# Solitary waves of matter in general relativity

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The possibility that potentials which describe matter in general relativity obey nonlinear differential equations with solitary waves as solutions is studied. A particular model wherein the metric as well as the matter can propagate as solitary waves is presented. Explicit solutions to this model are exhibited. In these solutions either the matter or the metric or both propagate as a single soliton. The use of soliton solutions to represent concentrations of matter is briefly discussed.

## I. INTRODUCTION

The soliton concept has been widely used in particle physics to represent particles, even at a semiclassical level, mainly owing to the localized behavior of a soliton type of waves but also to their stability.[1] It appears to us that solitons of matter in general relativity might be a good representation of massive objects, e.g., galaxies. The collision of two galaxies shares some common features with the collision of two solitons—for instance, the collision of two galaxies does not destroy the galaxies.

In general relativity, equations having a soliton type of behavior appear in the study of the vacuum Einstein equations for axially symmetric waves with two degrees of freedom.[2,3] Also the vacuum Einstein equations for stationary axially symmetry space-times have a soliton[3–5] type of solution (Ernst equations). Because of the close relation between the vacuum Einstein equations for cylindrically symmetry space-times and the Einstein equations coupled with irrotational perfect fluids with $p=w$ equations of state for the same type of space-times, the soliton concept also has appeared in this context.[6] However, in this last case the matter does not have soliton behavior because the potential that describes the matter obeys the usual *linear* wave equation in cylindrical coordinates.

The methods used to solve the Einstein equations in the above-mentioned cases are the inverse scattering[2,4] and Bäcklund transformation methods.[3,5,7]

In this paper we want to study the possibility that matter propagates as solitary waves or rather that the potentials which describe the matter obeying nonlinear differential equations admit solitary waves as solutions.

In Sec. II we present some basic aspects of the author's model of anisotropic fluid described by two-perfect-fluid components.[8] In the next section (Sec. III) we specialize the above-mentioned model to obtain matter evolution equations general enough to include, as a particular case, a known system of equations with soliton solutions. In Sec. IV we study the Einstein equations coupled to the above-mentioned anisotropic fluid for a cylindrically symmetric space-time. In Sec. V we analyze the possibility of having solitons of matter. We present a particular example wherein the matter, as well as the metric, can propagate as solitary waves.

In Sec. VI we study three particular solutions to the Einstein equations. The solutions obtained describe cylindrically symmetric space-times or cosmological models, depending on the value of an integration constant. In these solutions either the matter or the metric or both propagate as a single soliton. Finally in Sec. VII we discuss some of the possible generalizations and applications of the model.

## II. A MODEL OF ANISOTROPIC FLUID

The main reason to study the model of anisotropic fluid with two-perfect-fluid components[8,9] in the present context is the appearance, in a very natural way, of two potentials that can be "forced" to obey a soliton type of equation.

The stress-energy tensor for the anisotropic fluid is formed from the sum of two tensors, each of which is the energy-momentum tensor (EMT) of a perfect fluid,[8,10] i.e.,

$$T^{\mu\nu}(u,v)=t^{\mu\nu}(u)+t^{\mu\nu}(v)\ , \tag{2.1}$$

$$t^{\mu\nu}(u)=(p+w)u^{\mu}u^{\nu}-pg^{\mu\nu}\ , \tag{2.2a}$$

$$t^{\mu\nu}(v)=(q+e)v^{\mu}v^{\nu}-qg^{\mu\nu}\ , \tag{2.2b}$$

 

$$u_\mu u^\mu = v_\mu v^\mu = 1, \quad u^\mu \neq v^\mu . \qquad (2.3)$$

The vectors $u^\mu$ and $v^\mu$ are the velocities associated with each fluid component, $p$ and $q$ are the pressures, and $w$ and $e$ the fluids' rest-energy densities.

With the transformation

$$u^\mu \to u^{*\mu} = \cos\alpha u^\mu + \left[ \frac{q+e}{p+w} \right]^{1/2} \sin\alpha v^\mu , \qquad (2.4a)$$

$$v^\mu \to v^{*\mu} = -\left[ \frac{p+w}{q+e} \right]^{1/2} \sin\alpha u^\mu + \cos\alpha v^\mu , \qquad (2.4b)$$

where

$$\tan(2\alpha) = \frac{[(p+w)(q+e)]^{1/2}}{p+w-q-e} 2u^\mu v_\mu \qquad (2.4c)$$

we find that the EMT (2.1) can be cast in the form

$$T^{\mu\nu} = (\rho+\pi) U^\mu U^\nu + (\sigma-\pi) X^\mu X^\nu - \pi g^{\mu\nu} . \qquad (2.5)$$

