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Use of hyperfunctions for classical radiation-reaction calculations

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It is shown that the use of hyperfunctions for the evaluation of radiation reaction in classical field theories leads to calculational simplifications compared to other methods. As illustrations, we calculate the radiation-reaction terms for systems of point particles in electrodynamics and in the lowest nontrivial order of the "fast motion" approximation of general relativity. Applications to other field theories are discussed briefly.

I. INTRODUCTION

In recent years, radiation calculations both in general relativity[1–5] and in gauge theories of strongly interacting particles[6,7] have become very important. In general relativity, theoretical calculations of the period change of a binary system due to radiation reaction can be compared to observations[8] of the binary pulsar PSR 1913 + 16 to provide a critical test of the theory. In strong-interaction physics, the results of radiation-reaction calculations for classical SU(3) Yang-Mills fields may be of help for the problem of quark confinement.

In a fully nonlinear theory such as general relativity, the simplest kind of source one can imagine is not a δ function, but rather a Schwarzschild black hole. Nevertheless, some approximation methods in general relativity use distributions as a representation of the energy-momentum tensor for the source, and for a large class of problems this simplified picture is adequate.[9,10] Although the distribution associated with a pointlike source is formally infinite at the origin, it can be used to idealize the center of mass of an extended object. Within the exact theory, no difficulty of the "self-energy" type or due to the singularity of the field at the origin arises for a single particle; nothing is known about exact solutions for several particles. Within the "fast-motion" approximation method,[11,9] which is based on Einstein's linear approximation method and leads to a series of Lorentz-invariant equations, one encounters a problem familiar from special-relativistic linear field theories such as electrodynamics: although it can be shown (most generally by Mathisson[12]) that the equations of motion are finite, special techniques are required to obtain the finite fields entering these equations, in particular the finite part of the self-field yielding the radiation-reaction terms. A method devised for this purpose in electrodynamics by Dirac[13] and generalized by Harish-Chandra[14] requires rather lengthy calculations involving limiting processes, as do a number of essentially equivalent methods. The mathematically most satisfactory method is that of Riesz,[15,16] which is based on analytic continuation.

The somewhat lengthy and delicate analytic continuation calculations can be considerably shortened by the use of hyperfunction techniques in the evaluation of the integrals appearing in the Riesz method. The electromagnetic case was described briefly in an earlier letter.[17] Here, we fill in and correct some details of the electromagnetic case and also calculate the leading radiation-reaction terms for the fast-motion approximation method of general relativity.

II. ELECTROMAGNETIC RADIATION REACTION AND HYPERFUNCTIONS

We first consider the equations of motion of point particles interacting with an electromagnetic field. For a point charge of strength e with coordinates $z^\mu(\tau)$, where τ is the proper time, one has to integrate the wave equation

$$\Box A^{\mu}=4\pi j^{\mu} \tag{1}$$

with the source term

$$j^{\mu}(x^{\rho})=e\int_{-\infty}^{\infty}v^{\mu}\delta^{4}(s^{\rho})d\tau . \tag{2}$$

Here x^{μ} is a point outside the world line of the particle,

$$d\tau^2\equiv\eta_{\mu\nu}dz^{\mu}dz^{\nu},\quad \eta_{\mu\nu}\equiv\text{diag}\,(1,-1,-1,-1)\,, \tag{3}$$

$$\Box\equiv\eta^{\mu\nu}\partial_{\mu}\partial_{\nu},\quad \eta_{\mu\rho}\eta^{\rho\nu}\equiv\delta_{\mu}^{\nu},\quad \partial_{\mu}\equiv\frac{\partial}{\partial x^{\mu}}\,, \tag{4}$$

$$v^{\mu}\equiv\frac{dz^{\mu}}{d\tau}=\dot{z}^{\mu}\,, \tag{5}$$

$$s^{\rho}\equiv x^{\rho}-z^{\rho}(t),\quad s^2\equiv\eta_{\rho\sigma}s^{\rho}s^{\sigma}\,, \tag{6}$$

and $\delta^4(s^{\rho})$ is a four-fold product of Dirac δ "functions." The Riesz potential for Eqs. (1) is taken to be[15] (in units such that $c=1$)

$$_{\alpha}A^{\mu}(x^{\rho})=\frac{4\pi e}{H(\alpha)}\int_{-\infty}^{\tau_0}v^{\mu}s^{\alpha-4}d\tau\,, \tag{7}$$

where $H(\alpha)$ is defined in terms of Γ functions as

$$H(\alpha)\equiv2^{\alpha-1}\pi\Gamma(\tfrac{1}{2}\alpha)\Gamma(\tfrac{1}{2}(\alpha-2))\,. \tag{8}$$

