Strong-coupling quantum gravity. I. Solution in a particular gauge

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(Received 6 November 1981)

The strong-coupling limit of general relativity is quantized in a fixed gauge. An exact solution to the quantum field theory is given (it does not require any artifice such as a lattice), and the dynamical properties of the theory are discussed. The configuration space is closely related to the symmetric space SL(3,R)/SO(3), and this connection is exploited.

I. INTRODUCTION

Much effort has been made in applying the standard methods of quantum field theory to the problem of quantizing the gravitational field. Although these methods have been spectacularly successful when applied to other field theories of physical interest, they have failed when applied to general relativity. The theory is nonrenormalizable and cannot be defined.

This failure has led to the consideration of many different theories of gravity in the hope that one of them will be better defined. In this paper we will study a strong-coupling limit of general relativity, and, in particular, we will quantize the theory in this limit, i.e., an exact solution on a continuous four-dimensional manifold will be found. This limit highlights properties of the quantum gravitational field that are complementary to those that are usually studied, and it is hoped that an understanding of this additional facet of the gravitational field will be of help in obtaining a complete quantum theory.

In studying this limit we will follow a suggestion of Isham[1] which is most easily motivated by looking at the Hamiltonian formulation of general relativity. The dynamical part of the Hamiltonian is given by[2,3]

$$\mathscr{H}_\perp = \kappa g^{-1/2} G_{ijkl}\pi^{ij}\pi^{kl} - \kappa^{-1} g^{1/2} R = 0 , \tag{1.1}$$

where

$$G_{ijkl} = \tfrac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) , \tag{1.2}$$

and $\kappa = 16\pi G/c^3$, π^{ij} is the canonical conjugate to the three-dimensional metric g_{ij}, $g = \det g_{ij}$, and R is the curvature scalar computed from g_{ij}. Notice that (1.1) has the form $H =$ "$p^2 + V$". The strong-coupling limit is obtained by dropping the V, i.e., by taking as the dynamical part of the Hamiltonian

$$\mathscr{H}_0 = g^{-1/2} G_{ijkl}\pi^{ij}\pi^{kl} = 0 . \tag{1.3}$$

It is expression (1.3) that we want to quantize exactly. The idea is this solution will provide the analog, for the strong-coupling limit, of Fock space, for usual quantum theory. The full theory of general relativity is then to be recovered by treating "V" $= -g^{1/2}R$ as a perturbation about this solution. Here, only the quantization of (1.3) is considered, and the development of the perturbation theory will be treated in subsequent papers.

Notice that the Hamiltonian (1.3) roughly has the form "$\phi^2\pi^2$", i.e., it is the Hamiltonian for a nonlinear field theory. Standard experience tells us that exact quantum field theories for such Hamiltonians are rather rare; but also notice that (1.3) has a property that might considerably simplify the construction of an exact solution. It is ultralocal; that is to say, by taking the $G \to \infty$ limit all spatial derivatives have disappeared from the dynamics. Nearby spatial points are uncoupled and the dynamics at different points are independent. The light cones have collapsed to "vertical" lines. Interacting field theories in the ultralocal limit have been extensively studied by Klauder,[4–6] and exact solutions to a wide class of such theories have been given by him. In this paper, we will apply his methods to general relativity. (It should be mentioned that the strong-coupling limit of Yang-

Mills theory is also ultralocal, and this will be discussed in a separate paper.)

The Hamiltonian (1.3) can also be considered (a) a $c\to 0$ limit of (1.1) or (b) a $\sigma\to 0$ limit, where $\sigma=n^{\mu}n_{\mu}$, with n_{μ} the normal to a generic spacelike hypersurface. This latter limit has been emphasized by Teitelboim,[7,8] and is particularly helpful in understanding the geometry of the four-dimensional manifold generated by $\mathscr{H}_0$. This geometry has been investigated in detail by Henneaux[9] who has shown that it corresponds to a four-dimensional manifold with degenerate metric ($\det{}^{(4)}g_{\mu\nu}=0$; roughly speaking, the timelike directions have become lightlike lines after the collapse of the light cone). Any three-dimensional manifold that is nowhere tangent to one of these lightlike lines is spacelike, and, classically, it is on such a manifold that the initial data are given.

General relativity is a gauge theory and it has, in addition to the generator of dynamics (1.1), a generator of gauge transformations [i.e., coordinate transformations (Lie derivatives) on the spacelike hypersurfaces] given by

$$\mathscr{H}_i=-2g_{ik}\pi^{kj}{}_{|j}=0\ , \tag{1.4}$$

where the vertical bar denotes the covariant derivative in the metric g_{ij}. In contrast to the Yang-Mills case, the coupling constant κ does not occur in the gauge generator $\mathscr{H}_i$, and the strong-coupling limit has the same properties under three-dimensional coordinate transformations as the full theory. This is what one would expect. The coordinate transformation properties of fields defined on a three-dimensional manifold are intrinsic properties of the fields and the manifold. They are independent of how the manifold is embedded in the four-manifold (which is determined by the dynamics[9,10] and depends on the strength of the coupling). In this paper the gauge freedom associated with (1.4) will be frozen out.

The present paper provides a quantization of the theory whose dynamics is determined by (1.3) with gauge invariance generated by (1.4). It is organized as follows. In Sec. II the problem of the proper choice of variables for general relativity is considered, and a noncanonical choice is made in order to preserve the positive-definite spectrum of g_{ij}. Section III discusses the gauge invariance and the gauge-fixing condition is given there. The Hamiltonian is then expressed in terms of the independent variables. In Sec. IV the ultralocal representation of the independent operators is given and some of its properties discussed. The expression for the Hamiltonian in this representation is given. In Sec. V the form of the ground state is found, and a proposal for extracting dynamical information discussed. The results of this paper have much in common with work done in the early Seventies on quantized cosmological models and this is also discussed in Sec. V. The main part of the paper concludes with a summary and discussion. It is found, in the course of developing the strong-coupling theory, that the symmetric space SL(3,R)/SO(3) is closely connected with the configuration space of general relativity. The geometry of this space and elementary analysis on it are described in Appendixes A and B. In addition, there is an Appendix illustrating a simple analog of the gauge condition used in Sec. III, another summarizing the work on quantum cosmology, and a final one providing motivation for our choice of representation.

Throughout this paper the coordinate label $\vec{x}$ will refer to a coordinate patch on a three-dimensional, *compact* (but otherwise arbitrary) manifold. The noncompact case is fundamentally different and the results of this paper do not apply. This is related to the fact that, in the noncompact case, the generator of time translations is given by a surface integral and not by (1.1) alone. While many interesting topological questions arise when developing the quantum theory for the strong-coupling limit they will not be discussed here.

II. CHOICE OF VARIABLES

Classically the standard choice for basic canonical variables is the three-metric g_{ij} and its conjugate π^{ij} which satisfy

$$\begin{aligned}\{\,g_{ij}(\vec{x}),\pi^{kl}(\vec{x}\,')\,\}&=\tfrac{1}{2}(\delta_i^k\delta_j^l+\delta_j^k\delta_i^l)\delta(\vec{x},\vec{x}\,')\\&\equiv\delta_{ij}^{kl}\delta(\vec{x}\vec{x}\,')\ ,\end{aligned} \tag{2.1}$$

where { , } denotes Poisson brackets. The field g_{ij} is not arbitrary, but must have signature $(+,+,+)$. Equivalently, the four-metric g must have signature $(-,+,+,+)$ [the minus sign here is determined in the canonical formalism[10] by the relative sign of the "p^2" and "V" terms in (1.1)]. This condition must be carried over into the quantum theory by requiring that the operator g_{ij} have positive-definite spectrum since if this condition was violated in the quantum theory it is hard to see how it could be maintained in the classical limit, thus destroying its physical interpretation.[1]

In the usual covariant quantization of gravity the metric is expanded in a perturbation series about $\eta_{\mu\nu}$, the Lorentz metric, so to low orders in perturbation theory, one might feel justified in ignoring the spectrum problem. For the strong-coupling theory such a perturbation expansion is inappropriate and the metric operator will be quantized as a whole without being broken into quantized and unquantized parts. In this case the problem of the restricted spectrum must be confronted head on.

It is well known that if q and p are canonically conjugate variables the spectrum of one cannot be restricted and at the same time have q and p both represented by self-adjoint operators. To see why, assume that q is restricted by $q>0$, and work in a q representation. The Hilbert space will be the square-integrable functions of a positive variable, i.e., $L^2(0,\infty)$. Take $\psi(q)\in L^2(0,\infty)$, then

$$e^{irp}\psi(q)=\psi(q+r) ,$$

and for the general ψ we see that e^{irp} is not a unitary operator on $L^2(0,\infty)$ $[\int_0^\infty \psi^*(q)\psi(q)dq \neq \int_0^\infty \psi^*(q+r)\psi(q+r)dq]$; hence p cannot be self-adjoint. Loosely, the variable p cannot be self-adjoint since it tries to translate q outside of its restrictions. The same thing will be true of[1] g_{ij} and π^{ij}.

One suggested solution to this problem[1] is to choose canonically conjugate variables (triads) e_i^a, p_a^i $(a=1,2,3)$ satisfying

$$\{ e_i^a(\vec{x}),p_b^j(\vec{x}\,') \} = \delta_b^a\delta_i^j\delta(\vec{x},\vec{x}\,') \tag{2.2}$$

and

$$e_i^a e_j^b \delta_{ab} = g_{ij}, \quad e_i^a e_{jb} g^{ij} = \delta_b^a . \tag{2.3}$$

Condition (2.3) guarantees that the derived quantity g_{ij} never has a negative eigenvalue. The metric g_{ij} is also required by (2.3) to be invertible, i.e., we must have that $\det e_i^a \neq 0$. As above, maintaining this classical restriction in the quantum theory is not compatible with a self-adjoint p_a^i. Some[11] would relax this restriction, i.e., allow $\det e_i^a=0$ because they desire a mechanism for the change of topology—the topology change proceeding through the singular geometries given by the noninvertible triads. We will not choose this solution since we prefer to stay as close as possible to the spirit of the classical theory.