The quantities $U^\mu$, $X^\mu$, $\rho$, $\sigma$, and $\pi$ are the fluid flux velocity, the direction of anisotropy, the fluid rest-energy density, the pressure along the anisotropy direction, and the pressure on the "perpendicular directions" to $X^\mu$, respectively. These quantities are related to the perfect-fluid components by

$$U^\mu = u^{*\mu} / (u^{*\alpha} u^*_\alpha)^{1/2} , \qquad (2.6)$$

$$X^\mu = v^{*\mu} / (-v^{*\alpha} v^*_\alpha)^{1/2} , \qquad (2.7)$$

$$\rho = \tfrac{1}{2}(w-p+e-q) + \tfrac{1}{2}\{(p+w+q+e)^2 + 4(p+w)(q+e)[(u^\mu v_\mu)^2 - 1]\}^{1/2} , \qquad (2.8)$$

$$\sigma = -\tfrac{1}{2}(w-p+e-q) + \tfrac{1}{2}[(p+w-q-e)^2 + 4(u_\mu v^\mu)^2 (p+w)(q+e)]^{1/2} , \qquad (2.9)$$

$$\pi = p+q . \qquad (2.10)$$

Also, we have

$$U^\mu U_\mu = -X^\mu X_\mu = 1, \quad U_\mu X^\mu = 0 , \qquad (2.11)$$

$$\rho = T^{\mu\nu} U_\mu U_\nu, \quad \sigma = T^{\mu\nu} X_\mu X_\nu . \qquad (2.12)$$

In general, it is necessary to add supplementary conditions to close the model; this point was treated in some detail in Ref. 8.

## III. THE MATTER EVOLUTION EQUATIONS

The matter evolution equations are obtained in the usual way, i.e., from the "conservation law"

$$T^{\mu\nu}{}_{;\nu} = 0 . \qquad (3.1)$$

We shall "project" Eq. (3.1) along the directions $u^\mu, v^\mu$ and the directions that are perpendicular to both $u^\mu$ and $v^\mu$. To perform this last projection we introduce the projection operator

$$H^\mu_\nu = \delta^\mu_\nu - [1-(u^\alpha v_\alpha)^2]^{-1} \times [u^\mu u_\nu + v^\mu v_\nu - u^\alpha v_\alpha (u^\mu v_\nu + u_\nu v^\mu)] . \qquad (3.2)$$

Some properties of this operator are

$$H^\mu_\nu u^\nu = H^\mu_\nu v^\nu = 0 , \qquad (3.3)$$

$$H^\mu_\alpha H^\alpha_\nu = H^\mu_\nu, \quad H_{\mu\nu} = H_{\nu\mu} , \qquad (3.4)$$

$$H^\alpha_\alpha = 2, \quad \det||H^\mu_\nu|| = 0 . \qquad (3.5)$$

Also, $H^\mu_\nu$ can be written in terms of the new operator

$$P_{\mu\nu} = -P_{\nu\mu} \equiv [1-(u^\lambda v_\lambda)^2]^{-1/2} \epsilon_{\mu\nu\alpha\beta} u^\alpha v^\beta \qquad (3.6)$$

as

$$H^\mu_\nu = P_{\nu\beta} P^{\beta\mu} , \qquad (3.7)$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita symbol. This representation of $H^\mu_\nu$ is particularly useful for computational purposes.

Transvecting (3.1) with $u_\mu$, $v_\mu$, and $H^\alpha_\mu$ we get, respectively,

$$(p+w)_{,\nu} u^\nu + (p+w) u^\nu{}_{;\nu} + (q+e)_{,\nu} v^\nu u^\alpha v_\alpha + (q+e) v^\mu{}_{;\nu} v^\nu u_\mu + (q+e) u^\alpha v_\alpha v^\nu{}_{;\nu} - (p+q)_{,\nu} u^\nu = 0 , \qquad (3.8)$$

$$(q+e)_{,\nu} v^\nu + (q+e) v^\nu{}_{;\nu} + (p+w)_{,\nu} u^\nu u^\alpha v_\alpha + (p+w) u^\mu{}_{;\nu} u^\nu v_\mu + (p+w) u^\alpha v_\alpha u^\nu{}_{;\nu} - (p+q)_{,\nu} v^\nu = 0 , \qquad (3.9)$$

$$(p+w) u^\mu{}_{;\nu} u^\nu H^\alpha_\mu + (q+e) v^\mu{}_{;\nu} v^\nu H^\alpha_\mu - (p+q)^{,\mu} H^\alpha_\mu = 0 . \qquad (3.10)$$