The upper limit τ_0 of the integral is the proper time of the particle when the past null cone of the field point x^{μ} intersects the particle's trajectory. The electromagnetic field strength following from the potential (7) is

$$_{\alpha}F^{\mu\nu}(x^{\rho})=-\frac{4\pi e(4-\alpha)}{H(\alpha)}\int_{-\infty}^{\tau_0}(s^{\mu}v^{\nu}-s^{\nu}v^{\mu})s^{\alpha-6}d\tau\,. \tag{9}$$

Analytic continuation of Eq. (7) to $\alpha=2$ produces the classical Liénard-Wiechart potential at events *not* located on the world line z^{μ}. In order to evaluate the effects of radiation reaction on the motion, however, one needs to evaluate the Riesz potentials at events *on* the world lines. We now show that a meaningful interpretation of the Riesz potentials evaluated on the world lines can be given by hyperfunction techniques.

The concept of hyperfunctions was introduced by Sato[18]; a brief review was given by Fujii.[19] It is a generalization of the concept of function very closely related to Schwartz's theory of distributions,[20] which defines a distribution space as the dual of a linear space of "test functions." Among other possible definitions of distributions the one most closely related to hyperfunctions (and more readily accessible than Sato's work) is that of Bremermann and Durand.[21] They associate with a distribution on the real axis a pair of holomorphic functions in the complex plane, one [which we shall denote by $F(x+i\epsilon)$] holomorphic in the (open) upper half plane, the other [similarly denoted by $F(x-i\epsilon)$] holomorphic in the lower half plane. The limit of the sum of these two functions at $x+i\epsilon$ and $x-i\epsilon$, $\epsilon\to0$, x on the real axis, represents the distribution, and it can be shown that any distribution with compact support (a sufficient but not necessary condition) can be so represented (see theorem 3 of Ref. 21). Thus in the following we consider a hyperfunction $f(x)$ related to a "defining function" $F(z)$ by

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)\phi(x)dx &= \lim_{\epsilon\to0}\int_{-\infty}^{\infty}[F(x+i\epsilon)-F(x-i\epsilon)]\phi(x)dx\\ &\equiv\int_{-\infty}^{\infty}[F(z)]\phi(x)dx\,,\end{aligned} \tag{10}$$

where ϕ is any test function $\phi\in(\mathscr{D})$, with $(\mathscr{D})$ being the vector space of all C^{∞} functions on R^n that have compact support.[22] It can then be shown (see theorem 4 of Ref. 21) that the definite integral of $f(x)$ is given by

$$\int_a^b f(x)dx\equiv-\int_C F(z)dz\,, \tag{11}$$

where $F(z)$ is the defining function that gives a hyperfunction $f(x)$ for $a<x<b$ and zero otherwise. The contour C must enclose a and b and may be deformed freely, without changing the result, within the region in which $F(z)$ is analytic. Similarly, the derivative of a hyperfunction is given by the derivative of the defining function

$$\frac{df}{dx}\equiv\left[\frac{dF(z)}{dz}\right] \tag{12}$$

[in the sense of Eq. (10)].

