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Stationary cylindrically symmetric cluster of charged particles in general relativity

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Raychaudhuri and Som have investigated the problem of a stationary cylindrically symmetric cluster of uncharged particles in general relativity. In this paper we have considered the same problem with charged particles and have obtained some new interesting results.

I. INTRODUCTION

Einstein[1] investigated the problem of a stationary spherically symmetric cluster of particles moving freely under the influence of the gravitational field produced by all of them. He showed that the particles at the boundary of the cluster, for a given gravitational mass, are constrained to move beyond a certain critical distance so that the Schwarzschild singularity is unattainable.

Later, in analogy with Einstein's investigation, Raychaudhuri and Som[2] studied the problem of a stationary cylindrically symmetric cluster of particles and, in contrast to the case of spherical symmetry, found that there is no limit as far as the radial concentration is concerned, but there is a limit to the concentration per unit length of the cylinder which is independent of the radius. Considering an infinitely thin cylindrical shell of particles they showed that unlike a thin spherical shell of particles, the gravitational mass per unit length is independent of the radius. This is not however generally correct as we have pointed out in Sec. III of this paper.

Recently Banerjee and Bhattacharjee[3] have shown that the Reissner-Nordström singularity is also unattainable in the case of a stationary spherically symmetric cluster of charged particles. They have further shown that photons and neutrinos moving radially outwards inside the system cannot be trapped within the cluster.

In this paper we investigate the problem corresponding to that studied by Raychaudhuri and Som[2] when the orbiting particles are charged. Obviously this work will be of little astrophysical interest. But we believe that our results will be of some interest from the theoretical point of view. It is interesting to note here that the solution of a cylindrically symmetric cluster of charged particles differs in some respects from that of uncharged particles. Unlike the cluster of uncharged particles, there is a lower limit to the radius in the case of a cluster of charged particles. In other words, like a spherically symmetric cluster, the charged particles at the boundary for given mass and charge per unit length are constrained to move beyond a certain critical distance. Of course there is also an upper limit to the mass per unit length of the cluster.

In Sec. III, we discuss the case of an extremely thin cylindrical shell of charged particles. We find that the gravitational mass per unit length of the shell is not independent of the radius. We also obtain a similar result for a cylindrical cluster of finite radius.

In Sec. V, we investigate the motion of photons and neutrinos. From the analysis it appears that there is a possibility of photons and neutrinos coming out from the interior of a cylindrical cluster of charged particles. This possibility exists in the case of a cluster of uncharged particles also.

II. BASIC EQUATIONS

Let us consider the static cylindrically symmetric line element in Weyl's canonical form,

$$ds^2 = e^{2\alpha}dt^2 - e^{2\beta-2\alpha}(dr^2+dz^2) - r^2e^{-2\alpha}d\phi^2 , \tag{1}$$

where α and β are functions of r alone. A purely circular motion in the (r,z) plane is characterized by

$$\frac{dr}{ds}=\frac{dz}{ds}=0\ . \qquad (2)$$

Since there is only a radial electric field, the only nonvanishing component of $F^{\mu\nu}$ is F^{14}. Then the Maxwell equation $F^{\mu\nu}{}_{;\nu}=4\pi J^{\mu}$ leads to

$$F^{14}=-\frac{A(r)}{r}e^{2\alpha-2\beta} \text{ for } \mu=1 \qquad (3)$$

and

$$J^{4}=\frac{[A(r)]'}{4\pi r}e^{2\alpha-2\beta} \text{ for } \mu=4\ , \qquad (4)$$

where $A(r)$ is an arbitrary function of r and J^{μ} is the four-current density. Here a prime denotes differentiation with respect to r. Since the charged particles have no radial motion and rotate perfectly at random we get $J^1=J^2=J^3=0$. J^2 vanishes owing to equal numbers of charged particles moving in opposite directions. Thus the only nonvanishing component of J^{μ} is J^4.