We shall specialize the velocity of each fluid component in the following way:

$$u_\mu = \phi_{,\mu} / (\phi_{,\alpha} \phi^{,\alpha})^{1/2} , \qquad (3.11a)$$

$$v_\mu=\psi_{,\mu}/(\psi_{,\alpha}\psi^{,\alpha})^{1/2} . \tag{3.11b}$$

In other words, we impose the condition of irrotationality.[8]

From (3.10), (3.11), and (3.3) we get

$$\left[\frac{p+w}{\phi_{,\alpha}\phi^{,\alpha}}(\phi_{,\beta}\phi^{,\beta})_{,\mu}+\frac{q+e}{\psi_{,\alpha}\psi^{,\alpha}}(\psi_{,\beta}\psi^{,\beta})_{,\mu}\right]H^\mu_\alpha=2(p+q)_{,\mu}H^\mu_\alpha . \tag{3.12}$$

This last equation will be satisfied identically if we choose

$$p+w=F\phi_{,\alpha}\phi^{,\alpha} , \tag{3.13}$$

$$q+e=H\phi_{,\alpha}\phi^{,\alpha} , \tag{3.14}$$

$$p+q=\tfrac{1}{2}(F\phi_{,\alpha}\phi^{,\alpha}+H\psi_{,\alpha}\psi^{,\alpha}-G) , \tag{3.15}$$

where $F$, $H$, and $G$ are *arbitrary* functions of the scalar potentials $\phi$ and $\psi$.

The evolution equations for $\phi$ and $\psi$ are obtained directly from (3.8) and (3.9). We find

$$\phi_{,\alpha}\phi^{,\alpha}F\Box\phi+\phi^{,\alpha}\psi_{,\alpha}H\Box\psi+\tfrac{1}{2}F_{,\beta}\phi^{,\beta}\phi_{,\alpha}\phi^{,\alpha}-\tfrac{1}{2}H_{,\alpha}\phi^{,\alpha}\psi_{,\beta}\psi^{,\beta}+H_{,\alpha}\psi^{,\alpha}\phi^{,\beta}\psi_{,\beta}+\tfrac{1}{2}G_{,\alpha}\phi^{,\alpha}=0 , \tag{3.16}$$

$$\phi_{,\alpha}\psi^{,\alpha}F\Box\phi+\psi^{,\alpha}\psi_{,\alpha}H\Box\psi+\tfrac{1}{2}H_{,\alpha}\psi^{,\alpha}\psi_{,\beta}\psi^{,\beta}-\tfrac{1}{2}F_{,\alpha}\psi^{,\alpha}\phi_{,\beta}\phi^{,\beta}+F_{,\alpha}\phi^{,\alpha}\phi^{,\beta}\psi_{,\beta}+\tfrac{1}{2}G_{,\alpha}\phi^{,\alpha}=0 , \tag{3.17}$$

where we have used the notation $\Box\phi\equiv\phi_{,\alpha}{}^{;\alpha}$. Now if we solve (3.16) and (3.17) for $F\Box\phi$ and $H\Box\psi$ we obtain

$$F\Box\phi=\tfrac{1}{2}H_\phi\psi_{,\alpha}\psi^{,\alpha}-\tfrac{1}{2}F_\phi\phi^{,\alpha}\phi_{,\alpha}-F_\psi\phi^{,\alpha}\psi_{,\alpha}-\tfrac{1}{2}G_\phi , \tag{3.18a}$$

$$H\Box\psi=\tfrac{1}{2}F_\psi\phi_{,\alpha}\phi^{,\alpha}-\tfrac{1}{2}H_\psi\psi^{,\alpha}\psi_{,\alpha}-H_\phi\phi^{,\alpha}\psi_{,\alpha}-\tfrac{1}{2}G_\psi \tag{3.18b}$$

where $H_\phi\equiv\partial H/\partial\phi$, $F_\psi\equiv\partial F/\partial\psi$, etc.

By using (3.13)–(3.15) the EMT (2.1) can be cast as

$$T_{\mu\nu}=F\phi_{,\mu}\phi_{,\nu}+H\psi_{,\mu}\psi_{,\nu} -\tfrac{1}{2}g_{\mu\nu}(F\phi_{,\alpha}\phi^{,\alpha}+H\psi_{,\alpha}\psi^{,\alpha}-G) . \tag{3.19}$$

It is interesting to point out that the evolution equations (3.18), as well as the EMT (3.19), can also be obtained, in the usual way, from the Lagrangian density

$$\mathscr{L}=\tfrac{1}{2}\sqrt{-g}\,(F\phi_{,\alpha}\phi^{,\alpha}+H\psi_{,\alpha}\psi^{,\alpha}-G) =\sqrt{-g}\,(p+q) . \tag{3.20}$$

In Refs. 8 and 9 we studied the particular case of a fluid in which the condition (3.1) was implemented by $t^{\mu\nu}(u)_{;\nu}=t^{\mu\nu}(v)_{;\nu}=0$, i.e., we had a kind of minimal coupling between the fluid components. This particular case is the specialization $F=H=1$ and $G=0$ of the present model. Now, with the specifications (3.13)–(3.15) we have a different kind of coupling, i.e., $t^{\mu\nu}(u)_{;\nu}=-t^{\mu\nu}(v)_{;\nu}\neq0$; therefore, the fluids interact through a force density $f^\mu=t^{\mu\nu}{}_{;\nu}$ different from zero.

