2(x)]\delta(x,x')\ . \tag{4.19}$$

The contributions to the two-loop effective action shown in Fig. 1 are seen to be

$$\Gamma^{(2)}_{(a)}[\hat{\phi}]=\tfrac{1}{2}\mu^{n-4}\int dv_x\left[\delta M^{2(1)}+\delta\xi^{(1)}R+\delta g^{(1)}\hat{\phi}(x)+\tfrac{1}{2}\delta\lambda^{(1)}\hat{\phi}^2(x)\right]\Delta(x,x)\ , \tag{4.20a}$$

$$\Gamma^{(2)}_{(b)}[\hat{\phi}]=\tfrac{1}{8}\lambda\mu^{n-4}\int dv_x\Delta^2(x,x)\ , \tag{4.20b}$$

$$\Gamma^{(2)}_{(c)}[\hat{\phi}]=-\tfrac{1}{12}\mu^{n-4}\int dv_x dv_{x'}\left[g+\lambda\hat{\phi}(x)\right]\left[g+\lambda\hat{\phi}(x')\right]\Delta^3(x,x')\ . \tag{4.20c}$$

The divergent parts of $\Gamma^{(2)}_{(a)}$ and $\Gamma^{(2)}_{(b)}$ follow easily from results given in Sec. III, using the heat-kernel expansion. From (3.24),

$$\text{P.P.}\{\Delta(x,x)\}=-2\epsilon^{-1}E_1(x,I_2^R)\ . \tag{4.21}$$

For Q given in (4.10) and E_1 given in (3.31) this becomes

$$\text{P.P.}\{\Delta(x,x)\}=2\epsilon^{-1}[M^2+(\xi-\tfrac{1}{6})R(x)+g\hat{\phi}(x)+\tfrac{1}{2}\lambda\hat{\phi}^2(x)]\ . \tag{4.22}$$

The expression for $\Delta(x,x)$ may, therefore, be written as

$$\Delta(x,x)=2\epsilon^{-1}[M^2+(\xi-\tfrac{1}{6})R+g\hat{\phi}+\tfrac{1}{2}\lambda\hat{\phi}^2]+\Delta_R(x)+O(\epsilon)\ , \tag{4.23}$$

where $\Delta_R(x)$ is finite as $\epsilon\to 0$. $\Delta_R(x)$ is in general a nonlocal expression involving the curvature and the background field. It also contains quantities like $ln(M^2/4\pi\mu^2)$ and the Euler constant which are familiar from flat space-time. The expression for $\Delta_R(x)$ is different for the inequivalent scalar fields which exist if the space-time is not simply connected;[8,19,20,38] thus, it also contains information about the type of bundle of which the field is a cross section.

Using (4.23) in (4.20a), with the one-loop counterterms given in (4.14) leads to

$$\text{P.P.}\{\Gamma^{(2)}_{(a)}[\hat{\phi}]\}=\mu^{n-4}\int dv_x\{-\epsilon^{-2}[(\lambda M^2+g^2)M^2+(2\lambda M^2+g^2)(\xi-\tfrac{1}{6})R+\lambda(\xi-\tfrac{1}{6})^2R^2+(4\lambda M^2+g^2)g\hat{\phi}(x)$$
$$+4\lambda g(\xi-\tfrac{1}{6})R\hat{\phi}(x)+(2\lambda M^2+\tfrac{7}{2}g^2)\lambda\hat{\phi}^2(x)$$
$$+2\lambda^2(\xi-\tfrac{1}{6})R\hat{\phi}^2(x)+3\lambda^2 g\hat{\phi}^3(x)+\tfrac{3}{4}\lambda^3\hat{\phi}^4(x)]$$
$$-\epsilon^{-1}[\tfrac{1}{2}\lambda M^2+\tfrac{1}{2}g^2+\tfrac{1}{2}\lambda(\xi-\tfrac{1}{6})R+\tfrac{3}{2}\lambda g\hat{\phi}(x)+\tfrac{3}{4}\lambda^2\hat{\phi}^2(x)]\Delta_R(x)\}\ . \quad (4.24)$$

Substituting (4.23) into (4.20b) gives

$$\text{P.P.}\{\Gamma^{(2)}_{(b)}[\hat{\phi}]\}=\mu^{n-4}\int dv_x\{\epsilon^{-2}[\tfrac{1}{2}\lambda M^4+\lambda M^2(\xi-\tfrac{1}{6})R+\tfrac{1}{2}\lambda(\xi-\tfrac{1}{6})^2R^2+\lambda gM^2\hat{\phi}(x)+\lambda g(\xi-\tfrac{1}{6})R\hat{\phi}(x)$$
$$+\tfrac{1}{2}(\lambda M^2+g^2)\lambda\hat{\phi}^2(x)+\tfrac{1}{2}\lambda^2(\xi-\tfrac{1}{6})R\hat{\phi}^2(x)+\tfrac{1}{2}\lambda^2 g\hat{\phi}^3(x)+\tfrac{1}{8}\lambda^3\hat{\phi}^4(x)]$$
$$+\epsilon^{-1}[\tfrac{1}{2}\lambda M^2+\tfrac{1}{2}\lambda(\xi-\tfrac{1}{6})R+\tfrac{1}{2}\lambda g\hat{\phi}(x)+\tfrac{1}{4}\lambda^2\hat{\phi}^2(x)]\Delta_R(x)\}\ . \quad (4.25)$$