The solution that we will choose involves the representation of a noncanonically conjugate set of variables as self-adjoint. In addition to the metric operator g_{ij}, variables denoted by π_j^i [heuristically $\pi_j^i = \frac{1}{2}(g_{jk}\pi^{ki} + \pi^{ki}g_{jk})$] and satisfying the commutation relations (we briefly use quantum-mechanical language to motivate the choice of the π_j^i),

$$[g_{ij}(\vec{x}),\pi_l^k(\vec{x}\,')] = \frac{i}{2}(g_{il}\delta_j^k + g_{jl}\delta_i^k)\delta(\vec{x},\vec{x}\,') , \tag{2.4}$$

$$[\pi_j^i(\vec{x}),\pi_l^k(\vec{x}\,')] = \frac{i}{2}(\pi_l^i\delta_j^k - \pi_j^k\delta_l^i)\delta(\vec{x},\vec{x}\,') , \tag{2.5}$$

are chosen. The action of π_l^k on g_{ij} is given by

$$\begin{aligned}\bar{g}_{ij} &\equiv \exp\left(i\int \lambda_k^l(\vec{x}\,')\pi_l^k(\vec{x}\,')dx'\right)g_{ij}(\vec{x})\exp\left[-i\int \lambda_k^l(\vec{x}\,')\pi_l^k(\vec{x}\,')d\vec{x}\,'\right] \\ &= \exp[-\tfrac{1}{2}\lambda_i^k(\vec{x})]g_{kl}(\vec{x})\exp[-\tfrac{1}{2}\lambda_j^l(\vec{x})] ,\end{aligned} \tag{2.6}$$

where λ_k^l are c-number elements of some suitable function space.

It is easy to check that if $|\gamma_{ij}\rangle$ is an eigenvector of g_{ij} with eigenvalue γ_{ij} of signature $(+,+,+)$ then $\exp(-i\int \lambda_k^l\pi_l^k)|\gamma_{ij}\rangle$ is an eigenvector with eigenvalue $\exp(-\frac{1}{2}\lambda_i^k)\gamma_{kl}\exp(-\frac{1}{2}\lambda_j^k)$, and this eigenvalue has signature $(+,+,+)$. The operator π_l^k can be taken as self-adjoint since its action respects the restriction on the spectrum of g_{ij}. This is an application to the specific case of general relativity of methods first suggested by Klauder.[12,13]

One should notice that the commutation relations (2.5) are the same as those for the generators of GL(3,R). The importance of GL(3,R) and especially of SL(3,R)/SO(3) will become obvious in the remainder of this paper. Closely related to this is the fact that the π_j^i correspond to Killing vectors for the metric of the configuration space of the theory (see Appendix A). This provides another motivation for using the variables π_j^i.

The discussion in this section has been independent of the strong-coupling limit or any other aspect of the dynamics, but, if one takes into account that we eventually want to quantize the ultralocal limit of gravity, the work of Klauder[4,5] provides yet another motivation for considering the π^i_j. He finds that for ultralocal theories canonically conjugate classical variables do not exist as quantum operators, but variables satisfying commutation relations of the type (2.4) do exist.

The operators g_{ij}, π^k_l have gauge as well as dynamical information in them. In the next section we fix the gauge freedom. The choice of π^k_l plays an important role in this.

III. FIXING A GAUGE

The classical constraint $\mathscr{H}_i=0$ (1.4) is, in the quantum theory, usually taken as the restriction $\mathscr{H}_i\Psi=0$ on the state functionals.[14] It is customary to work in a metric representation, i.e., $\Psi=\Psi[g_{ij}]$, and one can argue[15] that this restriction implies that Ψ is a functional only of the coordinate-independent information in g_{ij}. The state is thus a functional only of the intrinsic geometry.[16] This sort of discussion usually stops at this point. While one can write simple examples of coordinate-invariant state functionals,[1] no one has come close to giving a complete Hilbert space of such functionals with a complete set of coordinate-invariant observables (a naive attempt was unsuccessfully made in Ref. 17). In this paper, we would like to develop a formalism in which computations can potentially be made in reality and not just in principle. To do this we cannot get away with merely saying that the states are coordinate-invariant functionals of the canonical variables. For this reason we will fix a gauge. The particular gauge that we choose is especially natural for the strong-coupling limit, but is very unnatural for the perturbation theory in $g^{1/2}R$. This defect will be corrected in a future publication.

The three constraints $\mathscr{H}_i=0$ imply that we must impose three gauge conditions in order to eliminate the three degrees of freedom (three q's and three p's) associated with changes of coordinates on a spacelike hypersurface. (To be absolutely clear, we are fixing the gauge in the classical formalism and will then quantize.) The metric tensor g_{ij} has six independent components, with three being associated with coordinate invariance and two representing independent degrees of freedom. We are thus left with one component that is neither gauge nor physical. This extra degree of freedom has been identified as an intrinsic time.[18] We thus have the problem of explicitly identifying a natural intrinsic time as a functional of the g_{ij}. We will do this as a first step in identifying the independent degrees of freedom.

In Appendix A the structure of the space of g_{ij}'s is considered. On that space the generator (1.1) implies a natural metric, namely expression (1.2), which has the hyperbolic signature $(-+++++)$. The quantity $\pi(x)=\pi^i_i(x)$ is found to be a timelike, hypersurface orthogonal Killing vector on the space of g_{ij}. The "timelike" [in metric (1.2)] coordinate conjugate to it is

$$\tau(\vec{x})=\tfrac{1}{3}\ln g(\vec{x})\ . \tag{3.1}$$

This we identify with the intrinsic time (this choice is also made in Refs. 19 and 20).

The dynamical generator is given by

$$\begin{aligned}\mathscr{H}_0(x)&=g^{-1/2}G_{ijkl}\pi^{ij}\pi^{kl}\\&=g^{-1/2}(\pi^i_j\pi^j_i-\tfrac{1}{2}\pi^2)\\&=e^{-3\tau/2}(P^i_jP^j_i-\tfrac{1}{6}\pi^2)\ ,\end{aligned} \tag{3.2}$$

where

$$P^i_j=\pi^i_j-\tfrac{1}{3}\pi\delta^i_j\ ,$$

which satisfies $P^i_i=0$. If one makes the identification $\pi\sim i\delta/\delta\tau$, then $\mathscr{H}_0$ looks like a Klein-Gordon operator. For a state functional Ψ the condition

$$\mathscr{H}_0\Psi=0 \tag{3.3}$$

determines the development of Ψ in the intrinsic time τ. We will write $\Psi=\Psi[\tilde{g}_{ij},\tau]$ where

$$\begin{aligned}\tilde{g}_{ij}(\vec{x})&=g^{-1/3}(\vec{x})g_{ij}(\vec{x})\\&=e^{-\tau(\vec{x})}g_{ij}(\vec{x})\ ,\end{aligned} \tag{3.4}$$

which satisfies $\det\tilde{g}_{ij}=1$. If it was possible to introduce a Hilbert-space structure on the Ψ then the inner product would be taken at some fixed time, although the particular fixed time could vary.

This particular way of identifying τ as a nondynamical variable and taking inner products at fixed τ has much in common with gauge fixation, but it is obviously much different since Ψ remains an explicit function of a varying τ. The existence of such a variable is a direct result of the general covariance of general relativity and is one of the features that distinguish theories of gravitation from other gauge theories. The gauge invariance for gravity is not an internal one, and one of the "directions" of the transformations is timelike (in spacetime). Associated with the four gauge degrees of freedom are the four generators $\mathscr{H}_\perp, \mathscr{H}_i$ that (for compact spaces) make up the Hamiltonian. Fixing τ completely (say, making it equal to 5) would entail the elimination of the constraint $\mathscr{H}_\perp = 0$ thus destroying the theory's dynamical information. The formalism would become "frozen" in time.[14,21]

This discussion applies only to the case of a compact three-manifold. For the noncompact case, time is measured by the Minkowski time at infinity, and associated with this is a surface integral which is included in the Hamiltonian. In this case the gauge is completely fixed by four conditions, and, after gauge fixation, the surface integral generates the dynamics.[22] Notice that the Hamiltonian formalism makes these considerations, which are necessary for properly treating compact manifolds, particularly clear. The fundamental distinction between the gauge invariance associated with $\mathscr{H}_i$ and the transformation in proper time generated by $\mathscr{H}_\perp$ has been treated in a path-integral formulation by Teitelboim.[8,20]

From the above discussion we can see why the fact that π is a timelike, hypersurface orthogonal Killing vector makes τ a particularly natural choice for an intrinsic time. The existence of such a vector enables one to define in a natural way positive- and negative-frequency solutions to the wave equation[23] (3.3). Put slightly differently, one is able to separate the τ dependence from the $\tilde{g}_{ij}$ dependence of Ψ. Notice in this context that increasing (decreasing) τ corresponds to an expanding (contracting) local geometry. Positive- (negative-) frequency states correspond, loosely, to expanding (contracting) universes.[8,20,24]

From the above considerations it can be seen that the quantum theory that we will develop will possess some nonstandard properties. Remember that π is a Killing vector on the space of g_{ij}, not on spacetime. In the above discussion we have been treating the quantum field theory of strong-coupling general relativity as if it was finite-dimensional quantum mechanics on the space of g_{ij}. We will find that this is roughly what happens. We reemphasize that nothing has been quantized yet and any discussion of quantum mechanics so far should be considered heuristic and motivational. In the actual quantum theory many of the details will be different.

So far we have singled out τ and π as being nondynamical and we have left $\tilde{g}_{ij}$ and P^i_j. These variables satisfy the bracket relations [we will drop factors of $\delta(\vec{x},\vec{x}\,')$]

$$\{\tau,\pi\}=1\,, \tag{3.5}$$

$$\{\tau,P^i_j\}=0,\quad \{\tau,\tilde{g}_{ij}\}=0\,, \tag{3.6}$$

$$\{\pi,\tilde{g}_{ij}\}=0\,, \tag{3.7}$$

$$\{\pi,P^i_j\}=0,\quad \{\pi,\pi\}=0\,, \tag{3.8}$$

$$\{\tilde{g}_{ij},P^k_l\}=\tfrac{1}{2}(\tilde{g}_{il}\delta^k_j+\tilde{g}_{jl}\delta^k_i-\tfrac{2}{3}\tilde{g}_{ij}\delta^k_l)\,, \tag{3.9}$$

$$\{P^i_j,P^k_l\}=\tfrac{1}{2}(P^i_l\delta^k_j-P^k_j\delta^i_l)\,. \tag{3.10}$$

The P^i_j, being traceless and satisfying (3.10), generate SL(3, R).

To eliminate the constraints $\mathscr{H}_i=0$ we need to impose three gauge-fixing conditions that are to be functionals of $\tilde{g}_{ij}$ and P^i_j. In terms of the new variables $\mathscr{H}_i$ has the form

$$\mathscr{H}_i(\vec{x})=-\tfrac{2}{3}\pi_{,i}(\vec{x})+\tfrac{1}{3}(e^{3\tau(\vec{x})})_{,i}\pi(\vec{x}) -2P^j_{i,j}(\vec{x})+\tilde{g}_{kj,i}\tilde{g}^{km}P^j_m(\vec{x}) =0\,. \tag{3.11}$$

Notice that the canonical pair τ,π is completely separate from $\tilde{g}_{ij}$ and P^i_j in (3.11).