The particular choice of the physical variables (3.11) and (3.13)–(3.15) tells us that the one-fluid variables are related to the potentials $\phi$ and $\psi$ by

$$u^*_\mu(\phi_{,\beta}\phi^{,\beta})^{1/2}=\cos\alpha\phi_{,\mu}+(H/F)^{1/2}\sin\alpha\psi_{,\mu} , \tag{3.21a}$$

$$v^*_\mu(\psi_{,\beta}\psi^{,\beta})=-(F/H)^{1/2}\sin\alpha\phi_{,\mu}+\cos\alpha\psi_{,\mu} , \tag{3.21b}$$

$$\tan(2\alpha)=\frac{2(FH)^{1/2}\phi^{,\mu}\psi_{,\mu}}{F\phi_{,\beta}\phi^{,\beta}-H\psi_{\beta,}\psi^{,\beta}} , \tag{3.21c}$$

$$2\pi=F\phi_{,\alpha}\phi^{,\alpha}+H\psi_{,\alpha}\psi^{,\alpha}-G , \tag{3.22}$$

$$2\rho=G+[(F\phi_{,\alpha}\phi^{,\alpha}-H\psi_{,\alpha}\psi^{,\alpha})^2 +4FH(\phi_{,\alpha}\psi^{,\alpha})^2]^{1/2} , \tag{3.23}$$

$$\sigma=\rho-G . \tag{3.24}$$

The special case $G=0$ is of particular interest due to the "mass term" character of $G$, as indicated by (3.20). Also when $G=0$ we have $\sigma=\rho$, i.e., we have the "stiff" equation of state along the anisotropy direction.

## IV. EINSTEIN EQUATIONS COUPLED TO MATTER

The Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-T_{\mu\nu} \tag{4.1}$$

coupled to the EMT (3.20) are equivalent to

$$R_{\mu\nu}=-(F\phi_{,\mu}\phi_{,\nu}+H\psi_{,\mu}\psi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}G)\ . \quad (4.2)$$

In order to study the system of equations (4.2) together with its integrability conditions (3.18) we choose the particular metric

$$ds^2=e^{\omega}(dt^2-dr^2)-tf(d\theta+hdz)^2-(t/f)dz^2\ , \quad (4.3)$$

where $\omega$, $f$, and $h$ are functions of $t$ and $r$ only. This metric has been used to study cylindrically symmetric waves with two degrees of freedom; also particular cases of (4.3) are homogeneous space-times[11] of the Bianchi types I through VII.

The component $R_{\theta\theta}$ of the Ricci tensor computed with the metric (4.3) is zero. Since the metric tensor associated with (4.3) depends only on $t$ and $r$, we have that $\phi$ and $\psi$ also depend on these two variables $(t,r)$. This can be proved easily using the fact that a symmetry of the metric is also a symmetry of the Ricci tensor. Therefore, Eq. (4.2) implies $G=0$. Using this fact we find that (4.2) for the metric (4.3) reduces to

$$\omega_{00}-\omega_{11}-\omega_0/t+f^{-2}f_0{}^2+f^2h_0{}^2-t^{-2}=-2(F\phi_0{}^2+H\psi_0{}^2)\ , \quad (4.4a)$$

$$-\omega_{00}+\omega_{11}-\omega_0/t+f^{-2}f_1{}^2+f^2h_1{}^2=-2(F\phi_1{}^2+H\psi_1{}^2)\ , \quad (4.4b)$$

$$-\omega_1/t+f^{-2}f_0f_1+f^2h_0h_1=-2(F\phi_0\phi_1+H\psi_0\psi_1)\ , \quad (4.4c)$$

$$f_{00}+f_0/t-f_{11}-(f_0{}^2-f_1{}^2)/f-f^3(h_0{}^2-h_1{}^2)=0\ , \quad (4.5a)$$

$$(tf^2h_0)_0-(tf^2h_1)_1=0\ , \quad (4.5b)$$

where $\omega_0=\partial\omega/\partial t$, $\phi_1=\partial\phi/\partial r$, etc. Also the evolution equations (3.18) can be written as

$$F(\phi_{00}+\phi_0/t-\phi_{11})=\tfrac{1}{2}H_\phi(\psi_0{}^2-\psi_1{}^2)-\tfrac{1}{2}F_\phi(\phi_0{}^2-\phi_1{}^2)-F_\phi(\phi_0\psi_0-\phi_1\psi_1)\ , \quad (4.6a)$$

$$H(\psi_{00}+\psi_0/t-\psi_{11})=\tfrac{1}{2}F_\psi(\phi_0{}^2-\phi_1{}^2)-\tfrac{1}{2}H_\psi(\psi_0{}^2-\psi_1{}^2)-H_\phi(\phi_0\psi_0-\phi_1\psi_1)\ . \quad (4.6b)$$