Unfortunately, it does not appear possible to calculate the pole part of $\Gamma^{(2)}_{(c)}$ using heat-kernel methods. Consider instead the following argument. $\Delta^3(x,x')$ has dimension six and transforms as a biscalar under coordinate transformations. Therefore, it necessarily takes the form

$$\Delta^3(x,x')=\mathscr{O}_2(x)\delta(x,x')+\mathscr{O}_6(x,x')\ , \quad (4.26)$$

where $\mathscr{O}_2(x)$ is an operator of dimension two which transforms as a scalar, and $\mathscr{O}_6(x,x')$ is an operator of dimension six, which transforms as a biscalar. Only the pole parts of these two operators need be considered. If $\mathscr{O}_6(x,x')$ contains any pole terms, from (4.20c) it is seen that there will be a contribution to the pole part of the two-loop effective action involving $\hat{\phi}(x)\hat{\phi}(x')$. This is impossible to remove in a theory with a local Lagrangian, and thus it would be necessary to conclude that the theory is nonrenormalizable. It is first argued that $\mathscr{O}_6(x,x')$ contains no pole terms.

The products of local curvature-independent quantities such as $M^2\hat{\phi}(x)\hat{\phi}(x')$ times pole terms may be immediately ruled out by the knowledge that the theory is renormalizable in flat space-time with an R^4 topology.[53] The presence of nonlocal curvature-independent divergences such as those arising from a nontrivial topology may be ruled out by the calculations done[8,19,20] in $S^1\times R^3$ where such terms would have occurred had they existed. This leaves only the possibility of nonlocal curvature-dependent divergences which may be ruled out by the calculations[13] done in S^4, or by Birrell and Ford[8] in a spatially flat Robertson-Walker universe with one of the spatial coordinates periodically identified to give a nontrivial topology. (Any of these types of divergences which one might think of [such as $R^2(x)R(x')$ times a dimensionless function of RM^{-2}] would appear to be pathological in certain limits.) The conclusion is that $\mathscr{O}_6(x,x')$ must be finite as $\epsilon\to 0$, and that any pole terms must come from $\mathscr{O}_2(x)$.

Since $\mathscr{O}_2(x)$ has dimension two and is a scalar, it must be a linear combination of the following dimension-two quantities:

$$\Box_x,\ M^2,\ R(x),\ g\hat{\phi}(x),\ \hat{\phi}^2(x),\ F(x)\ , \quad (4.27)$$

where $F(x)$ is any nonlocal expression of dimension two which may occur as a result of curvature or a nontrivial topology. [Terms like $M\hat{\phi}(x)$ which are linear in the mass are forbidden because $\Delta(x,x')$ is symmetric under $M\to -M$.] The combination of references quoted in the previous paragraph is sufficient to rule out any other possibilities, to show that $F(x)=\Delta_R(x)$ and to fix the pole term coefficients of (4.27). The coefficient of $g\hat{\phi}+\frac{1}{2}\lambda\hat{\phi}^2$ must be the same as that of M^2 because derivatives of $\hat{\phi}$ do not appear so that the coefficient of this term may be fixed by repeating the flat space-time calculation with a constant background field. This results in the replacement of M^2 in the result obtained by Collins[53] for example, with $M^2+g\hat{\phi}+\frac{1}{2}\lambda\hat{\phi}^2$. This leads to

$$\text{P.P.}\{\mathscr{O}_2(x)\}=[-6\epsilon^{-2}+3(4\pi)^{-2}\epsilon^{-1}][M^2+(\xi-\tfrac{1}{6})R(x)+g\hat{\phi}(x)+\tfrac{1}{2}\lambda\hat{\phi}^2(x)]$$
$$-\tfrac{1}{2}(4\pi)^{-2}\epsilon^{-1}\Box_x-6\epsilon^{-1}\Delta_R(x)-\tfrac{1}{12}(4\pi)^{-2}\epsilon^{-1}R(x)\ . \quad (4.28)$$

From (4.20c), using (4.26) and (4.28) it follows that

$$\text{P.P.}\{\Gamma^{(2)}_{(c)}[\hat{\phi}]\}=\mu^{n-4}\int dv_x\{\tfrac{1}{24}(4\pi)^{-2}\epsilon^{-1}\lambda^2\hat{\phi}(x)\Box_x\hat{\phi}(x) + \tfrac{1}{4}[2\epsilon^{-2}-(4\pi)^{-2}\epsilon^{-1}][g+\lambda\hat{\phi}(x)]^2[M^2+(\xi-\tfrac{1}{6})R(x)+g\hat{\phi}(x)+\tfrac{1}{2}\lambda\hat{\phi}^2(x)] + \tfrac{1}{144}(4\pi)^{-2}\epsilon^{-1}R(x)[g+\lambda\hat{\phi}(x)]^2+\epsilon^{-1}[\tfrac{1}{2}g^2+\lambda g\hat{\phi}(x)+\tfrac{1}{2}\lambda^2\hat{\phi}^2(x)]\Delta_R(x)\}\ . \tag{4.29}$$