In choosing a gauge condition our goal is to keep the strong-coupling limit as simple as possible. This goal essentially implies that the gauge condition should be a functional only of P^i_j. To see why this is so assume the condition is only a functional of $\tilde{g}_{ij}$. One would then have to use (3.11) to solve for the dependent P^i_j in terms of π, τ, and the independent degrees of freedom. This expression for the dependent P^i_j would then be plugged back into expression (3.2) for $\mathscr{H}_0$, giving a tremendously complicated Hamiltonian to be quantized. Making the gauge condition a functional of the P^i_j puts this complication into the dependent parts of $\tilde{g}_{ij}$ [$\mathscr{H}_0$ is determined by just plugging the gauge condition into (3.2)]; but (3.2) is independent of $\tilde{g}_{ij}$, so this complication will not bother us in the strong-coupling limit. It of course becomes a problem in developing the perturbation theory since R is a functional of $\tilde{g}_{ij}$. The fact that $\{P^i_j,\mathscr{H}_0\}=0$ means that any gauge condition that is only a functional of P^i_j will imply no additional restrictions on the Lagrange multipliers such as occur in Yang-Mills theories[25] or various gauge conditions in general relativity.[26]

We again see the utility of the π^i_j variables. In fact, even if there were no other good reasons for using π^i_j, these simple gauge conditions would provide ample motivation. Gauge conditions of a type similar to the ones chosen here have also been useful in Yang-Mills theory.[27,28]

When considering gauge conditions formed from the P^i_j the first thing that one must come to grips with is the fact that there are eight P^i_j while one wants to have three gauge conditions with two independent degrees of freedom left over. There seem to be three components of P^i_j too many. The P^i_j generate SL(3,R), but the space of $\tilde{g}_{ij}$ is, effectively, SL(3,R)/SO(3). The "extra" three components of P^i_j are just the generators of SO(3). Put another way, the tangent space of SL(3,R)/SO(3) is five dimensional and the eight P^i_j can be identified with elements of the tangent space; so not all of P^i_j are linearly independent [when considered as vectors on SL(3,R)/SO(3)].

The quantities

$$\begin{aligned} P_1(\vec{x}) &= 2[P^1_2(\vec{x}) - P^2_1(\vec{x})] , \\ P_2(\vec{x}) &= 2[P^1_3(\vec{x}) - P^3_1(\vec{x})] , \\ P_3(\vec{x}) &= 2[P^3_2(\vec{x}) - P^2_3(\vec{x})] \end{aligned} \tag{3.12}$$

satisfy

$$\{ P_a, P_b \} = \epsilon_{abc} P_c ,$$

[i.e., the commutation relations of SO(3)]. In addition define

$$\begin{aligned} R_1(\vec{x}) &= P^1_2(\vec{x}) , \\ R_2(\vec{x}) &= P^2_3(\vec{x}) , \\ R_3(\vec{x}) &= P^3_1(\vec{x}) . \end{aligned} \tag{3.13}$$

The $P_a, R_A, \Pi_1(\vec{x}) \equiv P^1_1(\vec{x}) - P^3_3(\vec{x})$, and $\Pi_2(\vec{x}) \equiv P^2_2(\vec{x}) - P^3_3(\vec{x})$ span the set of P^i_j with Π_1, Π_2 corresponding to the generators of the Cartan subgroup of SL(3,R). They satisfy

$$\{ \Pi_1, \Pi_2 \} = 0 . \tag{3.14}$$

We want to choose as a combination of gauge-fixing and SO(3)-eliminating conditions,

$$R_A = 0$$

and

$$P_a = 0 , \tag{3.15}$$

where $\Pi_2(\vec{x})$ and $\Pi_2(\vec{x})$ with their conjugates $t_1(\vec{x})$ and $t_2(\vec{x})$ are to represent the two independent degrees of freedom.

What we would like to do is take $R_A = 0$ as the gauge conditions with $P_a = 0$ being the automatic result of the P_a's being generators of SO(3). This cannot strictly be done since the P_a generate the action of SO(3) only at the origin of SL(3,R)/SO(3). For us to be sure that (3.15) can be consistently imposed we must find a two-dimensional submanifold E of SL(3,R)/SO(3) such that Π_1 and Π_2 represent tangent vectors to E that span its tangent space, and such that the gauge conditions $R_A = 0$ also guarantee that the generators of the action of SO(3) are zero at all points of E. This manifold exists and is just the manifold generated by the action of Π_1 and Π_2 on the identity δ_{ij}, i.e., it is the manifold of matrices of the form

$$\begin{bmatrix} e^{t_1(\vec{x})} & 0 & 0 \\ 0 & e^{t_2(\vec{x})} & 0 \\ 0 & 0 & e^{-[t_1(\vec{x})+t_2(\vec{x})]} \end{bmatrix} . \tag{3.16}$$

We have used our knowledge of the geometry of SL(3,R)/SO(3) to find a gauge-fixing condition and to identify the manifold of independent coordinates with a submanifold of SL(3,R)/SO(3). The close connection between $\mathscr{H}_0$ after fixing this gauge and the Laplace-Beltrami operator on E will be shown below.

One must be careful not to confuse (3.16) with $\tilde{g}_{ij}$. The matrix (3.16) is an expression for a generic point of the manifold of independent degrees of freedom. The actual relation between t_1 and t_2 and the components of $\tilde{g}_{ij}$ is rather complicated. In addition, some of the components of $\tilde{g}_{ij}$ are to be expressed as functions of π, τ, t_1, t_2, Π_1, and Π_2 by solving the constraints (3.11). The simplicity of many of our expressions can be traced to the fact that they are written in terms of $t_1(x)$ and $t_2(x)$ and not in terms of $\tilde{g}_{ij}$.

Now we wish to compute the expression for $\mathscr{H}_0$ in this gauge. First of all we will drop the $e^{-3\tau/2}$ from expression (3.2). The quantity $\mathscr{H}_0$ is constrained to be zero; this can make no difference classically, but it has been found to be a very convenient thing to do in various treatments of canonically quantized gravity.[8,17,26] We also mention in passing that this modified $\mathscr{H}_0$ expressed in terms of P^i_j does not have a factor-ordering problem.

Equation (3.2) becomes

$$\mathscr{H}_0(\vec{x})=(P^1_1)^2+(P^2_2)^2+(P^3_3)^2+2P^2_1R_1+2P^3_2R_2+2P^1_3R_3-\tfrac{1}{6}\pi^2=0\,, \tag{3.17}$$

and, after imposing $R_A=0$ and using $P^1_1=\frac{1}{3}(2\Pi_1-\Pi_2)$, $P^2_2=\frac{1}{3}(2\Pi_2-\Pi_1)$, $P^3_3=-\frac{1}{3}(\Pi_1+\Pi_2)$, we get the expression

$$\mathscr{H}_0(\vec{x})=\tfrac{4}{3}[(\Pi_1)^2+(\Pi_2)^2-\Pi_1\Pi_2]-\tfrac{1}{6}\pi^2=0 \tag{3.18}$$

in terms of Π_1, Π_2, and π. Comparison with (B1) shows the close connection between (3.18) and the Laplace-Beltrami operator on E. Following Misner[24] we define

$$\pi_+(\vec{x})=\sqrt{2}(\Pi_1+\Pi_2)\,,\qquad \pi_-(\vec{x})=\sqrt{6}(\Pi_1-\Pi_2)\,, \tag{3.19}$$

in terms of which $\mathscr{H}_0$ becomes

$$\mathscr{H}_0=\tfrac{1}{6}({\pi_+}^2+{\pi_-}^2-\pi^2)=0 \tag{3.20}$$

and from now on we will drop the factor of $\frac{1}{6}$ from (3.20). Now we take $t_+(x),\pi_+(x)$ and $t_-(x),\pi_-(x)$ with

$$t_+(\vec{x})=\frac{1}{\sqrt{8}}(t_1+t_2)\,,\qquad t_-(\vec{x})=\frac{1}{\sqrt{24}}(t_1-t_2) \tag{3.21}$$

as our independent degrees of freedom.

Expressions (3.16) and (3.20) establish a close relationship between the strong-coupling limit of general relativity in this gauge and previous work on quantum cosmology for a matterless Bianchi type-I (Kasner) universe.[24,26] In these simplified models the complications associated wtih solving $\mathscr{H}_i=0$ do not exist and expression (3.16) is directly related to the metric (although t_1 and t_2 are no longer functions of x) and similarly for (3.20). This work provides motivation for much of what we do in the remainder of this paper, and it is summarized in Appendix D.

As was mentioned above, the manifold $\mathscr{E}=\prod_{\vec{x}}E_{\vec{x}}$ (where $E_{\vec{x}}$ is a copy of E at the point $\vec{x}$) on which we are going to do the quantum theory is not the same as the manifold $\mathscr{D}$ of independent degrees of freedom obtained by imposing the constraints $R_A=0$ and $\mathscr{H}_i=0$. The form of the expression (3.18) and its close connnection with (B1) and (B3) make $\mathscr{E}$ the obvious place to construct the quantum theory. This is not the normal way of doing things, and the relevance of the results obtained here to a quantum theory of general relativity will have to be established (although we make no attempt to do it).

To see why working on $\mathscr{D}$, as is conventional, is something to be avoided we outline the usual approach to quantization after a gauge has been fixed. An object of primary importance is $\{R_A,\mathscr{H}_i\}$ which is given by

$$\begin{aligned}
&\{R_2(\vec{x}),\mathscr{H}_1(\vec{x}\,')\}=[P^1_1(\vec{x})-P^2_2(\vec{x})]\delta_{,2}(\vec{x},\vec{x}\,')-P^3_2(\vec{x})\delta_{,3}(\vec{x},\vec{x}\,')\,,\\
&\{R_1,\mathscr{H}_2\}=0,\quad \{R_1(\vec{x}),\mathscr{H}_3(\vec{x}\,')\}=P^1_3(\vec{x})\delta_{,2}(\vec{x},\vec{x}\,')\,,\\
&\{R_2(\vec{x}),\mathscr{H}_1(\vec{x}\,')\}=P^2_1(\vec{x})\delta_{,3}(\vec{x},\vec{x}\,'),\quad \{R_2,\mathscr{H}_3\}=0\,,\\
&\{R_2(\vec{x}),\mathscr{H}_2(\vec{x}\,')\}=[P^2_2(\vec{x})-P^3_3(\vec{x})]\delta_{,3}(\vec{x},\vec{x}\,')-P^1_3(\vec{x})\delta_{,1}(\vec{x},\vec{x}\,')\,,\\
&\{R_3,\mathscr{H}_1\}=0,\quad \{R_3(\vec{x}),\mathscr{H}_2(\vec{x}\,')\}=P^3_2(\vec{x})\delta_{,1}(\vec{x},\vec{x}\,')\,,\\
&\{R_3(\vec{x}),\mathscr{H}_3(\vec{x}\,')\}=[P^3_3(\vec{x})-P^1_1(\vec{x})]\delta_{,1}(\vec{x},\vec{x}\,')-P^2_1(\vec{x})\delta_{,2}(\vec{x},\vec{x}\,')\,.
\end{aligned} \tag{3.22}$$

The quantities P^3_2, P^1_3, and P^2_1 in (3.22) are dependent variables that are determined as functionals of τ, π, and the independent degrees of freedom through $\mathscr{H}_i=0$, $R_A=0$, and the definition $\pi^i_j=g_{jl}\pi^{li}$. The gauge condition is a good one when[25] $\det\{R_A,\mathscr{H}_i\}\neq 0$. Expressions (3.22) and the brackets of $\mathscr{H}_i$ (and R_A) with themselves are sufficient to determine the brackets of all independent quantities when restricted to the surface $\mathscr{D}$ (these are given by the Dirac brackets[25]).