The equation resulting from the difference of Eqs. (4.4a) and (4.4b) follows from the other field equations. From the sum of (4.4a) and (4.4b), and (4.4c) we get

$$d\omega=\frac{t}{2}\{[-t^{-2}+f^{-2}(f_0{}^2+f_1{}^2)+f^2(h_0{}^2+h_1{}^2)+2F(\phi_0{}^2+\phi_1{}^2)+2H(\psi_0{}^2+\psi_1{}^2)]dt$$
$$+2[f^{-2}f_0f_1+f^2h_0h_1+2(F\phi_0\phi_1+H\psi_0\psi_1)]dr\}\ . \quad (4.7a)$$

Thus,

$$\omega=-\tfrac{1}{2}\ln t+\Omega\ , \quad (4.7b)$$

where $\Omega$ is related to $f$, $h$, $F$, $H$, $\phi$, and $\psi$ by

$$\Omega=\Sigma[f,h]+\Lambda[F,H,\phi,\psi] \quad (4.8)$$

and

$$\Sigma[f,h]\equiv\int\frac{t}{2}\{[f^{-2}(f_0{}^2+f_1{}^2)+f^2(h_0{}^2+h_1{}^2)]dt+2(f^{-2}f_0f_1+f^2h_0h_1)dr\}\ , \quad (4.9)$$

$$\Lambda[F,H,\phi,\psi]\equiv\int t\{[F(\phi_0{}^2+\phi_1{}^2)+H(\psi_0{}^2+\psi_1{}^2)]dt+2(F\phi_0\phi_1+H\psi_0\psi_1)dr\}\ . \quad (4.10)$$

The integrability conditions for $\Sigma$ and $\Lambda$ are exactly Eqs. (4.5) and (4.6), respectively. Thus, any solution to (4.5) and (4.6) will generate the line element

$$ds^2=\frac{e^{\Omega}}{\sqrt{t}}(dt^2-dr^2)-tf(d\theta+h\,dz)^2-(t/f)dz^2 \quad (4.11)$$

whose metric coefficients are a solution to (4.2). We can also get the same type of solution by setting $G=0$ in (4.2) and choosing the metric that is obtained by letting $t\to T(t,r)$ in (4.3), where $T$ is a function of the indicated arguments. It happens that the Einstein equations, in this case, tell us that $T$ obeys the usual wave equations. This fact can be used to perform a change of variables, and finally, to end up with (4.11).

A variety of particular solutions to (4.5) are known. Also, particular cases of the metric (4.11), when $H=0$ and $F=1$, have been widely studied.[12,13] Recently we studied the case $H=F=1$ that corresponds to a massless complex scalar field.[9,14]

## V. SOLITONS

Particularly interesting are the soliton solutions to (4.5) obtained using the "inverse scattering" technique or Bäcklund transformations.[7] The starting point of the inverse scattering technique is the fact that Eqs. (4.5) can be written as the matrix equation[15]

$$(t\gamma_{,0}\gamma^{-1})_{,0}-(t\gamma_{,1}\gamma^{-1})_{,1}=0\ , \tag{5.1a}$$

with

$$\gamma[f,h]=t\begin{vmatrix} f & fh \\ fh & h^2f+f^{-1}\end{vmatrix}\ . \tag{5.1b}$$

To a known solution of (5.1), say $\gamma_s$, we can associate another solution $\gamma$ as follows:

$$\gamma=(\chi\,|_{\lambda=0})\gamma_s\ , \tag{5.2}$$

where $\chi$ is a $2\times2$ matrix function of $t$, $r$, and the complex spectral parameter $\lambda$. Solitons in this context are related to single poles of the "scattering matrix" $\chi$. The known solution $\gamma_s$ is called the "background" or "seed" solution. The general formalism used to find soliton solutions to (5.1) is presented in Ref. 2; also the particular cases of 1- and 2-soliton solutions are presented in some detail.