The pole part of the two-loop contribution from the graphs shown in Fig. 1 is obtained by adding (4.24), (4.25), and (4.29). This gives

$$\begin{aligned}\text{P.P.}\{\Gamma^{(2)}_{(a)+(b)+(c)}[\hat{\phi}]\}=\mu^{n-4}\int dv_x\{&\tfrac{1}{24}(4\pi)^{-2}\epsilon^{-1}\lambda^2\hat{\phi}(x)\Box_x\hat{\phi}(x)\\ &+\epsilon^{-2}[-\tfrac{1}{2}\lambda M^4-\tfrac{1}{2}g^2M^2-(\lambda M^2+\tfrac{1}{2}g^2)(\xi-\tfrac{1}{6})R(x)-\tfrac{1}{2}\lambda(\xi-\tfrac{1}{6})^2R^2(x)\\ &\quad-(2M^2\lambda g+\tfrac{1}{2}g^3)\hat{\phi}(x)-2\lambda g(\xi-\tfrac{1}{6})R\hat{\phi}(x)-(\lambda^2M^2+\tfrac{7}{4}\lambda g^2)\hat{\phi}^2(x)\\ &\quad-\lambda^2(\xi-\tfrac{1}{6})R(x)\hat{\phi}^2(x)-\tfrac{3}{2}\lambda^2g\hat{\phi}^3(x)-\tfrac{3}{8}\lambda^3\hat{\phi}^4(x)]\\ &-\tfrac{1}{4}(4\pi)^{-2}\epsilon^{-1}[M^2g^2+(\xi-\tfrac{7}{36})g^2R(x)+(2M^2\lambda g+g^3)\hat{\phi}(x)\\ &\quad+2(\xi-\tfrac{7}{36})\lambda gR(x)\hat{\phi}(x)+(\lambda^2M^2+\tfrac{5}{2}\lambda g^2)\hat{\phi}^2(x)\\ &\quad+\lambda^2(\xi-\tfrac{7}{36})R(x)\hat{\phi}^2(x)+2\lambda^2g\hat{\phi}^3(x)+\tfrac{1}{2}\lambda^3\hat{\phi}^4(x)]\}\ .\end{aligned} \tag{4.30}$$

Finally, all that is required to complete the two-loop calculation is the $\mathscr{O}(\hbar^2)$ term in (2.12). This is (noting that $\delta Z^{(1)}=0$)

$$\begin{aligned}I^{(2)}[\hat{\phi}]=\mu^{n-4}\int dv_x\Big[&-\tfrac{1}{2}\delta Z^{(2)}\hat{\phi}(x)\Box_x\hat{\phi}(x)+\tfrac{1}{2}(\delta M^{2(2)}+M^2\delta Z^{(2)})\hat{\phi}^2(x)+\tfrac{1}{2}(\delta\xi^{(2)}+\xi\delta Z^{(2)})R(x)\hat{\phi}^2(x)\\ &+\frac{1}{3!}(\delta g^{(2)}+\tfrac{3}{2}g\delta Z^{(2)})\hat{\phi}^3(x)+\frac{1}{4!}(\delta\lambda^{(2)}+2\lambda\delta Z^{(2)})\hat{\phi}^4(x)\\ &+(\delta h^{(2)}+\tfrac{1}{2}h\delta Z^{(2)})\hat{\phi}(x)+(\delta\eta^{(2)}+\tfrac{1}{2}\eta\delta Z^{(2)})R(x)\hat{\phi}(x)-\delta\Lambda^{(2)}\\ &-\delta\kappa^{(2)}R(x)-\delta\alpha_1^{(2)}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-\delta\alpha_2^{(2)}R^{\mu\nu}R_{\mu\nu}-\delta\alpha_3^{(2)}R^2\Big]\ .\end{aligned} \tag{4.31}$$

The divergences occurring in (4.30) are seen to be of types which are removable by the counterterms in (4.35). This is because the nonlocal divergences containing $\Delta_R(x)$ which have arisen separately among the graphs of Fig. 1 have cancelled among each other. The counterterms are fixed by the requirement that

$$\lim_{\epsilon\to 0}\Gamma^{(2)}[\hat{\phi}] \tag{4.32}$$

be finite. The two-loop counterterms are therefore

$$\delta Z^{(2)}=\tfrac{1}{12}(4\pi)^{-2}\lambda^2\epsilon^{-1}\ , \tag{4.33a}$$

$$\delta\alpha_1^{(2)}=\delta\alpha_2^{(2)}=0\ , \tag{4.33b}$$

$$\delta\alpha_3^{(2)}=-\tfrac{1}{2}\lambda(\xi-\tfrac{1}{6})^2\epsilon^{-2}\ , \tag{4.33c}$$

$$\delta\kappa^{(2)}=-(\lambda M^2+\tfrac{1}{2}g^2)(\xi-\tfrac{1}{6})\epsilon^{-2} -\tfrac{1}{4}(4\pi)^{-2}g^2(\xi-\tfrac{7}{36})\epsilon^{-1}\ , \tag{4.33d}$$

$$\delta\Lambda^{(2)}=-\tfrac{1}{2}(\lambda M^4+g^2M^2)\epsilon^{-2} -\tfrac{1}{4}(4\pi)^{-2}g^2M^2\epsilon^{-1}\ , \tag{4.33e}$$