There are two (equivalent) methods of quantizing this system. The first is to find explicit functional forms for the independent quantities such that their classical brackets (restricted to $\mathscr{D}$) are satisfied as commutators (i.e., the analog of $\hat{x}=x$, $\hat{p}=-i\partial/\partial x$). The second is to include ghosts into the Hamiltonian with couplings determined[29,30] by (3.22) while using all of the $\tilde{g}_{ij}$, P^i_j with their ordinary brackets (3.5)–(3.10). These methods are extremely difficult to implement in the case that we are interested in. For example, Π_1 and Π_2 no longer commute when restricted to $\mathscr{D}$, but, instead, satisfy a very complicated bracket, which makes the first method difficult. In Yang-Mills theory the inclusion of ghosts is possible only because their coupling to the gauge fields can be treated as a perturbation. The ghost terms that are implied by (3.22) have no coupling constant (we are looking at the strong-coupling limit after all); so the inclusion of ghosts here requires the solution of a very complicated, nonlinear problem. In the analogous situation of zero coupling Yang-Mills theory the theory reduces to a set of uncoupled vector gauge fields whose solution is well known.

The problems are avoided in this paper by simply quantizing the theory on $\mathscr{C}$. This quantization is given in the subsequent sections. While there is a good possibility that the quantum theory presented here is in some sense isomorphic to the one that would be obtained on $\mathscr{D}$, it is clear that it is at least a necessary first step towards a final solution.

IV. THE REPRESENTATION

The first step to finding the quantum theory of the Hamiltonian (3.20) is to construct a Hilbert space and a representation of the canonical commutation relations on that space. In fact, this is effectively the last step. The representation to a large extent determines the Hamiltonian.[31] The type of representation that has been found to be of most use for ultralocal Hamiltonians is the exponential representation.[32] Rather than derive the representation from first principles we will just state the most important properties and refer the reader to Refs. 4, 5, 6, 12, and 32 for more comlete details.

The Hilbert space will be taken to be a Fock space with fiducial vector $|0\rangle$ and creation and annihilation operators

$$A^\dagger(\vec{x},\beta_+,\beta_-,\Omega),A(\vec{x},\beta_+,\beta_-,\Omega).$$

The variables β_+,β_-, and Ω are auxiliary variables and $\vec{x}$ is a label for a point in a generic coordinate patch of the three-dimensional compact manifold that we are working on. The quantities $|0\rangle$, $A^\dagger$, and A satisfy

$$[A(\vec{x},\beta_+,\beta_-,\Omega),A^\dagger(\vec{x}\,',\beta'_+,\beta'_-,\Omega')] =\delta(\vec{x},\vec{x}\,')\delta(\beta_+-\beta'_+)\delta(\beta_--\beta'_-)\delta(\Omega-\Omega')\,, \tag{4.1}$$

$$[A,A]=[A^\dagger,A^\dagger]=0\,,$$

and

$$A\,|0\rangle=0\,. \tag{4.2}$$

The Hilbert space of states is found by taking the span of the action of all powers of $A^\dagger$ on $|0\rangle$ as is usual for a Fock representation. The variables β_+, β_-, and Ω will be associated with t_+, t_-, and τ, respectively.

The representation is developed further by translating it, i.e., by working in terms of the operators

$$B(\vec{x},\beta_+,\beta_-,\Omega)\equiv A(\vec{x},\beta_+,\beta_-,\Omega)+C(\beta_+,\beta_-)\,. \tag{4.3}$$

The function $C(\beta_+,\beta_-)$ labels the representations and determines many of their properties. It is chosen to be a function of β_+ and β_- only since C is closely connected with the "vacuum" and it is desired that it be a zero-frequency state, i.e., that it be independent of the variable Ω associated with the intrinsic time. This function C will be further determined below. Notice that B and $B^\dagger$ also satisfy the commutation relations of creation and annihilation operators.

It was emphasized in Sec. III that the variable $\tau(\vec{x})$ was to be associated with an intrinsic time, i.e., is nondynamical. In Appendix E we argue that, in spite of this, there is no way to avoid representing $\tau(\vec{x})$ as an operator on the Hilbert space. Explicit "Lorentz" covariance in the time and space variables τ, t_+, and t_- seems to be required.

The fields τ, t_+, and t_- with their conjugates are represented on the Hilbert space by the (non-Fock) expressions

$$\tau(\vec{x})=\int d\beta_+d\beta_-d\Omega B^\dagger(\vec{x},\beta_+,\beta_-,\Omega)\Omega B(\vec{x},\beta_+,\beta_-,\Omega)\,, \tag{4.4a}$$

$$\pi(\vec{x})=-i\int d\beta_+ d\beta_- d\Omega B^\dagger \frac{\partial}{\partial\Omega}B \;, \tag{4.4b}$$

$$t_+(\vec{x})=\int d\beta_+ d\beta_- d\Omega B^\dagger \beta_+ B \;, \tag{4.5a}$$

$$\pi_+(\vec{x})=-i\int d\beta_+ d\beta_- d\Omega B^\dagger \frac{\partial}{\partial\beta_+}B \;, \tag{4.5b}$$

$$t_-(\vec{x})=\int d\beta_+ d\beta_- d\Omega B^\dagger \beta_- B \;, \tag{4.6a}$$

$$\pi_-(\vec{x})=-i\int d\beta_+ d\beta_- d\Omega B^\dagger \frac{\partial}{\partial\beta_-}B \;. \tag{4.6b}$$

These operators are not necessarily all well defined and self-adjoint. This depends on the choice of $C(\beta_+,\beta_-)$.[4] Notice that we have represented all of the operators in terms of the same B and $B^\dagger$ while the form of the Hamiltonian (3.20) might indicate that they should be treated as independent and the Hilbert space should be chosen as the tensor product of those associated with each of the fields. The choice in (4.4)–(4.6) is motivated by the knowledge that τ, t_+, t_- represent different components of a single field g_{ij} to which a gauge condition has been applied, and by the closeness of the results that we find below to those of quantum cosmology.

It is convenient to define the following overcomplete family of states:

$$|p_+,p_-,\omega\rangle=\exp\left[i\int d\vec{x}[p_+(\vec{x})t_+(\vec{x})+p_-(\vec{x})t_-(\vec{x})-\omega(\vec{x})\tau(\vec{x})]\right]|0\rangle \;, \tag{4.7}$$

where p_+, p_-, and ω are elements of suitable function spaces. These states are eigenstates of the annihilation operator, i.e.,

$$A(\vec{x},\beta_+,\beta_-,\Omega)|p_+,p_-,\omega\rangle=\phi_{p_+p_-\omega}(\vec{x},\beta_+,\beta_-,\Omega)|p_+,p_-,\omega\rangle \;, \tag{4.8}$$

with

$$\phi_{p_+p_-\omega}(\vec{x},\beta_+,\beta_-,\Omega)=\exp\{i[p_+(\vec{x})\beta_+ + p_-(\vec{x})\beta_-(\vec{x})-\omega(\vec{x})\Omega)\}C(\beta_+,\beta_-)-C(\beta_+,\beta_-) \;. \tag{4.9}$$

Equation (4.9) is a simple result of (4.4)–(4.7) and the properties of creation and annihilation operators. Of course, $\phi_{p_+,p_-,\omega}=0$ when $p_+=p_-=\omega=0$. Formally, the inner product on the Hilbert space is given by

$$\langle p'_+,p'_-,\omega'|p_+,p_-,\omega\rangle=\exp[-\tfrac{1}{2}||\phi_{p_+p_-\omega}||^2-\tfrac{1}{2}||\phi_{p'_+p'_-\omega'}||^2+(\phi_{p'_+p'_-\omega'},\phi_{p_+p_-\omega})] \;, \tag{4.10}$$

where (,) is the inner product on the space of functions of β_+,β_-,Ω defined by $\int d\beta_+ d\beta_- d\Omega\phi^*\psi$, and $||\psi||^2=(\psi,\psi)$. Notice from (4.9) the emerging connection between what is being done here and quantum cosmology as described in Appendix D.

In field theory the proper definition of products of operators evaluated at the same spatial point must always be given. We want to find the analog for this representation of normal ordering for the usual Fock representation. To motivate the answer (which is more rigorously obtained in Ref. 33) consider the simple example of

$$t_+(\vec{x})t_+(\vec{x}')=\delta(\vec{x},\vec{x}')\int d\beta_+ d\beta_- d\Omega B^\dagger \beta_+^2 B + :t_+(\vec{x})t_+(\vec{x}): \;, \tag{4.11}$$

where : : denotes normal ordering of the operators A,$A^\dagger$. Taking $\vec{x}\to\vec{x}'$ and keeping the most singular part gives

$$z^{-1}t_+^2(\vec{x})=\int d\beta_+ d\beta_- d\Omega B^\dagger \beta_+^2 B\equiv(t_+^2)_r \;, \tag{4.12}$$

with $Z=\delta(\vec{x},\vec{x})$ and the subscript r standing for "renormalized." In general,

$$(F(t_+,t_-,\tau,\pi_+,\pi_-,\pi))_r=\int d\beta_+ d\beta_- d\Omega B^\dagger F\left[\beta_+,\beta_-,\Omega,-i\frac{\partial}{\partial\beta_+},-i\frac{\partial}{\partial\beta_-},-i\frac{\partial}{\partial\Omega}\right]B \;, \tag{4.13}$$

where the resolution of any factor-ordering ambiguities must be found for the particular F that is of interest. Expression (4.13) is what one would intuitively expect of an ultralocal field theory.