The soliton solutions to (5.1) are metric solitons, since they are independent of the matter content of the space-time. The solitary-wave solutions to (4.6) are solitons associated with matter, since the potentials $\phi$ and $\psi$ are velocity potentials from which are derived the velocities of each fluid component. That for some particular choice of the functions $F$ and $H$ the system of equations (4.6) admits a soliton type of solutions can be proved by noticing that the specialization

$$F=\tfrac{1}{2}a^2\phi^{-2},\ \ H=\tfrac{1}{2}a^2\phi^2\ , \tag{5.3}$$

where $a$ is a real constant, makes the system (4.6) equivalent to the system (4.5), or by noticing that (4.6) with (5.3) can be cast as (5.1a) with $\gamma=\gamma[\phi,\psi]$. We believe that other specializations different from (5.3) might make the system (4.6) admit solitons as solutions. This point is under active consideration by the author.

Now we shall examine the "gauge" freedom that we have in the choice of the scalar functions $\phi$ and $\psi$. First we notice that in order to maintain the character of velocity potential of $\phi$ and $\psi$, the "gauge" freedom of these quantities reduces according to

$$\phi\to\phi'=\phi'(\phi),\ \ \partial\phi'/\partial\phi\neq0\ , \tag{5.4a}$$

$$\psi\to\psi'=\psi'(\psi),\ \ \partial\psi'/\partial\psi\neq0\ . \tag{5.4b}$$

In other words, (5.4) is the most general transformation of potentials that leaves (3.11) invariant.[16] It is rather apparent that the "contact" transformation (5.4) cannot change the soliton character of a particular set $(\phi,\psi)$.

To analyze the physical consequences of the choice (5.3), let us compute $\pi$, $\sigma$, and $\rho$; from (3.22)–(3.24) and (4.11) we get

$$\pi=(a/2)^2\sqrt{t}\,e^{-\Omega}[(\phi_0{}^2-\phi_1{}^2)\phi^{-2}+(\psi_0{}^2-\psi_1{}^2)\phi^2]\ , \tag{5.5}$$

$$\sigma=\rho=(a/2)^2\sqrt{t}\,e^{-\Omega}\{[(\phi_0{}^2-\phi_1{}^2)\phi^{-2}-(\psi_0{}^2-\psi_1{}^2)\phi^2]^2+4(\phi_0\psi_0-\phi_1\psi_1)^2\}^{1/2}\ . \tag{5.6}$$

The function $\Omega$ in this case can be cast as

$$\Omega=\Sigma[f,h]+a^2\Sigma[\phi,\psi]\ , \tag{5.7}$$

since

$$\Lambda[\phi^{-2},\phi^2,\phi,\psi]=2\Sigma[\phi,\psi]\ . \tag{5.8}$$

One of the properties of the solitary waves is that they are localized, i.e., the functions that characterize the wave are different from a constant in a finite range of the space variable at any given value of the time variable. We see that the localized character of $\phi$ and $\psi$ for solitons can be propagated to $\pi$ and $\sigma=\rho$ as indicated by (5.5) and (5.6). We shall return to this point in the next sections.

## VI. PARTICULAR CASES

The particular cases that we shall consider are particular solutions to Eq. (5.1) for the metric coefficients $f$ and $h$, i.e., solutions to the Einstein equations (4.5), and particular solutions to the same Eq. (5.1) for the scalar fields $\phi$ and $\psi$, i.e., solutions to the integrability condition of $\Lambda$ with $F$ and $H$ given by (5.3). The solution for the metric coefficients $f$ and $h$ and the solutions for the velocity potentials $\phi$ and $\psi$ do not need to be equal, e.g., we can have the metric propagating as an $n$ soliton and the matter as an $m$ soliton. We shall call such a solution an $n$-$m$

soliton. If the metric coefficients (potentials) do not propagate as a soliton we shall assign to $n$ ($m$) the value $n=0$ ($m=0$).

### A. 0-1-soliton solution

A solution to (5.1) is given by

$$f=t^{2\alpha}, \quad h=0\,, \tag{6.1}$$

where $\alpha$ is an arbitrary constant. The simplest soliton solution[2] to (5.1) for $\gamma=\gamma[\phi,\psi]$ is the 1-soliton solution obtained from the seed solution $\phi_s=t^{2\beta}$ and $\psi_s=0$, i.e.,

$$\phi=t^{2\beta}\frac{\cosh(kx+\epsilon)}{\cosh(\beta x+\epsilon)}\,, \tag{6.2a}$$

$$\psi=t^{2\beta}\frac{\sinh(x/2)}{\cosh(\beta x+\epsilon)}\,, \tag{6.2b}$$

where $k=\beta+\frac{1}{2}$, $\beta$ and $\epsilon$ are arbitrary constants, and

$$x=2\ln\{[r+(r^2-t^2)^{1/2}]/t\}\,. \tag{6.3}$$

From (4.11), (4.9), (5.7), and (6.1)–(6.3) we get

$$ds^2=\frac{t^{2(\alpha^2+a^2\beta^2)+(a^2-1)/2}}{(r^2-t^2)^{a^2/2}}[\cosh(\beta x+\epsilon)]^{a^2}(dt^2-dr^2)-t(t^{2\alpha}d\theta^2+t^{-2\alpha}dz^2)\,. \tag{6.4}$$