$$\delta M^{2(2)}=(2\lambda^2M^2+\tfrac{7}{2}\lambda g^2)\epsilon^{-2} +\tfrac{5}{12}(4\pi)^{-2}(\lambda^2M^2+3\lambda g^2)\epsilon^{-1}\ , \tag{4.33f}$$

$$\delta\xi^{(2)}=2\lambda^2(\xi-\tfrac{1}{6})\epsilon^{-2} +\tfrac{5}{12}(4\pi)^{-2}\lambda^2(\xi-\tfrac{7}{30})\epsilon^{-1}\ , \tag{4.33g}$$

$$\delta g^{(2)}=9\lambda^2g\epsilon^{-2} +\tfrac{23}{8}(4\pi)^{-2}\lambda^2g\epsilon^{-1}\ , \tag{4.33h}$$

$$\delta\lambda^{(2)}=9\lambda^3\epsilon^{-2} +\tfrac{17}{6}(4\pi)^{-2}\lambda^3\epsilon^{-1}\ , \tag{4.33i}$$

$$\delta h^{(2)}=\tfrac{1}{2}(g^3+4M^2\lambda g)\epsilon^{-2}+\tfrac{1}{4}(4\pi)^{-2}(g^3+2M^2\lambda g-\tfrac{1}{6}\lambda^3 h)\epsilon^{-1}\,, \tag{4.33j}$$

$$\delta\eta^{(2)}=2\lambda g(\xi-\tfrac{1}{6})\epsilon^{-2}+\tfrac{1}{2}(4\pi)^{-2}[(\xi-\tfrac{7}{36})\lambda g-\tfrac{1}{12}\lambda^2\eta]\epsilon^{-1}\,. \tag{4.33k}$$

This completes the proof of renormalizability and computation of the two-loop counterterms.

V. APPLICATION TO ϕ^3 THEORY IN SIX DIMENSIONS

Consider a single scalar field with a cubic self-interaction in a six-dimensional space-time. Such a theory has been discussed in flat space-time by Macfarlane and Woo,[54] in S^4 by Drummond,[13,14] and in a spatially flat Robertson-Walker universe by Gass.[16] The theory is discussed in a general curved space-time in this section and some errors in Ref. 16 are corrected.

The bare matter action must contain, as in flat space-time,

$$\int dv_x\Big(-\tfrac{1}{2}\phi_B\Box\phi_B+\tfrac{1}{2}M_B{}^2\phi_B{}^2+\frac{1}{3!}g_B\phi_B{}^3+h_B\phi_B\Big)\,. \tag{5.1}$$

If only these terms were present, this would represent the minimally coupled theory, which, as shall be seen below, would be nonrenormalizable in curved space-time. It would also be expected to have a nonrenormalizable stress-energy tensor in flat space-time. Extra nonminimal terms, similar to the familiar $\xi R\phi^2$ term in four dimensions must be added. Attention may be restricted to local expressions involving the curvature and the fields which are of dimension six and which are independent up to a total divergence.

The only terms in the field such that invariants representing a coupling to the background geometry may be formed are

$$\phi,\nabla_\mu\phi,\phi^2,\nabla_\mu\nabla_\nu\phi,\Box\phi\,.$$

This restricts the allowed expressions in the curvature to be a dimension-two second-rank tensor, a dimension-three vector, or a scalar of dimension two, three, or four. The only terms which are independent up to a total divergence are

$$\phi\Box R,\phi R^2,\phi R^{\mu\nu}R_{\mu\nu},\phi R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma},\phi^2 R,\phi R\,.$$

Therefore, there are six allowed nonminimal terms in contrast to the two which appeared in the last section. The bare scalar-field matter action is taken to be

$$I_M[\phi]=\int dv_x\Big[-\tfrac{1}{2}\phi_B\Box\phi_B+\tfrac{1}{2}M_B{}^2\phi_B{}^2+\frac{1}{3!}g_B\phi_B{}^3+h_B\phi_B+\tfrac{1}{2}\xi_B R\phi_B{}^2+\eta_{1B}R\phi_B+\eta_{2B}\phi_B\Box R+\eta_{3B}\phi_B R^2+\eta_{4B}\phi_B R^{\mu\nu}R_{\mu\nu}+\eta_{5B}\phi_B R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\Big]\,. \tag{5.2}$$

An expression for the bare Einstein-Hilbert gravitational action in six dimensions is also required. In addition to the terms appearing in (4.2) it is necessary to add on those curvature invariants of dimension six with dimensionless coupling constants, which are independent up to a total divergence. There are ten of these which may be found from Sakai[45] or Gilkey.[46,47] The bare gravitational action is taken to be

$$\begin{aligned}I_G=-\int dv_x(&\Lambda_B+\kappa_B R+\alpha_{1B}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\alpha_{2B}R^{\mu\nu}R_{\mu\nu}+\alpha_{3B}R^2\\&+\gamma_{1B}R\Box R+\gamma_{2B}R^{\mu\nu}\Box R_{\mu\nu}+\gamma_{3B}R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma}+\gamma_{4B}R^3\\&+\gamma_{5B}RR^{\mu\nu}R_{\mu\nu}+\gamma_{6B}RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\gamma_{7B}R_{\mu\nu}R^{\mu}{}_{\sigma}R^{\nu\sigma}+\gamma_{8B}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma}\\&+\gamma_{9B}R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^{\nu}{}_{\lambda\rho\sigma}+\gamma_{10B}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\tau}R^{\rho\sigma\lambda\tau})\,.\end{aligned} \tag{5.3}$$