The charge density σ is given by

$$\sigma^2=J^4J_4=\frac{[A'(r)]^2}{16\pi^2r^2}e^{6\alpha-4\beta}$$

and

$$4\pi\sigma=\pm\frac{[A(r)]'}{r}e^{3\alpha-2\beta}\ . \qquad (5)$$

In view of Eq. (2), from consideration of the Lorentz force equation with the line element (1), we get

$$r(r\alpha'-1)\left[\frac{d\phi}{ds}\right]^2+\alpha' e^{4\alpha}\left[\frac{dt}{ds}\right]^2=\frac{[A^2(r)]'}{8\pi\rho r^2}e^{6\alpha-2\beta}\ , \qquad (6)$$

$$\frac{d^2\phi}{ds^2}=\frac{d^2t}{ds^2}=0\ , \qquad (7)$$

where ρ is the proper density of matter. From Eqs. (1) and (6) we get

$$e^{2\alpha}\left[\frac{dt}{ds}\right]^2=\frac{1}{1-2r\alpha'}\left[(1-r\alpha')-\frac{[A^2(r)]'}{8\pi\rho r}e^{4\alpha-2\beta}\right] .(8)$$

The geodesic will be timelike if and only if

$$r\alpha'<\tfrac{1}{2} \text{ and } r\alpha'+\frac{[A^2(r)]'}{8\pi\rho r}e^{4\alpha-2\beta}<1\ . \qquad (9)$$

These equations are confirmed from Eq. (22) later also.

For the exterior static field the line element of the form (1) is[4]

$$ds^2=-H^2e^{2C^2}(C_1r^{-C}+C_2r^{C})^2(dr^2+dz^2)-r^2(C_1r^{-C}+C_2r^{C})^2d\phi^2+(C_1r^{-C}+C_2r^{C})^{-2}dt^2\ , \qquad (10)$$

where H, C, and C_1, and C_2 are constants, $C/2$ being the mass per unit length of the cylinder. Further, C_1 is positive and C_2 is negative and is related to the charge per unit length.[5] Hence $r\alpha'<1$ gives for particles at the boundary of our distribution[6]

$$r>\left[\frac{2C-1}{2C+1}\frac{C_1}{C_2}\right]^{1/2C}\ . \qquad (11)$$

Since C_2 is negative one must have

$$C<\tfrac{1}{2}\ . \qquad (12)$$

Equation (11) shows that the particles at the boundary for a given mass and charge per unit length are constrained to move beyond a certain critical distance. Hence we can conclude that there is a lower limit to the radius of the cluster. It is interesting to note here the difference between the clusters of charged and uncharged particles. Raychaudhuri and Som obtained only Eq. (12) in the uncharged case. This shows that for clusters of uncharged particles the restriction is not on the radius of the orbit.

In view of the static cylindrically symmetric nature of the metric, the matter and the electromagnetic field may be represented by

$$\tau_1^1=\tau_3^3=0\ ,$$

$$\tau_2^2=-\rho\left[\frac{r\alpha'}{1-2r\alpha'}-\frac{[A^2(r)]'}{8\pi\rho}\frac{e^{4\alpha-2\beta}}{1-2r\alpha'}\right], \qquad (13)$$

$$\tau_4^4=\frac{\rho}{1-2r\alpha'}\left[(1-r\alpha')-\frac{[A^2(r)]'}{8\pi\rho r}e^{4\alpha-2\beta}\right],$$

and

$$8\pi E_1^1=-8\pi E_2^2=-8\pi E_3^3=8\pi E_4^4=\frac{A^2(r)}{r^2}e^{4\alpha-2\beta}\ , \qquad (14)$$

where τ_ν^μ and E_ν^μ represent energy-momentum tensors for matter and electromagnetic fields, respectively. τ_4^2 vanishes owing to equal numbers of particles moving in clockwise and anticlockwise directions.