And from (5.5)–(5.7), (4.9), and (6.1)–(6.3) we find

$$\pi=\frac{a^2(r^2-t^2)^{a^2/2}[R_2+R_1\cosh^2(\beta x+\epsilon)]}{t^{2(\alpha^2+a^2\beta^2)+(3+a^2)/2}[\cosh(\beta x+\epsilon)]^{2+a^2}}\,, \tag{6.5}$$

$$\rho=\sigma=\frac{a^2(r^2-t^2)^{a^2/2}\{[R_2-R_1\cosh(\beta x+\epsilon)]^2+[2R_3\cosh(\beta x+\epsilon)]^2\}}{t^{2(\alpha^2+a^2\beta^2)+(3+a^2)/2}[\cosh(\beta x+\epsilon)]^{2+a^2}}\,, \tag{6.6}$$

where the functions $R_1$, $R_2$, and $R_3$ are defined as

$$R_1=-\frac{\beta^2t^2}{r^2-t^2}+\left\{k\tanh(kx+\epsilon)-\beta\left[\frac{r}{(r^2-t^2)^{1/2}}+\tanh(\beta x+\epsilon)\right]\right\}^2, \tag{6.7a}$$

$$R_2=\frac{-\beta^2t^2}{r^2-t^2}+\left[\frac{\beta r}{(r^2-t^2)^{1/2}}\sinh(x/2)+\tfrac{1}{2}\cosh(x/2)-k\tanh(kx+\epsilon)\right]^2, \tag{6.7b}$$

$$\begin{aligned}-R_3=&\sinh(x/2)[\beta^2+k^2\tanh^2(kx+\epsilon)-\beta k\tanh(\beta x+\epsilon)\tanh(kx+\epsilon)]\\&+\tfrac{1}{2}\cosh(x/2)[\beta\tanh(\beta x+\epsilon)-k\tanh(kx+\epsilon)]\\&+\beta r(r^2-t^2)^{-1/2}[\tfrac{1}{2}\cosh(x/2)-2k\sinh(x/2)\tanh(kx+\epsilon)+\beta\sinh(x/2)\tanh(\beta x+\epsilon)]\,.\end{aligned} \tag{6.7c}$$

First we notice that the timelike character of $t$ and the spacelike character of $r$ are determined by the sign of $t^{-1/2}e^{\Omega}$ in (4.11), i.e., the factor of $dt^2-dr^2$ in (6.4). This sign can be chosen to be either positive or negative, since we can always add the integration constant $i\pi$ to $\Omega$. Thus, the role of $t$ and $r$ can be interchanged. For a discussion of this point see Refs. 2 and 14. When $t$ is a time coordinate, the metric (6.4) represents a "perturbation" to the Kasner space-time. We recover the Kasner metric letting $a^2\to 0$ in (6.4). The space-time represented by (6.4) has a big-bang type of singularity, since the pressures $\pi$ and $\sigma$ and the matter density $\rho$ blow up at $t=0$ as indicated by Eqs. (6.5) and (6.6). The soliton character of the solution can be noticed by the appearance of the factor $[\cosh(\beta x+\epsilon)]^{-(2+a^2)}$ in (6.5) and (6.6). This factor makes the pressures and the density localized. Thus, this solution describes a solitary wave of matter propagating in a Kasner "background." When $t$ represents a space coordinate, the metric (6.4) describes a space-time with cylindrical symmetry and the singularity at $t=0$ describes an infinite wire of matter. Equations (6.5) and (6.6) tell us that we have localized pressures and density, as before. Thus, this solution represents a cylindrical solitary wave of matter incident on an infinite wire and reflected from it.[2]

### B. 1-0-soliton solution

By using the solution (6.1) as a seed solution we find[17]

$$f=-t^{2\alpha}\frac{\cosh(ly+\delta)}{\cosh(\alpha y+\delta)}\,, \tag{6.8a}$$

$$h=-t^{-2\alpha}\frac{\sinh(y/2)}{\cosh(\alpha y+\delta)}\,, \tag{6.8b}$$

where

$$y=2\ln(\{r-r_0+[(r-r_0)^2-t^2]^{1/2}\}/t)\,, \tag{6.9}$$

$r_0$ and $\delta$ are integration constants, and $l=\alpha+\frac{1}{2}$.