Define the counterterms and renormalized quantities by

$$\mu^{3-n/2}\hat{\phi}_B(x)=Z^{1/2}\hat{\phi}(x)\,, \tag{5.4a}$$

$$Z=1+\delta Z\,, \tag{5.4b}$$

$$M_B{}^2=M^2+\delta M^2\,, \tag{5.4c}$$

$$\xi_B=\xi+\delta\xi\,, \tag{5.4d}$$

$$\mu^{n/2-3}g_B=g+\delta g\,, \tag{5.4e}$$

$$\mu^{3-n/2}h_B = h + \delta h \ , \tag{5.4f}$$

$$\mu^{3-n/2}\eta_{iB} = \eta_i + \delta\eta_i \quad (i=1,\ldots,5) \ , \tag{5.4g}$$

$$\mu^{6-n}\Lambda_B = \Lambda + \delta\Lambda \ , \tag{5.4h}$$

$$\mu^{6-n}\kappa_B = \kappa + \delta\kappa \ , \tag{5.4i}$$

$$\mu^{6-n}\alpha_{iB} = \alpha_i + \delta\alpha_i \quad (i=1,2,3) \ , \tag{5.4j}$$

$$\mu^{6-n}\gamma_{iB} = \gamma_i + \delta\gamma_i \quad (i=1,\ldots,10) \ . \tag{5.4k}$$

All counterterms have loop expansions of the form (4.5).

A. The One-Loop Effective Action

It is clear that $I_2(x,x')$ may be obtained from that of the last section by setting $\lambda=0$. Thus,

$$I_2^R(x,x') = [-\Box_x + M^2 + \xi R(x) + g\hat{\phi}(x)]\delta(x,x') \ . \tag{5.5}$$

From (3.20) and (3.21),

$$\text{P.P.}\{\tfrac{1}{2}\ln \text{Det} I_2^R\} = \epsilon^{-1} a_3(I_2^R) \ , \tag{5.6}$$

where $a_3(I_2^R)$ is given in (3.33). The quantity Q which enters the expression for a_3 is

$$Q = M^2 + \xi R(x) + g\hat{\phi}(x) \ . \tag{5.7}$$

After some algebra, discarding terms which are total divergences, it is found that

$$\begin{aligned}
\text{P.P.}\{\tfrac{1}{2}\ln \text{Det} I_2^R\} = \mu^{n-6}\epsilon^{-1}\int dv_x \{ & -\tfrac{1}{6}M^6 - \tfrac{1}{2}M^4(\xi-\tfrac{1}{6})R - \tfrac{1}{2}M^2(\xi-\tfrac{1}{6})^2R^2 + \tfrac{1}{180}M^2(R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}) \\
& + \tfrac{1}{12}[(\xi-\tfrac{1}{5})^2 - \tfrac{23}{9450}]R\Box R + \tfrac{13}{22680}R^{\mu\nu}\Box R_{\mu\nu} + \tfrac{1}{6480}R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} \\
& - \tfrac{1}{6}(\xi-\tfrac{1}{6})^3R^3 + \tfrac{1}{180}(\xi-\tfrac{1}{6})RR^{\mu\nu}R_{\mu\nu} - \tfrac{1}{180}(\xi-\tfrac{1}{6})RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\
& - \tfrac{1}{1260}R_{\mu\nu}R^{\mu}{}_{\sigma}R^{\nu\sigma} + \tfrac{1}{2268}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} - \tfrac{1}{5670}R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^{\nu}{}_{\lambda\rho\sigma} \\
& + \tfrac{1}{1890}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\tau}R^{\rho\sigma\lambda\tau} + \tfrac{1}{180}gR^{\mu\nu}R_{\mu\nu}\hat{\phi}(x) - \tfrac{1}{180}gR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\
& + \tfrac{1}{6}(\xi-\tfrac{1}{5})g\hat{\phi}(x)\Box R - \tfrac{1}{2}M^4g\hat{\phi}(x) - M^2g(\xi-\tfrac{1}{6})R\hat{\phi}(x) - \tfrac{1}{2}g(\xi-\tfrac{1}{6})^2R^2\hat{\phi}(x) \\
& + \tfrac{1}{12}g^2\hat{\phi}(x)\Box\hat{\phi}(x) - \tfrac{1}{2}M^2g^2\hat{\phi}^2(x) - \tfrac{1}{2}(\xi-\tfrac{1}{6})g^2R\hat{\phi}^2(x) - \tfrac{1}{6}g^3\hat{\phi}^3(x)\} \ .
\end{aligned} \tag{5.8}$$