Finally we represent the Hamiltonian (3.20). The expression is given by

$$\mathscr{H}_0(\vec{x}) = -\int d\beta_+ d\beta_- d\Omega B^\dagger \left[\frac{\partial^2}{\partial\beta_+{}^2} + \frac{\partial^2}{\partial\beta_-{}^2} - \frac{\partial^2}{\partial\Omega^2} + V(\beta_+,\beta_-) \right] B \ , \tag{4.14}$$

where V is a renormalization term that depends on $C(\beta_+,\beta_-)$ and is determined by the requirement

$$\mathscr{H}_0(\vec{x})\,|\,0\rangle = 0 \ ; \tag{4.15}$$

V is thus the analog of the zero-point $\frac{1}{2}$ that has been taken out of the normally ordered harmonic-oscillator Hamiltonian. The operator $\mathscr{H}_0(\vec{x})$ (4.14) is formally self-adjoint.

The expression (4.14) given above for $\mathscr{H}_0$ would be tremendously complicated by the inclusion of the Faddeev-Popov ghost terms[29,30] appropriate to (3.22). There is some evidence[34] that the net effect of the Faddeev-Popov determinant is to guarantee that all of the operators of the theory are formally self-adjoint. The formalism given above already has that property. The problem with doing things in the way that we have chosen is not whether the formalism is internally consistent, but whether it is in any sense quantum gravity.

V. DYNAMICS

The operator $\mathscr{H}_0$ (4.14) should be investigated in more detail. The first thing that we need to do is determine the renormalization term $V(\beta_+,\beta_-)$. The requirement that $\mathscr{H}_0\,|\,0\rangle = 0$ implies the following relation between V and C:

$$V(\beta_+,\beta_-) = -\frac{1}{C}\left[\frac{\partial^2}{\partial\beta_+{}^2} + \frac{\partial^2}{\partial\beta_-{}^2} \right] C(\beta_+,\beta_-) \ . \tag{5.1}$$

Renormalization terms are usually determined through contact with experiment (which is out of the question for our $1/G \to 0$ theory) or invariance requirements. The studies of quantum cosmology give a possible candidate for an invariance requirement, i.e., conformal invariance.[24] We will impose the condition that

$$\frac{\partial^2}{\partial\beta_+{}^2} + \frac{\partial^2}{\partial\beta_-{}^2} + V(\beta_+,\beta_-) - \frac{\partial^2}{\partial\Omega^2}$$

be a conformally covariant wave operator for the space Ω,β_+,β_-. Since the metric on this space $(-d\Omega^2 + d\beta_+{}^2 + d\beta_-{}^2)$ is flat the conformally covariant wave operator is just the d'Alembertian, and this implies that $V(\beta_+,\beta_-) = 0$. To attain this condition we will take as the representative for the ground state

$$C(\beta_+,\beta_-) = 1 \ . \tag{5.2}$$

The theory with more general $C(\beta_+,\beta_-)$ can be developed in a manner analogous to what will follow, and (5.2) can be taken as an illustrative example (although there must be a way to choose from all of the inequivalent theories the proper one).

The Hamiltonian also has another conformal invariance, i.e., it is invariant under rescalings of the spatial metric g_{ij} [this fact is independent of the choice of $C(\beta_+,\beta_-)$]. These rescalings are generated by π, and this conformal invariance is simply a statement of the fact that (3.18) is τ independent or

$$\frac{\partial^2}{\partial\beta_+} + \frac{\partial^2}{\partial\beta_-{}^2} - \frac{\partial^2}{\partial\Omega^2} + V(\beta_+,\beta_-)$$

is Ω independent. That is to say, the Hamiltonian is (intrinsic) time independent.

With the choice (5.2), the states $|p_+p_-\omega\rangle$ satisfy

$$B\,|\,p_+,p_-,\omega\rangle = e^{i(p_+\beta_+ + p_-\beta_- - \omega\Omega)}\,|\,p_+,p_-,\omega\rangle \ , \tag{5.3}$$

and the connection with quantum cosmology becomes stronger. So far there is no connection between $p_+(x)$, $p_-(x)$, and $\omega(x)$. The physical states are to be determined by $\mathscr{H}_0\,|\,\psi\rangle = 0$. Applying this condition to the states $|p_+p_-\omega\rangle$ and using (5.3) implies

$$p_+{}^2(\vec{x}) + p_-{}^2(\vec{x}) = \omega^2(\vec{x}) \ . \tag{5.4}$$

The states $|p_+,p_-,\omega\rangle$ span the full Hilbert space; so the span of the subset satisfying (5.4) will be defined as the physical subspace. A state satisfying (5.4) is, loosely speaking, a coherent state of quantum Kasner universes, an independent universe at each spatial point.

It is tempting to now restrict the theory so that we only talk about physical states. This cannot be done since none of the operators (4.4)–(4.6) is self-adjoint on this space [if $\phi(\beta_+\beta_-\Omega)$ satisfies $(\partial^2/\partial\beta_+{}^2+\partial^2/\partial\beta_-{}^2-\partial^2/\partial\Omega^2)\phi=0$, then $\beta_+\phi$, say, does not]. We must work in the full Hilbert space of our theory and somehow obtain physical results. The way that we will choose to do this is to follow Stückelberg,[35] introduce a "fake" time parameter, and use the heat-equation-like expression

$$\mathscr{H}_0(x)\,|\,\psi\rangle = i\left[\int d\beta_+ d\beta_- d\Omega B^\dagger \frac{\partial}{\partial\lambda} B\right]|\,\psi\rangle\ . \tag{5.5}$$

We are treating $\partial^2/\partial\beta_+{}^2+\partial^2/\partial\beta_-{}^2-\partial^2/\partial\Omega^2$ as if it were a Hamiltonian operator generating the dynamics in the time λ and (5.5) is the associated Schrödinger equation that determines $|\psi\rangle$ as a function of λ. The parameter λ always increases while Ω both increases for positive frequency states (expanding "local universe") and decreases for negative frequency states (contraction).[8,24,20] Equation (5.5) is what we wish to build the scattering theory on. Stückelberg[35] used a similar equation to reduce the problem of pair creation to a potential-well scattering problem in ordinary quantum mechanics.

The above discussion with the time parameter λ might appear to be artificial. The physics that our formalism must describe has solutions that evolve both forward and backward in the natural timelike parameter τ, and this must be described in a unified manner. The usual way of handling this problem for the Klein-Gordon equation is to second quantize with the backward-evolving particles being identified with forward-evolving antiparticles. This is not an appropriate solution for the gravitational case described above. This theory is already second quantized with the ultralocality of the theory making the formalism reduce to something that looks like a first quantization of a Klein-Gordon-type operator. To quantize once more, i.e., "third quantize," is a step that we would prefer not to take, and we do not have to take it if we use the formalism of Stückelberg. Notice that even classically positive and negative frequencies mix in gravity, i.e., expanding solutions stop and contract. The full implications of (5.5) will appear in a subsequent paper.

It should be restated that the choice $C=1$ (5.2) is just one of many possible choices. This choice has the bad property that with it none of the expressions (4.4)–(4.6) is a well-defined operator. The example (5.2) has the advantage of making the connection between the contents of this paper and the previous work on quantum cosmology particularly transparent. This connection will continue to exist, with some modifications of detail, no matter what C is. One can hope that the development of the perturbation theory will determine the form of C, i.e., that for only a specific C will the perturbation theory be possible; for this reason a more detailed discussion of C is not given here.

VI. SUMMARY AND DISCUSSION

In this paper we have developed a formalism for treating the quantum theory the of the strong-coupling Hamiltonian for general relativity in a fixed gauge. The need to preserve the positive definiteness of the metric g_{ij} led us to the generators π^i_j of GL(3, R). This, combined with the identification of timelike, hypersurface orthogonal Killing vectors led to the symmetric space SL(3, R)/SO(3). The geometry of the symmetric space provided motivation for the gauge-fixing condition which enabled us to show that the quantum theory of the strong-coupling limit can be considered as an independent Kasner universe at each spatial point just as the classical theory can. In order to have a hope of developing a scattering theory the formalism forced us to revive methods developed by Stückelberg for treating the Klein-Gordon operator as the Hamiltonian in a Schrödinger equation. We see that what might have been a very complicated theory is in fact very simple.

This simplicity is, to a large extent, the result of our decision to work on a simple configuration space (i.e., $\mathscr{C}$), a choice that was the result of our looking at the strong-coupling (i.e., ultralocal) limit. The formulation of the theory without the complicating ghost terms is what allowed the exact quantum solutions to be explicitly written. We do not want to imply that the correctness of this simplifying assumption is anything more than a hope at this stage, but it is something that occurs naturally within the formalism. In giving a continuum quantization of a strong-coupling gauge theory we are entering virgin territory and much work is required before appropriate computational methods for treating this type of problem are discovered and confidence in them is gained. As was mentioned before, the formalism given here is, at worst, a first step towards the complete solution.

We have not yet mentioned what physical regime we expect the strong-coupling limit to be applicable to. In the covariant approach to quantum gravity the dimensionless parameter Gp^2 appears. If strong coupling means having this parameter be infinite, then we see that strong coupling corresponds to $p^2 \to \infty$, i.e., to short distances. Quantizing the strong-coupling limit to gravity is an attempt to develop a quantal description of physics inside the Planck length. Notice that the strong-coupling limit of general relativity naturally provides a conformally invariant description of this physics. It is amusing to note that we are using quantum cosmology to describe physics at short distances, i.e., we have the mystical unity of microcosm and macrocosm.

Before being carried away by mysticism remember that the formalism that we have developed is only a preliminary to making a perturbation expansion in $1/G$. Many of the simplifying features of the strong-coupling limit will be destroyed upon adding in the perturbation R, e.g., the intrinsic-time independence of the Hamltonian.[36] There is, in fact, little reason to believe that the perturbation theory that we want to construct will be any better defined than that of the covariant approach, but it will possibly give us a glimpse of the physics inside the Planck length. This, combined with what we already know about large distance, will hopefully help us to find a viable quantum theory of gravity.

ACKNOWLEDGMENTS

The author would like to thank R. Beig, M. Henneaux, C. Isham, J. Klauder, K. Kuchař, and C. Teitelboim for helpful discussions. He would also like to thank Professor P. Aichelburg, The Einstein Memorial Foundation, and the Institute for Theoretical Physics of the University of Vienna for their kind hospitality and support while many of the ideas presented in this paper were developed. In addition, he thanks the Science Research Council for its support. This work was supported by the Einstein Memorial Foundation and SRC Grant No. GR/A61463.