The Einstein-Maxwell field equations are

$$R^\mu_\nu - \tfrac{1}{2} g^\mu_\nu R = -8\pi T^\mu_\nu \ , \tag{15}$$

with

$$T^\mu_\nu = \tau^\mu_\nu + E^\mu_\nu \ . \tag{16}$$

Now the field equations can be written down in detail,

$$\frac{\beta'}{r} - \alpha'^2 = -\frac{A^2(r)}{r^2} e^{2\alpha} \ , \tag{17}$$

$$e^{2\alpha-2\beta}\left[2\alpha'' - \beta'' + \frac{2\alpha'}{r} - \alpha'^2\right] = 8\pi\rho \frac{1-r\alpha'}{1-2r\alpha'} - \frac{[A^2(r)]'}{r}\frac{e^{4\alpha-2\beta}}{1-2r\alpha'} + \frac{A^2(r)}{r^2} e^{4\alpha-2\beta} \ , \tag{18}$$

$$e^{2\alpha-2\beta}(\beta'' + \alpha'^2) = 8\pi\rho \frac{r\alpha'}{1-2r\alpha'} - \frac{[A^2(r)]'}{1-2r\alpha'} e^{4\alpha-2\beta} + \frac{A^2(r)}{r^2} e^{4\alpha-2\beta} \ . \tag{19}$$

Equations (17)—(19) are not independent, there being an identical relation between them. This arises from the existence of the Bianchi identity and the fact that the energy-momentum tensor T^μ_ν already satisfies the conservation relation. Therefore one can take Eq. (17) and the sum of Eqs. (18) and (19) as equations of the problem. The sum of Eqs. (18) and (19) is

$$e^{2\alpha-2\beta}\left[\alpha'' + \frac{\alpha'}{r}\right] = \frac{4\pi\rho}{1-2r\alpha'} - \frac{1}{2}\frac{[A^2(r)]'}{1-2r\alpha'}\left[1 + \frac{1}{r}\right] e^{4\alpha-2\beta} + \frac{A^2(r)}{r^2} e^{4\alpha-2\beta} \ . \tag{20}$$

If m be the particle mass and n be their number density per unit coordinate volume, then one may have the relation[2]

$$mn = \rho\sqrt{-g}\,\frac{dt}{ds} = \rho r e^{2\beta-3\alpha}\left[\frac{1 - r\alpha' - \dfrac{[A^2(r)]'}{8\pi\rho r} e^{4\alpha-2\beta}}{1-2r\alpha'}\right]^{1/2} \ . \tag{21}$$

Equations (20) and (21) together give

$$\begin{aligned} 4\pi m n e^{\alpha} = \Bigg[& (r\alpha'' + \alpha')(1-2r\alpha')^{1/2} + \frac{1}{2}\frac{[A^2(r)]'}{(1-2r\alpha')^{1/2}}(r+1)e^{2\alpha} \\ & - \frac{A^2(r)}{r}(1-2r\alpha')^{1/2} e^{2\alpha}\Bigg]\left[1 - r\alpha' - \frac{[A^2(r)]'}{8\pi\rho r} e^{4\alpha-2\beta}\right]^{1/2} \ . \end{aligned} \tag{22}$$

For m and n to be real and positive, one must have $r\alpha' < \frac{1}{2}$ and

$$r\alpha' + \frac{[A^2(r)]'}{8\pi\rho r} e^{4\alpha-2\beta} < 1 \ ,$$

which are the conditions already obtained in Eq. (9).

Thus, finally we have three equations (5), (17), and (22) to determine α, β, n, $A(r)$, and σ. Mathematically the easiest way to construct a solution is to choose two quantities arbitrarily. Let us choose α and $A(r)$ as arbitrary. Equation (17) will determine β and (22) will give n. Finally, one may determine σ from Eq. (5).

III. ILLUSTRATIVE EXAMPLES

We shall now consider some special solutions for an extremely thin cylindrical shell and a finite cylindrical distribution. In constructing the solutions we start with a suitable form of α and $A(r)$ and a cylinder with inner radius r_1 and outer radius r_2.

Let us take

$$\alpha' = br^k \text{ and } A(r) = hr^l \ , \tag{23}$$

where b, h, k, and l are constants.