The term of order $\hbar$ in the expansion (2.12) of the classical action is

$$\begin{aligned}
I^{(1)}[\hat{\phi}] = \mu^{n-6}\int dv_x[& -\tfrac{1}{2}\delta Z^{(1)}\hat{\phi}(x)\Box\hat{\phi}(x) + \tfrac{1}{2}(\delta M^{2(1)} + M^2\delta Z^{(1)})\hat{\phi}^2(x) + \tfrac{1}{2}(\delta\xi^{(1)} + \xi\delta Z^{(1)})R\hat{\phi}^2(x) \\
& + \frac{1}{3!}(\delta g^{(1)} + \tfrac{3}{2}g\delta Z^{(1)})\hat{\phi}^3(x) + (\delta h^{(1)} + \tfrac{1}{2}h\delta Z^{(1)})\hat{\phi}(x) + (\delta\eta_1^{(1)} + \tfrac{1}{2}\eta_1\delta Z^{(1)})R\hat{\phi}(x) \\
& + (\delta\eta_2^{(1)} + \tfrac{1}{2}\eta_2\delta Z^{(1)})\hat{\phi}(x)\Box R + (\delta\eta_3^{(1)} + \tfrac{1}{2}\eta_3\delta Z^{(1)})\hat{\phi}(x)R^2 + (\delta\eta_4^{(1)} + \tfrac{1}{2}\eta_4\delta Z^{(1)})\hat{\phi}(x)R^{\mu\nu}R_{\mu\nu} \\
& + (\delta\eta_5^{(1)} + \tfrac{1}{2}\eta_5\delta Z^{(1)})\hat{\phi}(x)R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - \delta\Lambda^{(1)} - \delta\kappa^{(1)}R \\
& - \delta\alpha_1^{(1)}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - \delta\alpha_2^{(1)}R^{\mu\nu}R_{\mu\nu} - \delta\alpha_3^{(1)}R^2 \\
& - \delta\gamma_1^{(1)}R\Box R - \delta\gamma_2^{(1)}R^{\mu\nu}\Box R_{\mu\nu} - \delta\gamma_3^{(1)}R^{\mu\nu\rho\sigma}\Box R_{\mu\nu\rho\sigma} \\
& - \delta\gamma_4^{(1)}R^3 - \delta\gamma_5^{(1)}RR^{\mu\nu}R_{\mu\nu} - \delta\gamma_6^{(1)}RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\
& - \delta\gamma_7^{(1)}R_{\mu\nu}R^{\mu}{}_{\sigma}R^{\nu\sigma} - \delta\gamma_8^{(1)}R_{\mu\nu}R_{\rho\sigma}R^{\mu\rho\nu\sigma} - \delta\gamma_9^{(1)}R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^{\nu}{}_{\lambda\rho\sigma} - \delta\gamma_{10}^{(1)}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\tau}R^{\rho\sigma\lambda\tau}] \ .
\end{aligned} \tag{5.9}$$

The expression (2.16) for the one-loop effective action is finite as $\epsilon \rightarrow 0$ provided that the counterterms are fixed by

$$\delta Z^{(1)} = \tfrac{1}{6} g^2 \epsilon^{-1} , \quad (5.10a)$$

$$\delta M^{2(1)} = \tfrac{5}{6} M^2 g^2 \epsilon^{-1} , \quad (5.10b)$$

$$\delta g^{(1)} = \tfrac{3}{4} g^3 \epsilon^{-1} , \quad (5.10c)$$

$$\delta h^{(1)} = \tfrac{1}{12} g (6M^4 - gh) \epsilon^{-1} , \quad (5.10d)$$

$$\delta \xi^{(1)} = \tfrac{5}{6} (\xi - \tfrac{1}{5}) g^2 \epsilon^{-1} , \quad (5.10e)$$

$$\delta \eta_1^{(1)} = \tfrac{1}{12} g [12 M^2 (\xi - \tfrac{1}{6}) - g \eta_1] \epsilon^{-1} , \quad (5.10f)$$

$$\delta \eta_2^{(1)} = -\tfrac{1}{12} g [2(\xi - \tfrac{1}{5}) + g \eta_2] \epsilon^{-1} , \quad (5.10g)$$

$$\delta \eta_3^{(1)} = \tfrac{1}{12} g [6(\xi - \tfrac{1}{6})^2 - g \eta_3] \epsilon^{-1} , \quad (5.10h)$$

$$\delta \eta_4^{(1)} = -\tfrac{1}{180} g (1 + 15 g \eta_4) \epsilon^{-1} , \quad (5.10i)$$

$$\delta \eta_5^{(1)} = \tfrac{1}{180} g (1 - 15 g \eta_5) \epsilon^{-1} , \quad (5.10j)$$

$$\delta \Lambda^{(1)} = -\tfrac{1}{6} M^6 \epsilon^{-1} , \quad (5.10k)$$

$$\delta \kappa^{(1)} = -\tfrac{1}{2} M^4 (\xi - \tfrac{1}{6}) \epsilon^{-1} , \quad (5.10l)$$

$$\delta \alpha_1^{(1)} = -\tfrac{1}{180} M^2 \epsilon^{-1} , \quad (5.10m)$$

$$\delta \alpha_2^{(1)} = \tfrac{1}{180} M^2 \epsilon^{-1} , \quad (5.10n)$$

$$\delta \alpha_3^{(1)} = -\tfrac{1}{2} M^2 (\xi - \tfrac{1}{6})^2 \epsilon^{-1} , \quad (5.10o)$$