APPENDIX A: SL(3,R)/SO(3)

In Appendixes A and B the close connections between the symmetric space SL(3,R)/SO(3) and the basic variables chosen for the strong-coupling limit are described. The geometrical and group theoretic structure of this symmetric space and analysis on it are summarized. The presentation is very condensed, the purpose being to familiarize the reader with the terminology and state results specifically for SL(3,R)/SO(3) that are given in the literature for general symmetric spaces. For any real understanding of the content of these two appendixes Refs. 37–42 should be consulted. We will closely follow the discussion of Ref. 38.

The importance for canonical quantum gravity of the manifold M of 3×3, symmetric, positive-definite matrices γ_{ij} was first emphasized by DeWitt.[14] This importance is expected to be even greater in the ultralocal theory. M is a six-dimensional manifold with coordinates x^{A} given by the γ_{ij}. The Hamiltonian (1.1) implies that M should be given the contravariant (in spite of the indices) metric

$$G_{ijkl}=\tfrac{1}{2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}-\gamma_{kl}) . \tag{A1}$$

This metric differs from that of Ref. 14 by a factor of $\gamma^{-1/2}$. The inverse of (A1) is

$$G^{ijkl}=\tfrac{1}{2}(\gamma^{ik}\gamma^{jl}+\gamma^{il}\gamma^{jk}-2\gamma^{ij}\gamma^{kl}) \tag{A2}$$

with

$$G^{ijmn}G_{mnkl}=\delta^{ij}_{kl}\equiv\tfrac{1}{2}(\delta^{i}_{k}\delta^{j}_{l}+\delta^{i}_{l}\delta^{j}_{k}) . \tag{A3}$$

DeWitt showed that (A1) has signature $(-+++++)$.[14] It is this hyperbolic nature of G_{ijkl}, a characteristic of symmetric tensor ("spin-two") fields,[10] that leads to much of the interesting structure that is detailed in the main text and below.

The tangent space to M is spanned by the vectors $\partial/\partial\gamma_{ij}$. The importance of $\pi^{i}_{j}(x)$ described in the main text leads to the definition of

$$\rho^{i}_{j}\equiv\gamma_{jk}\frac{\partial}{\partial\gamma_{ki}} \tag{A4}$$

satisfying [in analogy with (2.4) and (2.5)]

$$\rho^{i}_{j}\gamma_{mn}=\tfrac{1}{2}(\gamma_{jm}\delta^{i}_{n}+\gamma_{jn}\delta^{i}_{m}) , \tag{A5}$$

$$[\rho^{i}_{j},\rho^{k}_{l}]=\tfrac{1}{2}(\rho^{k}_{j}\delta^{i}_{l}-\rho^{i}_{l}\delta^{k}_{j}) , \tag{A6}$$

where [,] here (and in the remainder of this appendix) denotes the Lie bracket. The vectors ρ^{i}_{j} are Killing vectors for the metric[26] (A1) as can be seen from

$$\mathscr{L}_{\rho^i_j}\left[G_{mnrs}\frac{\partial}{\partial\gamma_{mn}}\frac{\partial}{\partial\gamma_{rs}}\right]=(\rho^i_j G_{mnrs})\frac{\partial}{\partial\gamma_{mn}}\frac{\partial}{\partial\gamma_{rs}}+G_{mnrs}\left[\rho^i_j,\frac{\partial}{\partial\gamma_{mn}}\right]\frac{\partial}{\partial\gamma_{rs}}+G_{mnrs}\frac{\partial}{\partial\gamma_{mn}}\left[\rho^i_j,\frac{\partial}{\partial\gamma_{rs}}\right]=0\ .$$

The vector $\rho=\rho^i_i$ has as its norm

$$\rho\cdot\rho=G^{ijkl}\gamma_{ij}\gamma_{kl}=-6\ , \tag{A7}$$

so it is a timelike Killing vector. Now label the points of M by τ, $\tilde{\gamma}_{ij}$ where[14,43]

$$\tau=\ln\gamma^{1/3},\ \ \tilde{\gamma}_{ij}=\gamma^{-1/3}\gamma_{ij}=e^{-\tau}\gamma_{ij} \tag{A8}$$

with

$$\det\tilde{\gamma}_{ij}=1\ .$$

Notice that $\rho\tau=1$, i.e.,

$$\rho=\frac{\partial}{\partial\tau}\ , \tag{A9}$$

and τ is the timelike coordinate conjugate to ρ. Note that

$$\rho\cdot\frac{\partial}{\partial\tilde{\gamma}_{ij}}=0\ , \tag{A10}$$

$$\frac{\partial}{\partial\tilde{\gamma}_{ij}}\cdot\frac{\partial}{\partial\tilde{\gamma}_{kl}}=\tilde{G}^{ijkl}\ , \tag{A11}$$

with

$$\tilde{G}^{ijkl}=\tfrac{1}{2}(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl}+\tilde{\gamma}^{il}\tilde{\gamma}^{jk}-\tfrac{2}{3}\tilde{\gamma}^{ij}\tilde{\gamma}^{kl})\ , \tag{A12}$$

$$\tilde{G}_{ijkl}=\tfrac{1}{2}(\tilde{\gamma}_{ik}\tilde{\gamma}_{jl}+\tilde{\gamma}_{il}\tilde{\gamma}_{jk}-\tfrac{2}{3}\tilde{\gamma}_{ij}\tilde{\gamma}_{kl})\ , \tag{A13}$$

and

$$\tilde{G}^{ijmn}\tilde{G}_{mnkl}=\tilde{\delta}^{ij}_{kl}=\delta^{ij}_{kl}-\tfrac{1}{3}\tilde{\gamma}^{ij}\tilde{\gamma}_{kl}\ . \tag{A14}$$

From (A7), (A10), and (A11) it is seen that the metric $G^{\Lambda\Gamma}$ is given by

$$\begin{bmatrix}-\frac{1}{6} & 0\\ 0 & \tilde{G}_{ijkl}\end{bmatrix}, \tag{A15}$$

and from (A15) it follows that ρ is a timelike, hypersurface orthogonal Killing vector.[19]

As for the P^i_j of Sec. III, define the trace-free part of ρ^i_j,

$$\tilde{\rho}^i_j\equiv\rho^i_j-\tfrac{1}{3}\rho\delta^i_j \tag{A16}$$

satisfying

$$[\rho,\tilde{\rho}^i_j]=0\ , \tag{A17}$$

$$\tilde{\rho}^i_j\tau=0\ , \tag{A18}$$

$$\rho\cdot\tilde{\rho}^i_j=0\ , \tag{A19}$$

$$\tilde{\rho}^k_l\tilde{\gamma}_{ij}=\tfrac{1}{2}(\tilde{\gamma}_{il}\delta^k_j+\tilde{\gamma}_{jl}\delta^k_i-\tfrac{2}{3}\tilde{\gamma}_{ij}\delta^k_l)\ , \tag{A20}$$

$$[\tilde{\rho}^i_j,\tilde{\rho}^k_l]=\tfrac{1}{2}(\tilde{\rho}^k_j\delta^i_l-\tilde{\rho}^i_l\delta^k_j)\ . \tag{A21}$$

The $\tilde{\rho}^i_j$ are thus perpendicular to ρ and they lie entirely within the space spanned by $\partial/\partial\tilde{\gamma}_{ij}$; in fact

$$\tilde{\rho}^i_j=\tilde{\gamma}_{jk}\frac{\partial}{\partial\tilde{\gamma}_{ik}} \tag{A22}$$

(remember that $\partial/\partial\tilde{\gamma}_{ij}$ is a nonholonomic basis and must be used with care). The five-dimensional manifold parametrized by $\tilde{\gamma}_{ij}$ (the 3×3, symmetric, positive-definite matrices with determinant equal to one) will be denoted by $\tilde{M}$.

The group GL(3,R) acts transitively on M, its action being given by [as in Eq. (2.6)]

$$\gamma_{ij}\rightarrow(X)_{im}\gamma_{mn}(X^t)_{nj}\equiv(X)\gamma(X^t)\ , \tag{A23}$$

where $\gamma_{ij}\in M$, $X_{ij}\in$ GL(3,R), and t denotes transpose. The vectors ρ^i_j (A4) are generators of the action of GL(3,R) on M and (A5) are just the commutation relations for GL(3,R). The group SL(3,R) acts transitively on $\tilde{M}$, its action the same as in (A23). The $\tilde{\rho}^i_j$ (which are traceless) are the generators of this action with (A21) being the commutation relations for SL(3,R).

The Lie algebra sl(3,R) of SL(3,R) can be represented by traceless 3×3 matrices, with the Lie brackets given by the commutator of matrix multiplication. The Killing form is given by

$$B(X,Y)=6\,\mathrm{Tr}(XY),\ \ X,Y\in \mathrm{sl}(3,R)\ . \tag{A24}$$

The Cartan decomposition of sl(3,R) is given by

$$\mathrm{sl}(3,\mathrm{R})=\mathrm{so}(3)+p\ , \tag{A25}$$

where SO(3) is the set of 3×3 antisymmetric matrices and p is the set of 3×3, traceless, symmetric matrices. The associated Cartan involution is $\theta(X)=-X^t$; $\theta(X)=X$, $X\in\mathrm{so}(3)$; $\theta(X)=-X$, $X\in p$. There is a global analog of (A25), i.e., the "polar" decomposition

$$\mathrm{SL}(3,R)=P\,\mathrm{SO}(3)\ , \tag{A26}$$

where P is the space of 3×3, symmetric, positive-definite matrices, of determinant one (i.e., $P=\widetilde{M}$). In addition, it is true that

$$\widetilde{M}=\mathrm{SL}(3,\mathrm{R})/\mathrm{SO}(3)\ . \tag{A27}$$

We note that SO(3) is the maximal compact subgroup of SL(3,R).

The space SL(3,R)/SO(3) is a noncompact symmetric space. By definition a symmetric space S is a manifold with metric such that for each $p\in S$ there is an isometry of S that corresponds to reversing the parametrization of the geodesics through p.[37,38] Locally, the covariant derivative of the Riemann tensor of a symmetric space vanishes. For $\widetilde{M}$ take the point $e=\delta_{ij}$ as the origin, which is invariant under SO(3). Equation (A23) with $\gamma_{ij}=\delta_{ij}$ provides a mapping from SL(3,R) on to $\widetilde{M}$. This mapping induces a mapping from sl(3,R) to the tangent space at e that has so(3) as its kernel. The tangent space at e is isomorphic to p in (A25). The Killing form (A24) induces an SL(3,R) invariant metric on $\widetilde{M}$ and, up to a factor, this is given (somewhat surprisingly) by (A12)[42,44] (remember that $\tilde{\gamma}^{ij}d\tilde{\gamma}_{ij}=0$), the metric that was put on $\widetilde{M}$ by the Hamiltonian. We see that $\widetilde{M}$ is a Riemannian manifold. The geodesic-reversing isometry is induced by the Cartan involution θ, thus showing $\widetilde{M}$ to be a symmetric space.