Examples of hyperfunctions are

$$\begin{aligned}\delta(x)&\equiv\frac{1}{2\pi i}\left\{\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon}\right\}\\ &=\left[-\frac{1}{2\pi i}\frac{1}{z}\right]\end{aligned} \tag{13}$$

and

$$\begin{aligned}x_{+}{}^{\lambda}&\equiv\begin{cases}x^{\lambda}, & x>0,\ \lambda\neq\text{integer}\,,\\ 0, & x<0\,,\end{cases}\\ &=\left[-\frac{1}{2i\sin(\lambda\pi)}(-z)^{\lambda}\right]\,.\end{aligned} \tag{14}$$

A useful illustration of the integration of a hyperfunction is

$$\int_0^1 x^{\lambda-1}dx=\frac{1}{\lambda}\ . \tag{15}$$

This integral is convergent in the usual sense only if the real part of λ is greater than zero. However, if $x^{\lambda-1}$ is interpreted as a hyperfunction $x_+{}^{\lambda-1}$ [Eq. (14)], the integral (15) can be written as

$$\int_0^1 x_+{}^{\lambda-1}dx=\frac{1}{2i\sin[(\lambda-1)\pi]}\times\int_C(-z)^{\lambda-1}dz=\frac{1}{\lambda}\ . \tag{16}$$

Here the contour C is chosen as the unit circle centered at the origin $z=0$. Thus, without appealing to analytic continuation, one can give a meaningful interpretation to the integral (15) for all $\lambda\neq 0$.

We now return to the problem of evaluating the integral of Eq. (9) for field points on the world line of the source, omitting for the moment the (well-known) contributions of other sources. To be able to make use of Eq. (16), we express the integrand in a power series in τ around the retarded point τ_0, using the Taylor series

$$s^\mu=-v_0^\mu\tau-\tfrac{1}{2}\dot v_0^\mu\tau^2-\tfrac{1}{6}\ddot v_0^\mu\tau^3-\cdots\ , \tag{17}$$

$$v^\mu=v_0^\mu+\dot v_0^\mu\tau+\tfrac{1}{2}\ddot v_0^\mu\tau^2+\cdots\ , \tag{18}$$

$$s=-\tau-\tfrac{1}{24}\ddot v_0^\rho v_{0\rho}\tau^3\ , \tag{19}$$

$$s^\mu v^\nu-s^\nu v^\mu=\tfrac{1}{2}(\dot v_0^\mu v_0^\nu-\dot v_0^\nu v_0^\mu)\tau^2+\tfrac{1}{3}(\ddot v_0^\mu v_0^\nu-\ddot v_0^\nu v_0^\mu)\tau^3+\cdots\ . \tag{20}$$

Making use of the properties of the Γ function, we can rewrite the factor in front of the integral as

$$\frac{4\pi e}{H(\alpha)}=\frac{2(\alpha-2)e}{2^{\alpha-1}[\Gamma(\frac{1}{2}\alpha)]^2}\ . \tag{21}$$

Because Eq. (9) has to be evaluated at $\alpha=2$, the factor $(\alpha-2)$ of (21) will eliminate the contributions of all powers of τ in the integral except those of $\tau^{\alpha-3}$, which by Eq. (16) give a factor $(\alpha-2)^{-1}$. Thus we only need the corresponding terms in the products of the expansions, and obtain

$${}_\alpha F^{\mu\nu}\overset{*}{=}\frac{(4-\alpha)(\alpha-2)e}{2^{\alpha-1}[\Gamma(\frac{1}{2}\alpha)]^2}\times\int_{-\infty}^{\tau_0}\tfrac{2}{3}(\ddot v_0^\mu v_0^\nu-\ddot v_0^\nu v_0^\mu)(-\tau)^{\alpha-3}d\tau\ , \tag{22}$$

where $\overset{*}{=}$ indicates that we have omitted the terms which vanish for $\alpha=2$. Thus, using Eq. (16),[23] then putting α equal to 2 and dropping the subscript zero, we obtain

$${}_2F^{\mu\nu}=\tfrac{2}{3}e(\ddot v^\mu v^\nu-\ddot v^\nu v^\mu)\ . \tag{23}$$

From Mathisson's results[12] mentioned earlier, or equivalently from Dirac's,[13] both of which are based on overall local conservation of energy-momentum, the equations of motion must have the form

$$m\dot v^\mu=eF^{\mu\nu}v_\nu\ . \tag{24}$$

Inserting the contribution (23) of the particle under consideration as well as those of the other particles, we thus obtain for the ith particle

$$m_i\dot v_i^\mu=e_i\sum_{j\neq i}F_{j\,\mathrm{ret}}^{\mu\nu}v_{i\nu}+\tfrac{2}{3}e_i{}^2(\ddot v_i^\mu-\ddot v_i^\nu v_{i\nu}v_i^\mu)\ , \tag{25}$$

the well-known Schott-Lorentz-Dirac equation.