$$\delta \gamma_1^{(1)} = \tfrac{1}{12} [(\xi - \tfrac{1}{5})^2 - \tfrac{23}{9450}] \epsilon^{-1} , \quad (5.10p)$$

$$\delta \gamma_2^{(1)} = \tfrac{13}{22680} \epsilon^{-1} , \quad (5.10q)$$

$$\delta \gamma_3^{(1)} = \tfrac{1}{6480} \epsilon^{-1} , \quad (5.10r)$$

$$\delta \gamma_4^{(1)} = -\tfrac{1}{6} (\xi - \tfrac{1}{6})^3 \epsilon^{-1} , \quad (5.10s)$$

$$\delta \gamma_5^{(1)} = \tfrac{1}{180} (\xi - \tfrac{1}{6}) \epsilon^{-1} , \quad (5.10t)$$

$$\delta \gamma_6^{(1)} = -\tfrac{1}{180} (\xi - \tfrac{1}{6}) \epsilon^{-1} , \quad (5.10u)$$

$$\delta \gamma_7^{(1)} = -\tfrac{1}{1260} \epsilon^{-1} , \quad (5.10v)$$

$$\delta \gamma_8^{(1)} = \tfrac{1}{2268} \epsilon^{-1} , \quad (5.10w)$$

$$\delta \gamma_9^{(1)} = -\tfrac{1}{5670} \epsilon^{-1} , \quad (5.10x)$$

$$\delta \gamma_{10}^{(1)} = \tfrac{1}{1890} \epsilon^{-1} . \quad (5.10y)$$

The presence of all of the nonminimal terms in (5.2) is seen to be required for renormalizability. The theory considered by Gass[16] will, therefore, not be renormalizable in a general space-time. The form the $\delta\xi$ counterterm differs from that given by Gass.[16]

B. Renormalization-Group Analysis

In this section, the asymptotic freedom of the theory above is discussed using the renormalization-group approach of Refs. 51,52.

Since the counterterms are given as a sum of pole terms, the following definitions may be made:

$$g_B = \mu^{3-n/2} \left[1 + \sum_{\nu=1}^{\infty} (n-6)^{-\nu} a_\nu(g) \right] g , \quad (5.11)$$

$$\xi_B = \xi + \sum_{\nu=1}^{\infty} (n-6)^{-\nu} b_\nu(\xi, g) . \quad (5.12)$$

Define the renormalization-group functions $\beta_g(g)$ and $\beta_\xi(g,\xi)$ by

$$\beta_g(g) = \mu \frac{\partial g}{\partial \mu} , \quad (5.13)$$

$$\beta_\xi(g,\xi) = \mu \frac{\partial \xi}{\partial \mu} . \quad (5.14)$$

Renormalization-group functions for the other coupling constants appearing in (5.2) and (5.3) may be defined in a similar manner (see Refs. 55–58 for the procedure in the four-dimensional case).

Bare quantities must be independent of how the renormalization mass is chosen; that is, $\mu(\partial/\partial\mu) g_B = 0$, $\mu(\partial/\partial\mu)\xi_B = 0$. By equating coefficients of powers of $(n-6)$ it is found that

$$\beta_g(g) = -\tfrac{1}{2} g^2 \frac{\partial}{\partial g} a_1(g) , \quad (5.15)$$

$$\beta_\xi(g,\xi) = -\tfrac{1}{2} g \frac{\partial}{\partial g} b_1(\xi, g) \quad (5.16)$$

when $n \rightarrow 6$. From (5.10c) and (5.11),

$$a_1(g) = \tfrac{3}{4} (4\pi)^{-3} g^2 + \cdots , \quad (5.17)$$

so that

$$\beta_g(g) = -\tfrac{3}{4} (4\pi)^{-3} g^3 + \cdots . \quad (5.18)$$

This result for the β_g function agrees with Macfarlane and Woo.[54] From (5.18) it is seen immediately that the theory is asymptotically free. Integration of (5.13) and (5.18) leads to

$$g^2(\mu) = g_0{}^2 \left[1 + \tfrac{3}{2} (4\pi)^{-3} g_0{}^2 \ln \left[\frac{\mu}{\mu_0} \right] \right]^{-1} , \quad (5.19)$$

where $g_0 = g(\mu_0)$. This describes how the coupling constant changes under a change of renormalization point. From (5.19), $g(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$. From (5.10e) and (5.12),

$$b_1(\xi,g)=\tfrac{5}{6}(4\pi)^{-3}(\xi-\tfrac{1}{5})g^2+\cdots \quad (5.20)$$

Therefore, from (5.14), (5.19), and (5.20), it follows that

$$\xi(\mu)=\tfrac{1}{5}+(\xi_0-\tfrac{1}{5})\left[1+\tfrac{3}{2}(4\pi)^{-3}g_0{}^2\left[\frac{\mu}{\mu_0}\right]\right]^{-5/9} \quad (5.21)$$

describes how ξ changes under a change of μ, where $\xi_0=\xi(\mu_0)$. As $\mu\to\infty$ it is observed that $\xi(\mu)\to\frac{1}{5}$, which is the conformal value. In Ref. 16 it is claimed that $\xi=\frac{6}{25}$ is the ultraviolet fixed point in contrast to what we have found here.