The geometry of $\widetilde{M}$ can be understood further by looking in more detail at the Lie algebra sl(3,R). The maximal Abelian subspace of p is given by the diagonal, traceless matrices; call this set a [the Cartan subalgebra of sl(3,R); the Cartan subgroup of SL(3,R) is e^a]. If α is a real-valued linear function on a, define

$$g_\alpha=\{\,X\in\mathrm{sl}(3,R)\,|\,[H,X]=\alpha(H)X,\forall H\in a\,\}\ . \tag{A28}$$

If $g_\alpha\neq\{0\}$ then α is called a restricted root. The restricted roots for sl(3,R) are given by $a_{ij}(H)=e_i(H)-e_j(H)$, $H\in a$, $e_i(H)$ being the ith diagonal element of H. Call a' the subset of a with all roots nonzero. For sl(3,R) a' is the set of H in a with $e_i(H)\neq e_j(H)$ for all $i\neq j$. (A point at which a root is zero is called a singular point.) A Weyl chamber a^+ is a is a connected component of a'; its boundary points in a are singular points. For sl(3,R) one Weyl chamber is $\{H\,|\,e_1(H)>e_2(H)>e_3(H)\}$. We call a root α positive if its values on a^+ are positive, and denote

$$n=\sum_{\alpha>0}g_\alpha\ . \tag{A29}$$

For sl(3,R) with the above choice of a^+, n is given by the matrices of the form

$$\begin{bmatrix}0&n_1&n_2\\0&0&n_3\\0&0&0\end{bmatrix}\ . \tag{A30}$$

This leads to the Iwasawa decomposition of a Lie group. For sl(3,R) it is clear that

$$\mathrm{sl}(3,\mathrm{R})=\mathrm{so}(3)+a+n\ , \tag{A31}$$

and the Iwasawa decomposition states that for any $g\in\mathrm{SL}(3,\mathrm{R})$,

$$g=OAN;\ \ O\in\mathrm{SO}(3),\ A\in e^a,\ N\in e^n\ . \tag{A32}$$

N is unipotent. From (A32) and (A23) with $\gamma_{ij}=\delta_{ij}$ we see that all elements of $\widetilde{M}$ can be written as

$$(\tilde{\gamma})=NAA^tN^t\ , \tag{A33}$$

and we denote by E that submanifold of $\widetilde{M}$ given by AA^t for all $A\in e^a$.

Define the normalizer and centralizer of e^a in SO(3) by

$$Q'=\{\,O\in\mathrm{SO}(3)\,|\,O^{-1}e^aO\subset e^a\,\}\ , \tag{A34}$$

$$Q=\{\,O\in\mathrm{SO}(3)\,|\,O^{-1}e^{a_1}O=e^{a_1},\forall a_1\in a\,\}\ . \tag{A35}$$

Q' is a 24-element, discrete subgroup of SO(3). Q is given by the matrices [remember, they are elements of SO(3) not so(3)]

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}. \tag{A36}$$

Define the Weyl group to be Q'/Q. It has the matrix representation

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix},\ \begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix},\ \begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix},\ \begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix},\ \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \tag{A37}$$

and its action on a is just that of the permutation group on three letters. The Weyl group corresponds to reflections through the singular points of a, and maps one Weyl chamber into another.

The boundary of $SL(3,R)/SO(3)$ can be considered to be $SO(3)/Q$. This is analogous to saying that the boundary of R^2 is the circle $S^1=SO(2)$. $SL(3,R)/SO(3)$ (minus the points corresponding to the singular points of a) is given by $e^{a^+}\times[SO(3)/Q]$. This is like saying that the R^2-origin is given by $R^2\times S^1$, R^+ corresponding to the values of the radius; so we see that a Weyl chamber corresponds to a measure of radius, the singular points to the origin of a polar coordinate system, and the Weyl group to reflections through the origin. Notice that the boundary $SO(3)/Q$ is three dimensional while $SL(3,R)/SO(3)$ is five dimensional. This means that there are two independent radial directions. These correspond to the two independent generators of the Cartan subgroup of $SL(3,R)$. It is to these that we have identified the independent degrees of freedom in the gauge condition of Sec. III.

Being slightly cavalier about factor ordering, the analog of $\mathscr{H}_0$ on M is

$$D=\frac{\partial}{\partial\widetilde{\gamma}_{ij}}\widetilde{G}_{ijkl}\frac{\partial}{\partial\widetilde{\gamma}_{kl}}-\frac{1}{6}\frac{\partial^2}{\partial\tau^2}, \tag{A38}$$

where we notice that, because $\det\widetilde{G}_{ijkl}=\text{const}$, the first term in (A38) is the Laplace-Beltrami operator on $\widetilde{M}$. This leads to a discussion of analysis on symmetric spaces that is given in the next appendix.

APPENDIX B: ANALYSIS ON SL(3,R)/SO(3)

Differential operators of fundamental interest on any symmetric space G/K are those that are invariant under the action of G on G/K, and annihilate the constants.[37,38] The algebra of such differential operators has r independent generators where r is the rank of the symmetric space,[37] i.e., the number of generators of the Cartan subgroup h of G. For $SL(3,R)/SO(3)$, r is equal to two. These independent generators can be identified with Weyl group-invariant polynomial functions of the generators of h.[38,30]

For our purposes the most important differential operator is the Laplace-Beltrami operator, which is related to the Casimir operator on the Lie algebra.[37] Note that the contribution $P_i^jP_j^i$ to the Hamiltonian (3.2) has zero brackets with P_l^k, and the expression $\widetilde{p}_j^i\widetilde{p}_i^j$ associated with it is the second-order Casimir invariant of $sl(3,R)$. The Hamiltonian for the strong-coupling limit is thus, essentially, the Laplacian on $SL(3,R)/SO(3)$. In addition, if $\widetilde{p}_1,\widetilde{p}_2$ are the generators

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

of the Cartan subgroup of $SL(3,R)$ the Weyl group-invariant polynomial that is associated with the Laplacian is just

$$(\widetilde{p}_1)^2+(\widetilde{p}_2)^2-\widetilde{p}_1\widetilde{p}_2\ . \tag{B1}$$

The close relationship between this and the Hamiltonian after gauge restriction (3.18) has already been pointed out. Note that all $SL(3,R)$-invariant differential operators on $SL(3,R)/SO(3)$ are generated by $\widetilde{p}_j^i\widetilde{p}_i^j$ and $\widetilde{p}_j^i\widetilde{p}_k^j\widetilde{p}_i^k$. The physical interpretation of the latter operator is as yet obscure.

The gauge condition adopted in Sec. III effectively restricts our attention to the submanifold E of $SL(3,R)/SO(3)$ obtained by the action of the Cartan subgroup on the identity. If β_1,β_2 are the canonical coordinates conjugate to $\widetilde{p}_1$ and $\widetilde{p}_2$, then E is given by the manifold of matrices

$$\begin{bmatrix} e^{\beta_1} & 0 & 0 \\ 0 & e^{\beta_2} & 0 \\ 0 & 0 & e^{-(\beta_1+\beta_2)} \end{bmatrix} . \tag{B2}$$

The Laplacian on E is just[40]

$$\Delta_E = \frac{\partial^2}{\partial\beta_1{}^2} + \frac{\partial^2}{\partial\beta_2{}^2} - \frac{\partial^2}{\partial\beta_1\partial\beta_2} \tag{B3}$$

whose relation to (B1) is obvious.

A crude way of looking at the effect of our gauge-fixing condition is that it restricts the domain of the theory from all of SL(3,R)/SO(3) to the two-dimensional submanifold E. The Hamiltonian starts off as the Laplacian on SL(3,R)/SO(3) and, through gauge fixing, is projected onto an operator on E. As a result of the ambiguities of quantization (in particular, those associated with the ability to make a canonical transformation, see Appendix C) there is no unique way of identifying this operator on E. In Appendix C we argue that, up to renormalization, the choices are equivalent, and we choose the simplest, i.e., the Laplacian (B3). Other choices are discussed in Refs. 40 and 41. Theorem (2.11) of Ref. 41 in particular gives the relation between the operator and the measure chosen on E complete with "renormalization" terms that result from different choices of operator and measure. It should be remembered that the importance of SL(3,R)/SO(3), E, and the measure on E is particularly enhanced by the exponential representation that we have chosen for the strong-coupling limit.

The subject of analysis on symmetric spaces is highly developed. The topics of harmonic functions, spherical functions, Fourier analysis, and group representations are particularly important. In addition the Green's function for the Laplacian has been computed.[40] While these things will be important in subsequent papers they do not apply to this paper and discussion of them has therefore been omitted.

APPENDIX C: DISCUSSION OF A SIMPLE GAUGE CONDITION

In this appendix we wish to discuss a simple gauge condition that bears a close resemblance to the one discussed in the main text. The theory is one with two sets of canonically conjugate variables x,p_x and y,p_y with the Hamiltonian

$$H = p_x{}^2 + p_y{}^2 , \tag{C1}$$

and the "gauge" constraint

$$L \equiv xp_y - yp_x = 0 . \tag{C2}$$

The constraint (C2) generates rotations and implies that the theory is cylindrically symmetric.

The gauge-fixing condition is taken to be

$$y = 0 , \tag{C3}$$

and the Faddeev-Popov factor is given by

$$\{ y,L \} = x ; \tag{C4}$$

so the gauge condition is (locally) a good one except at $x=0$. The breakdown at $x=0$ is a signal that there is possibly a Gribov ambiguity.[45] We know that there is a Gribov ambiguity. Any point on the negative x axis is the image under a global "gauge" transformation generated by (C2) of a point on the positive x axis. The constrained surface defined by (C3) is twice as large as it should be.

Using the constraints (C2) and (C3) the canonically conjugate pair y,p_y may be eliminated from the theory. The Hamiltonian becomes

$$H = p_x{}^2 . \tag{C5}$$

Notice the strong analogy between (C4) and (3.22), and (C5) and (3.18). Upon quantization (C5) becomes

$$H = -\frac{\partial^2}{\partial x^2} , \tag{C6}$$

which is self-adjoint in the measure $\int dx$.