III. EQUATIONS OF MOTION IN GENERAL RELATIVITY

In this section, we essentially follow the notation and procedure of Refs. 11 and 2. One wishes in general to determine a spacetime model which satisfies the Einstein field equation

$$G^{\mu\nu}=8\pi GT^{\mu\nu} \tag{26}$$

and thus also its consequence

$$T^{\mu\nu}{}_{;\nu}=0\ . \tag{27}$$

We assume that the metric can be written in the form

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\ , \tag{28}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and where $h_{\mu\nu}\ll 1$. For calculational ease, we represent the center of mass of an extended body as a distribution. The stress-energy tensor can now be written formally as

$$T^{\mu\nu}=m\int_{-\infty}^{\infty}\frac{v^\mu v^\nu\delta^4(s^\rho)d\tau}{[(-g)g_{\alpha\beta}v^\alpha v^\beta]^{1/2}}\ , \tag{29}$$

where $d\tau$ and v^μ are given by Eqs. (3) and (5), and

$$g=\det g_{\alpha\beta}\ . \tag{30}$$

The form (29) of $T^{\mu\nu}$ actually can be *derived* from the field equations (26) and the assumption that the particles are monopole singularities of the field.[11] For n particles, $T^{\mu\nu}$ in Eq. (26) is a sum of terms of the form (29).

The conservation law (27) restricts the motion of the sources. It is convenient to rewrite the system of (1) a field equation and (2) energy-momentum conservation as (1′) a reduced field equation of the form

$$\Box_g\gamma^{\mu\nu}=16\pi GT^{\mu\nu}+\Lambda^{\mu\nu}(g_{\alpha\beta})\ , \tag{31}$$

where $\gamma^{\mu\nu}$ is a certain combination of the components of $g^{\mu\nu}$, and (2′) the coordinate condition

$$\partial_\nu[(-g)^{1/2}g^{\mu\nu}]=0 \ . \qquad (32)$$

The reduced field equation (31) itself imposes no restrictions on the motion of the sources. Substituting expansion (28) into this equation one obtains

$$\Box_\eta \gamma^{\mu\nu}=16\pi G T_{\mu\nu}(\eta_{\alpha\beta}+h_{\alpha\beta}) + \Lambda^{\mu\nu}(\eta_{\alpha\beta}+h_{\alpha\beta})+\mu^{\mu\nu}(h_{\alpha\beta}) \ , \qquad (33)$$

where $\mu(h)$ arises from the substitution of the flat-space wave operator $\Box_\eta$ for $\Box_g$ on the left-hand side of Eq. (31).

We seek an approximate solution of Eq. (33) by the following iteration procedure: Put $h=0$ on the right-hand side of Eq. (33) and use the retarded Green's function of flat space to obtain the linearized field ${}_1h$ associated with $T(\eta)$ for arbitrary orbits. The metric correction ${}_1h$ is of order $k=GM/L$, where M is a typical mass, L is a typical separation, and we assume that $k \ll 1$. We next insert ${}_1h$ into the right-hand side of Eq. (33) and keep terms of second order in k. These terms so far are functionals of the unspecified orbits.

The use of point particles implies that the metric contains formally divergent terms (and that our approximation is not uniformly valid in the sense of an asymptotic expansion as the world lines are approached). We will regularize the fields by a procedure to be elaborated after we have obtained the laws of motion.