Equations (23) and (17), on using the inner and outer boundary conditions, give solutions of the type

$$\alpha=-\frac{br_2{}^{k+1}}{k+1}\left[1-\left[\frac{r}{r_2}\right]^{k+1}\right]-\ln(C_1r_2{}^{-C}+C_2r_2{}^{C})\ , \qquad (24)$$

$$\beta=-\frac{b^2r_2{}^{2k+2}}{2k+2}\left[1-\left[\frac{r}{r_2}\right]^{2k+2}\right]-\epsilon(r)+\epsilon(r_2)+\ln Hr_2{}^{C^2}\ , \qquad (25)$$

where

$$\epsilon(r)=h^2\int r^{2l-1}e^{2\alpha}dr \qquad (26)$$

and

$$\frac{br_2{}^{k+1}}{k+1}\left[1-\left[\frac{r_1}{r_2}\right]^{k+1}\right]+\ln(C_1r_2{}^{-C}+C_2r_2{}^{C})=0\ . \qquad (27)$$

$$\frac{b^2r_2{}^{2k+2}}{2k+2}\left[1-\left[\frac{r_1}{r_2}\right]^{2k+2}\right]-\epsilon(r_2)+\epsilon(r_1)=\ln Hr_2{}^{C^2}\ . \qquad (28)$$

If N denotes the total number of particles per unit proper length of the cylinder, then N is given by

$$N=\int_{r_1}^{r_2}\int_0^{e^{\alpha-\beta}}\int_0^{2\pi}n\,dr\,dz\,d\phi\ . \qquad (29)$$

Then putting $Nm=M$, the total mass of the particles per unit proper length of the cylinder, we get

$$M=2\pi\int_{r_1}^{r_2}mne^{\alpha-\beta}dr\ . \qquad (30)$$

We shall now consider two cases.

A. A thin cylindrical shell of particles

Let us consider a very thin cylindrical shell of thickness Δ and inner radius $r_1=r_0$.

(i) Uncharged shell

We first find M for a very thin uncharged shell. From Eqs. (30) and (22) on putting $A(r)=0$ we get

$$M=\tfrac{1}{2}\int_{\gamma_0}^{\gamma_0+\Delta}(r\alpha')'(1-3r\alpha'+2r^2\alpha'^2)^{1/2}dr\ , \qquad (31)$$

where we have taken $e^{\beta}=1$ since Δ is very small. Then we have from the inner and outer boundary conditions,

$$M=\frac{C}{2}\left[1-\frac{\gamma_0(\alpha')_0}{C}\right](1-3C+2C^2)^{1/2}\ , \qquad (32)$$

where $C/2$ is the mass per unit length of the shell and $(\alpha')_0$ is the value of α' at the inner boundary of the shell. Obviously $\gamma_0(\alpha')_0/C$ is comparable to 1 for a thin cylindrical shell. Hence it cannot be neglected in comparison with 1. Only when $(\alpha')_0$ is of the form $f(C)/\gamma_0$ will M be independent of r_0. One possible value of α', viz., $f(C)/r$, is, however, ruled out, because from the outer boundary condition $f(C)$ comes out to be equal to C. This makes $M=0$ in Eq. (32).

Raychaudhuri and Som[2] neglected the term α' in the factor $(r\alpha')'$ of the integrand of Eq. (31) which, they argued, is small in comparison with $r\alpha''$; and so they obtained the result that M is independent of r_0 in Eq. (32). Obviously this conclusion cannot be generally true.

(ii) Charged shell

From Eqs. (30) and (22), with $e^{\alpha}=e^{\beta}=1$ for a very thin charged shell, we have, using Eq. (23),

$$M=\frac{1}{2}\left[b(k+1)r_0{}^k(1-2br_0{}^{k+1})^{1/2}+\left[\frac{l(r_0+1)}{1-2br_0{}^{k+1}}-1\right]h^2r_0{}^{2l-1}(1-2br_0{}^{k+1})^{1/2}\right]\times\left[1-br_0{}^{k+1}-\frac{lh^2}{4\pi\rho}r_0{}^{2l-2}\right]^{1/2}\Delta\ . \qquad (33)$$

Equation (33) shows that the gravitational mass per unit length of the shell depends on the radius r_0.