VI. CONCLUSIONS AND DISCUSSION

In the preceding sections, the renormalization of interacting scalar field theories in general curved space-times has been discussed. The background-field method was used to compute the effective action and the heat-kernel method was used to analyze the divergences. It is evident from the calculations of Secs. IV and V that this approach is much shorter than the usual diagrammatic calculations which involve an evaluation of the n-point functions separately for each n. Only a few vacuum bubbles need to be considered at each order in the loop expansion.

In Sec. IV, renormalization of the effective action was shown at the two-loop level for a scalar field with cubic and quartic self-interactions in curved space-time. The counterterms, including the gravitational ones, were computed to this order using dimensional regularization. They agree with previous flat-space-time results where they are known. The gravitational counterterms can still be fixed by flat-space-time calculations involving multiple insertions of the stress tensor in flat-space-time diagrams.[56,58] This does not prove that the theory is renormalizable in curved space-time however. The extension of the background-field method (or even the more traditional approach) beyond the two-loop level in curved space-time would be quite difficult. Also, except perhaps in special space-times, the finite parts of the higher-loop contributions will not be easy to obtain.

In Sec. V, renormalization of the effective action at the one-loop level was shown for a scalar field with a cubic self-interaction in a six-dimensional curved space-time. It was demonstrated using a renormalization-group analysis that the coupling constant ξ had an ultraviolet fixed point given by the conformal value of $\frac{1}{5}$. This corrects a recent result of Gass.[16]

The extension of the calculation in Sec. V to two loops is considerably more difficult than the four-dimensional one in Sec. IV B. In place of (4.26) there is

$$\Delta^3(x,x')=\mathscr{O}_6(x)\delta(x,x')+\mathscr{O}_{12}(x,x') , \quad (6.1)$$

where $\mathscr{O}_6(x)$ is a scalar operator of dimension six and $\mathscr{O}_{12}(x,x')$ is a biscalar of dimension twelve. The argument presented in Sec. IV B cannot be repeated to show that $\mathscr{O}_{12}(x,x')$ must be finite as $\epsilon\to 0$ since not enough calculations have been done. (It is still presumably true however.) It would be of interest to have a more direct way of evaluating $\Delta^3(x,x')$. An additional complication is that there are many operators of dimension six which can contribute to $\mathscr{O}_6(x)$.

In the case of a noncompact manifold, it is necessary to impose boundary conditions on the fields. Boundary effects, which have been ignored in the preceding sections, can alter the analysis even in the free-field case.[59,60] For an n-dimensional manifold M with $(n-1)$-dimensional boundary ∂M, the asymptotic expansion of the heat kernel is changed from that given in Sec. III to[61–64]

$$\int_M dv_x trK(t,x,x,I_2)\sim(4\pi t)^{-n/2}\sum_{k=0}^{\infty}t^{2k/2}C_k(I_2) , \quad (6.2)$$

where

$$C_k(I_2)=\begin{cases}\int_M dv_x a_{k/2}(x,I_2)+\int_{\partial M}d\sigma_x b_{k/2}(x,I_2), & k \text{ even} , \quad (6.3a)\\ \int_{\partial M}d\sigma_x b_{k/2}(x,I_2), & k \text{ odd} . \quad (6.3b)\end{cases}$$

($d\sigma_x$ is the induced volume element on ∂M.) Here $a_{k/2}(x,I_2)$ is as given in Sec. III and $b_{k/2}(x,I_2)$ arises because $\partial M\neq\varnothing$. $b_0(x,I_2)=0$ so that the leading term in the asymptotic expansion is independent of whether or not a boundary is present.

Repeating the analysis of Sec. III using (6.2), in place of (3.16), there is (assuming n_0 is even)

$$\text{P.P.}\{\Gamma^{(1)}[\hat{\phi}]\}=\epsilon^{-1}\left[\int_M dv_x a_{n_0/2}(x,I_2)+\int_{\partial M} d\sigma_x b_{n_0/2}(x,I_2)\right]. \tag{6.4}$$

Only the last term in (6.4) is present if n_0 is odd, so that the one-loop effective action is not necessarily one-loop finite in odd-dimensional space-times if $\partial M \neq \varnothing$

From (6.4), it is clear that there may be additional pole terms present which are caused by boundary effects. This is true even in the noninteracting case.[59,60] It is also known that surface terms may be required in the Einstein gravitational action.[65,66] It is not possible to be more explicit at this stage, since the $b_k(x,I_2)$ are not known for operators of the form (3.29). Expressions for $b_{1/2}(x,I_2)$, $b_1(x,I_2)$, and $b_{3/2}(x,I_2)$ may be found in Ref. 60 in the case where $I_2=-\Box+\xi R$ acts on scalars. In order to discuss renormalization in four dimensions, $b_2(x,I_2)$ is required. This coefficient has been computed by Kennedy[67] in the case $I_2=-\Box$ acting on scalars for M flat. The extension of the analysis of the present paper to the case when boundary effects may not be ignored merits further attention (see also Ref. 68).

Application of the methods of this paper to Yang-Mills theory will be given elsewhere.[69]

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