Placing the sum of the first- and second-order metric into the coordinate condition (32) leads to the first-order law of motion

$$m_i \frac{d}{d\tau_i}[(\eta_{\mu\rho}+{}_1g_{\mu\rho})v_i^\rho - \tfrac{1}{2}\eta_{\mu\rho}v_i^\rho \, {}_1g_{\alpha\beta}v_i^\alpha v_i^\beta] = \tfrac{1}{2}m_i v_i^\rho v_i^\sigma \partial_\mu {}_1g_{\rho\sigma} \ , \qquad (34)$$

where

$${}_1g_{\rho\sigma} \equiv -{}_1\gamma_{\rho\sigma}+\tfrac{1}{2}\eta_{\rho\sigma}\,{}_1\gamma_\alpha^\alpha, \quad {}_1g^{\rho\sigma}={}_1\gamma^{\rho\sigma}-\tfrac{1}{2}\eta^{\rho\sigma}\,{}_1\gamma^\alpha{}_\alpha \ , \qquad (35)$$

$${}_1\gamma_{\rho\sigma}=\eta_{\rho\alpha}\eta_{\sigma\beta}\,{}_1\gamma^{\alpha\beta}, \quad {}_1g_{\rho\nu}\,{}_1g^{\nu\sigma}=\delta_\rho^\sigma \ ,$$

$${}_1\gamma^{\rho\sigma} \equiv 16\pi G \sum_j \int_{-\infty}^{\infty} D(s^2) m_j v_j^\rho v_j^\sigma d\tau_j \ , \qquad (36)$$

$${}_1\gamma_\alpha^\alpha \equiv \eta^{\alpha\beta}\,{}_1\gamma_{\alpha\beta} \ . \qquad (37)$$

In Eqs. (36) and (37) $D(s^2)$ represents the retarded Green's function of the wave equation (33). Equation (34) agrees to first order in k with the geodesic law

$$m_i \frac{d}{d\tau_i}\frac{g_{\mu\rho}v_i^\rho}{(g_{\alpha\beta}v_i^\alpha v_i^\beta)^{1/2}} = \frac{1}{2}\frac{m_i v_i^\rho v_i^\sigma}{(g_{\alpha\beta}v_i^\alpha v_i^\beta)^{1/2}}\partial_\mu g_{\rho\sigma} \ , \qquad (38)$$

where the particle trajectories are parametrized by the Minkowski proper time. Note that, to obtain an equation of motion containing the first-order metric ${}_1h$, one must iterate the relaxed field equation (33) to the second order.

When the fields are placed in our law of motion (34), divergent terms occur. However, as shown in Ref. 11, one may remove the infinite terms coming from the first iteration of Eq. (33) and calculate the finite terms by the methods of Harish-Chandra or Riesz. Here, we will obtain finite terms by combining Riesz potentials with our hyperfunction techniques. The Riesz potential now does not have the form (7) which followed from the form (2) of the source term in the inhomogeneous wave equation. Instead we now have from Eq. (29), in the order considered, restricting for the moment our attention to a single particle,

$$T^{\mu\nu}=m\int_{-\infty}^{\infty} v^\mu v^\nu \delta^4(s^\rho)d\tau \ , \qquad (39)$$

and thus

$${}_{\alpha 1}\gamma_{\mu\nu}=\frac{16\pi G m}{H(\alpha)}\int_{-\infty}^{\tau_0} v_\mu v_\nu s^{\alpha-4}d\tau \ , \qquad (40)$$

or, making use of Eq. (35),

$${}_{\alpha 1}g_{\mu\nu}=-\frac{16\pi G m}{H(\alpha)}\int_{-\infty}^{\tau_0}(v_\mu v_\nu - \tfrac{1}{2}\eta_{\mu\nu})s^{\alpha-4}d\tau \ , \qquad (41)$$

and therefore

$${}_\alpha(\partial_\rho\,{}_1g_{\mu\nu})=\frac{16\pi G m(4-\alpha)}{H(\alpha)} \times \int_{-\infty}^{\tau_0}(v_\mu v_\nu-\tfrac{1}{2}\eta_{\mu\nu})s_\rho s^{\alpha-6}d\tau \ . \qquad (42)$$

In Eqs. (40)–(42) all indices were lowered with the Minkowski metric.