B. Finite charged cylindrical distribution

From Eqs. (22) and (30) we get for a charged cylindrical distribution of finite radius a, using Eqs. (23),

$$M=\frac{a}{2}\int_0^1 e^{-\beta}\left[b(k+1)a^k u^k+\left[\frac{l(au+1)}{1-2ba^{k+1}u^{k+1}}-1\right]h^2a^{2l-1}u^{2l-1}e^{2\alpha}\right]$$
$$\times(1-2ba^{k+1}u^{k+1})^{1/2}\left[1-ba^{k+1}u^{k+1}-\frac{lh^2}{4\pi\rho}a^{2l-2}u^{2l-2}e^{4\alpha-2\beta}\right]^{1/2}du\ . \tag{34}$$

To avoid a singularity at $r=0$, we take $k\geq 0$ and $l\geq 1$. Equation (34) shows that, as in the case of charged cylindrical shell, here also the gravitational mass per unit length depends on the radius a.

IV. TANGENTIALLY STRESSED CHARGED CYLINDER

The field equations for a tangentially stressed charged cylinder are

$$\frac{\beta'}{r}-\alpha'^2=-\frac{A^2(r)}{r^2}e^{2\alpha}\ , \tag{35}$$

$$e^{2\alpha-2\beta}(\beta''+\alpha'^2)=8\pi p_\phi+\frac{A^2(r)}{r^2}e^{4\alpha-2\beta}\ , \tag{36}$$

$$e^{2\alpha-2\beta}\left[2\alpha''-\beta''+\frac{2\alpha'}{r}-\alpha'^2\right]=8\pi\rho_0+\frac{A^2(r)}{r^2}e^{4\alpha-2\beta}\ , \tag{37}$$

where p_ϕ is the tangential stress and ρ_0 is the mass density. Assuming that the interior field for the stressed cylinder is the same as that of a cylindrical cluster and comparing the field equations (17)–(19) with Eqs. (35)–(37) we find that the following conditions must be satisfied:

$$\begin{aligned}\rho_0&=\rho\frac{1-r\alpha'}{1-2r\alpha'}-\frac{[A^2(r)]'}{8\pi r}\frac{e^{4\alpha-2\beta}}{1-2r\alpha'}\ ,\\ p_\phi&=\rho\frac{r\alpha'}{1-2r\alpha'}-\frac{[A^2(r)]'}{8\pi}\frac{e^{4\alpha-2\beta}}{1-2r\alpha'}\ .\end{aligned} \tag{38}$$

It then follows that a solution of Sec. II may be interpreted as the field inside a tangentially stressed charged cylinder whose density and stress are given by Eqs. (38).

V. EMISSION OF NEUTRINOS AND PHOTONS

Let us investigate the motion of neutrinos and photons inside a cylindrical cluster. The equations of motion of these particles are given by null geodesics and such paths within the cluster under consideration are given by

$$e^{2\alpha}\dot t^2-e^{2\beta-2\alpha}\dot r^2-r^2e^{-2\alpha}\dot\phi^2=0\ , \tag{39}$$

$$r^2e^{-2\alpha}\dot\phi=\lambda \text{ and } e^{2\alpha}\dot t=\lambda\mu\ , \tag{40}$$

where λ and μ are constants of motion. The dot represents differentiation with respect to some affine parameter. Thus from Eqs. (39) and (40) we get

$$e^{2\beta-2\alpha}\left[\frac{dr}{d\phi}\right]^2=\mu^2-r^{-2}e^{-12\alpha}\ . \tag{41}$$

Such paths have apses where $r^{-2}e^{-12\alpha}=\mu^2$ for some r.

Again

$$\frac{d}{dr}(r^{-2}e^{-12\alpha})=-2e^{-12\alpha}r^{-3}(1+6r\alpha')\ . \tag{42}$$

From Eq. (42) it follows that

$$\frac{d}{dr}(r^{-2}e^{-12\alpha})<0\ , \tag{43}$$

for example, in the case of the solution in Sec. III B. Thus there is a possibility of neutrinos and photons coming out from the interior of a cylindrical cluster of charged particles. This possibility exists in the uncharged case also.

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