This is not what we would get if we first quantized (C1) and then restricted ourselves to a rotationally invariant theory. Doing this we obtain

$$H = -\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} \right] , \tag{C7}$$

which is self-adjoint in the measure $\int |x| \, dx$. There seems to be an ambiguity in that gauge fixation and quantization do not seem to commute. One might be afraid that the theory developed in this paper (which is just a more complicated version of this simple example) possesses this same ambiguity. We will argue that, up to "renormalization," there is no ambiguity. If one has a principle that determines the renormalization of the Hamiltonian then there is no need to worry about these ambiguities. Such a principle, applicable to the strong-coupling limit of general relativity, is proposed in Sec. V.

To see that (C6) can be transformed into (C7) take

$$\tilde{\psi} = |x|^{1/2}\psi , \tag{C8}$$

with $\tilde{\psi}$ a member of the Hilbert space of functions square integrable in the measure $\int dx$. Notice that $\int dx \tilde{\psi}^* \tilde{\phi} = \int |x| dx \psi^* \phi$. Now

$$\frac{\partial^2}{\partial x^2}\tilde{\psi} = |x|^{1/2}\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} - \frac{1}{4x^2}\right]\psi ; \tag{C9}$$

so, up to the "renormalization" term $-1/4x^2$, the combination $\tilde{\psi}$, $\int dx, \partial^2/\partial x^2$ has been taken into

$$\psi, \int |x| dx, \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} .$$

As long as there is some way to determine this renormalization term, the two ways of formulating (C1) plus (C2) are equivalent. Also notice that if, instead of choosing $p_x = -i\partial/\partial x$ in going from (C5) to (C6), the choice $p_x = -i(\partial/\partial x + 1/2x)$ (which corresponds to a canonical transformation) is made, then similar results are obtained,[46] i.e.,

$$P_x{}^2 = -\left[\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} - \frac{1}{4x^2}\right] .$$

APPENDIX D: QUANTUM COSMOLOGY

In an effort to gain a better understanding of the possible structure of a quantum theory of gravity the quantization of simplified models of gravity was considered in the early Seventies.[24,26] The simplification was obtained by assuming homogeneity as well as other symmetries, i.e., only the modes satisfying the symmetries were quantized. In this way, the quantum field theory of general relativity was reduced to finite-dimensional quantum models.

In the simplest examples, the spacetime metric was assumed to take the form

$$ds^2 = -N^2(t)dt^2 + g_{ij}(t)d\vec{x}^i d\vec{x}^j , \tag{D1}$$

with g_{ij} a diagonal matrix given by

$$\begin{aligned} g_{11} &= \exp[2(-\Omega + \beta_+ + \sqrt{3}\beta_-)] , \\ g_{22} &= \exp[2(-\Omega + \beta_+ - \sqrt{3}\beta_-)] , \\ g_{33} &= \exp[2(-\Omega - 2\beta_+)] . \end{aligned} \tag{D2}$$

On the space determined by the variables Ω, β_+, β_- (minisuperspace) the metric is given by

$$d\mathscr{S}^2 = -d\Omega^2 + d\beta_+{}^2 + d\beta_-{}^2 , \tag{D3}$$

and this leads to the study of the Klein-Gordon equation

$$-\frac{\partial^2\psi}{\partial\Omega^2} + \frac{\partial^2\psi}{\partial\beta_+{}^2} + \frac{\partial^2\psi}{\partial\beta_-{}^2} + \mathscr{R}(\beta_+,\beta_-)\psi = 0 . \tag{D4}$$

Equation (D4) is the Hamiltonian constraint $\mathscr{H}_\perp = 0$ with $\mathscr{R} = gR$. The spatial constraints $\mathscr{H}_i = 0$ do not play a role as a result of the assumption of homogeneity. The case that is of most interest to us is that of Bianchi type I without matter, for which (D4) is

$$-\frac{\partial^2\psi}{\partial\Omega^2} + \frac{\partial^2\psi}{\partial\beta_+{}^2} + \frac{\partial^2\psi}{\partial\beta_-{}^2} = 0 . \tag{D5}$$

This represents a quantum Kasner universe.[24,26]

The Hilbert space that is usually taken is the space of $\psi(\beta_+,\beta_-,\Omega)$ with one of two inner products. The first is

$$\langle\psi,\phi\rangle = i\int_{\Omega=\text{const}} d\beta_+ d\beta_- \times\left[\psi^*\frac{\partial\phi}{\partial\Omega} - \frac{\partial\psi^*}{\partial\Omega}\phi\right] , \tag{D6}$$

which is independent of Ω, but does not give a positive-definite norm. The second requires a timelike, hypersurface-orthogonal Killing vector (although it can be generalized[47]). This inner product is given by[47,19]

$$\langle\psi,\phi\rangle = \tfrac{1}{2}\int_{\Omega=\text{const}} d\beta_+ d\beta_- \times\left[\psi^*\frac{\partial J\phi}{\partial\Omega} - \frac{\partial\psi^*}{\partial\Omega}J\phi\right] , \tag{D7}$$

where $J\phi$ is defined by first using the Killing vector to decompose into positive- and negative-frequency parts, $\phi = \phi^{(+)} + \phi^{(-)}$, and taking

$$J\phi = i\phi^{(+)} + (-i)\phi^{(-)} . \tag{D8}$$

This inner product is Ω independent and gives a positive-definite norm.

The wavelike solutions to (D5) have the form

$$\psi = e^{i(p_+\beta_+ + p_-\beta_- - \omega\Omega)} , \tag{D9}$$

with

$$p_+{}^2 + p_-{}^2 = \omega^2 . \tag{D10}$$

Many models other than the one represented by

(D5) were studied.[24,26] A feature of many of them that provides us with motivation is that they take the form of a scattering problem in minisuperspace. The asymptotic states are given by wave functions of the form (D9), i.e., waves propagating on the light cone of metric (D3), with the various models, having different potentials, scattering one of these states into another. The Kasner wave equation (D5) is closely related to the strong-coupling limit of the full theory, and we visualize the perturbation in $1/G$ as providing a similar type of scattering theory.

APPENDIX E: OTHER CHOICES OF REPRESENTATION

In Sec. IV an unorthodox choice of representation for the canonical commutation relations of $\tau,\pi,t_+,\pi_+,t_-,\pi_-$ was made. In this appendix that choice is motivated by eliminating the other obvious choices.

All of the choices will be discussed within the formalism of exponential representations.[32] Briefly, what this formalism does is take a "small" Hilbert space h and associate with each element of h an element in the "large" Hilbert space H on which one is representing the canonical commutation relations of the fields. The relation of basic importance is the following: if $\psi,\psi'\in h$ are associated with $|\psi\rangle,|\psi'\rangle\in H$, then

$$\langle\psi'\mid\psi\rangle=\exp[-\tfrac{1}{2}\|\psi'\|^2-\tfrac{1}{2}\|\psi\|^2+(\psi',\psi)]\ . \quad \text{(E1)}$$

where (,) is the inner product on h and $\|\psi\|^2=(\psi,\psi)$. This is a generalization of coherent states. For ultralocal theories one intuitively feels that, because the dynamics at different spatial points are independent, the theory should in some sense reduce to the finite-dimensional quantum mechanics of the theory at one spatial point. In this rough way of thinking, h represents the Hilbert space of the finite-dimensional quantum mechanics and the process of exponentiating it "spreads" this out into a field theory.

Classically, the ultralocal limit of general relativity has as a general solution an independent Kasner universe at every spatial point.[48–50] We thus expect the quantum theory to somehow be an independent quantum Kasner universe (see Appendix D) at each spatial point. As a result, the most natural choice for h is the Hilbert space associated with the quantum Kasner universe with one of the two inner products discussed in Appendix D [notice that even if (,) for h is not positive definite $\langle\ ,\ \rangle$ given by (E1) is], but problems occur when this is exponentiated. The elements of h are functions of Ω as well as β_+ and β_-, but the inner products have integrations only over $d\beta_+$ and $d\beta_-$. There is no way to get rid of the Ω dependence, and instead of representing $t_+(x)$, say, one ends up being forced to represent $t_+(x,\Omega)$, an explicit function of Ω. It is hard to make sense of this, and we reject this approach.

To correct the above deficiencies we are led to take the spacetime Hamiltonian decomposition that we have already made and make an additional Hamiltonian decomposition with respect to τ,t_+,t_-. In this approach, the Hilbert space h is taken to be the space of square-integrable functions $\psi(\beta_+,\beta_-)$ with the inner product $\int d\beta_+d\beta_-\psi^*\phi$. The operator $\partial^2/\partial\beta_+{}^2+\partial^2/\partial\beta_-{}^2$ is positive definite and self-adjoint on this space, and it possesses a completely, orthonormal set of eigenstates u_ω such that

$$\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}\right]u_\omega=-\omega^2u_\omega\ .$$

In using this to treat the pure quantum cosmology case, with each u_ω, $\omega>0$ is associated two time-dependent states $u_\omega e^{-i\omega\Omega}$ and $u_\omega e^{i\omega\Omega}$ that are elements of the space of functions of β_+, β_-, and Ω. On that space we have the Ω-independent, positive-definite inner product

$$(\phi_1,\phi_2)=\int d\beta_+d\beta_-\left[\phi_1^*\frac{\partial}{\partial\Omega}J\phi_2-\left[\frac{\partial}{\partial\Omega}\phi_1^*\right]J\phi_2\right]\ ,$$

where J is defined as in Appendix D.

To construct the exponential representation associated with h we do the analog of what was done in Sec. IV. We have operators $B^\dagger(x,\beta_+,\beta_-)$ and $B(x,\beta_+,\beta_-)$ which satisfy standard commutation relations, and, in terms of them,

$$t_+(\vec{x})=\int d\beta_+d\beta_-B^\dagger\beta_+B,$$

$$\pi_+(\vec{x})=-i\int d\beta_+d\beta_-B^\dagger\frac{\partial}{\partial\beta_+}B\ ,$$

$$t_-(\vec{x})=\int d\beta_+d\beta_-B^\dagger\beta_-B,$$

$$\pi_-(\vec{x})=-i\int d\beta_+d\beta_-B^\dagger\frac{\partial}{\partial\beta_-}B\ ,$$

and

$$\hat{H}=-\int d\beta_+d\beta_-B^\dagger\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}\right]B\ .$$

In addition we have the overcomplete set of states

$$|p_+,p_-\rangle = \exp\left[i\int [p_+(\vec{x})t_+(\vec{x}) + p_-(\vec{x})t_-(\vec{x})]\right] d\vec{x}\,|0\rangle\ .$$

The problem is now to include the "time" dependence in these states, i.e., to do the analog of what was done above for the u_ω. There seems to be no reasonable way of doing this. Finding the expressions for the eigenstates of $\hat{H}$ would be difficult and they would not have a simple connection with the space h that one would need to motivate following the treatment of $\partial^2/\partial\beta_+{}^2 + \partial^2/\partial\beta_-{}^2$ and u_ω given above.

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