We now proceed as in the electromagnetic case, expanding about the retarded point and inserting the expansions (17)–(20) into Eqs. (41) and (42). We can rewrite the factors in front of the integrals as in Eq. (21) and thus can restrict ourselves to the terms in the integrands proportional to $\tau^{\alpha-3}$. Therefore we have (omitting the subscripts zero)

$$ {}_{\alpha}{}_{1}g_{\mu\nu} \overset{*}{=} \frac{8(\alpha-2)Gm}{2^{\alpha-1}[\Gamma(\frac{1}{2}\alpha)]^2} \int_{-\infty}^{\tau_0} (-\tau)^{\alpha-3} \frac{dv_\mu v_\nu}{d\tau} d\tau \,, \tag{43} $$

$$ {}_{\alpha}(\partial_\rho\, {}_{1}g_{\mu\nu}) \overset{*}{=} \frac{4(4-\alpha)(\alpha-2)Gm}{2^{\alpha-1}[\Gamma(\frac{1}{2}\alpha)]^2} \int_{-\infty}^{\tau_0} (-\tau)^{\alpha-3} \eta_{\rho\sigma} \left[\tfrac{1}{3}(v_\mu v_\nu - \tfrac{1}{2}\eta_{\mu\nu}) \left[\dddot{v}^{\sigma} - \frac{6-\alpha}{4} \ddot{v}^{\beta} v_\beta v^\sigma \right] + \dot{v}^{\sigma} \frac{dv_\mu v_\nu}{d\tau} + v^\sigma \frac{d^2 v_\mu v_\nu}{d\tau^2} \right] d\tau \,, \tag{44} $$

and thus for $\alpha=2$ (omitting the left subscript 2)

$$ {}_{1}g_{\mu\nu} = 4Gm(v_{0\mu}\dot{v}_{0\nu} + \dot{v}_{0\mu} v_{0\nu}) \,, \tag{45} $$

$$ \partial_\rho\, {}_{1}g_{\mu\nu} = 4Gm\eta_{\rho\sigma} \left[\tfrac{1}{3}(v_{0\mu}v_{0\nu} - \tfrac{1}{2}\eta_{\mu\nu})(\dddot{v}_0^{\sigma} - \ddot{v}_0^{\beta} v_{0\beta} v_0^{\sigma}) + \dot{v}_0^{\sigma} \frac{dv_{0\mu}v_{0\nu}}{d\tau} + v_0^{\sigma} \frac{d^2 v_{0\mu} v_{0\nu}}{d\tau^2} \right] . \tag{46} $$

Inserting the contributions (45) and (46) of the particle under consideration as well as those of the other particles into the law of motion (34), we thus obtain for the ith particle

$$ m_i \frac{d}{d\tau_i} \left[\left[\eta_{\mu\rho} + \sum_{j\neq i} {}^{j}_{1}g_{\mu\rho\mathrm{ret}} \right] v_i^{\rho} - \tfrac{1}{2}\eta_{\mu\rho} v_i^{\rho} \sum_{j\neq i} {}^{j}_{1}g_{\alpha\beta\mathrm{ret}} v_i^{\alpha} v_i^{\beta} \right] = \tfrac{1}{2} m_i \sum_{j\neq i} \partial_\mu\, {}^{j}_{1}g_{\alpha\beta\mathrm{ret}} v_i^{\alpha} v_i^{\beta} - \tfrac{11}{3} G m_i^{2} \eta_{\mu\rho} (\dddot{v}_i^{\rho} - \ddot{v}_i^{\sigma} v_{i\sigma} v_i^{\rho}) \,, \tag{47} $$

which is identical with the result of Havas and Goldberg.[11]

IV. DISCUSSION

We have calculated the radiation-reaction terms both in classical electromagnetism and in the first nontrivial order of the fast-motion approximation in general relativity by using techniques involving hyperfunctions. These techniques lead to great calculational simplifications over the methods pioneered by Dirac, and greatly simplify Riesz's method of analytic continuation. In particular, when derivatives of field quantities are needed in the equations of motion, it is possible to evaluate these quantities on the world line of the particle by bringing the derivative inside the integral as in Eq. (42). These techniques can also be employed in calculating radiation-reaction terms in higher iterations of the Einstein field equation, as well as in Yang-Mills theories.[24] In addition, because hyperfunctions are defined in the complex plane, there is a deep connection to the theory of twistors, which will be discussed elsewhere.

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