Equation (5.1) for $\gamma=\gamma[\phi,\psi]$ admits the solution[13]

$$\phi=At^{-1/2}\,, \tag{6.10a}$$

$$\psi=\psi(t\pm r), \tag{6.10b}$$

where $A$ is an arbitrary constant and $\psi$ an arbitrary function of either $t-r$ or $t+r$.

From (4.11), (4.9), (5.7), and (6.8)–(6.10) we get

$$ds^2=e^{\omega}(dt^2-dr^2)-t^{2l}\frac{\cosh(ly+\delta)}{\cosh(\alpha y+\delta)}d\theta^2+2t\frac{\sinh(y/2)}{\cosh(\alpha y+\delta)}d\theta\,dz-t^{2(1-l)}\frac{\cosh[(1-l)y-\delta]}{\cosh(\alpha y+\delta)}dz^2, \tag{6.11a}$$

$$e^{\omega}=\frac{t^{2\alpha^2+a^2/8}}{[(r-r_0)^2-t^2]^{1/2}}\cosh(\alpha y+\delta)\exp\left[(aA)^2\int(\psi')^2d(t\mp r)\right], \tag{6.11b}$$

where the prime means derivative with respect to the argument. And from (5.5)–(5.7), (4.9), and (6.8)–(6.10) we obtain

$$\pi=\frac{a^2[(r-r_0)^2-t^2]^{1/2}\exp\left[-(aA)^2\int(\psi')^2d(t\mp r)\right]}{16t^{2(1+\alpha^2)+a^2/8}\cosh(\alpha y+\delta)}\,, \tag{6.12}$$

$$\rho=\sigma=\frac{a^2[(r-r_0)^2-t^2]^{1/2}[1+(2A\psi')^2/t]^{1/2}\exp\left[-(aA)^2\int(\psi')^2d(t\mp r)\right]}{8t^{1+2\alpha^2+a^2/8}\cosh(\alpha y+\delta)}\,. \tag{6.13}$$

First we want to point out that Eqs. (6.10b) and (6.11) imply $\psi_{,\alpha}\psi'^{\alpha}=0$. Therefore, Eq. (3.11b) is meaningless and the two-fluid interpretation of the solution breaks down. But, the anisotropic-fluid interpretation of this solution is still valid as a limit, since in (2.7) the term $\psi_{,\alpha}\psi'^{\alpha}$ for a $v^*_\mu$ given by (3.21b) cancels out.

The metric (6.11), the pressure (6.12), and the density (6.13) have a singularity at $t=0$. This is a big-bang type of singularity or a wire located at $t=0$ depending on the timelike or the spacelike character of $t$. When $a^2=0$ the metric (6.11) describes a solitary gravitational wave propagating in a Kasner background or a cylindrically symmetric solitary gravitational wave incident on an infinite wire and reflected from it.[2] The localized character of the gravitational wave is propagated to the quantities $\pi$ and $\rho=\sigma$ as the presence of the factor $[\cosh(\alpha y+\delta)]^{-1}$ in (6.12) and (6.13) indicates. One of the matter potentials propagates as a wave, but this is not a soliton type of solution, despite the fact that $\psi$ can be chosen to represent a localized wave. Furthermore, a localized behavior of $\psi$ does not propagate to $\pi$ and $\rho=\sigma$, because of the exponent appearing in (6.12) and (6.13). Loosely speaking we can say that the pressures and the density propagate as waves modulated by a gravitational single soliton.

### C. 1-1-soliton solution

From (4.11), (4.9), (5.7), (6.8), (6.9), (6.2), and (6.3) we get the line element (6.11a) with

$$e^{\omega}=\frac{t^{2(\alpha^2+a^2\beta^2)+a^2/2}}{(r^2-t^2)^{1/2}[(r-r_0)^2-t^2]^{a^2/2}}\cosh(\alpha y+\delta)[\cosh(\beta x+\epsilon)]^{a^2}\,. \tag{6.14}$$

And from (5.5)–(5.7), (4.9), (6.8), (6.9), (6.2), and (6.3) we find

$$\pi=\frac{a^2(r^2-t^2)^{1/2}[(r-r_0)^2-t^2]^{a^2/2}[R_2+R_1\cosh(\beta x+\epsilon)]}{t^{2(1+\alpha^2+a^2\beta^2)+a^2/2}\cosh(\alpha y+\delta)[\cosh(\beta x+\alpha)]^{2+a^2}}\,, \tag{6.15}$$

$$\rho=\sigma=\frac{a^2(r^2-t^2)^{1/2}[(r-r_0)^2-t^2]^{a^2/2}}{t^{2(1+\alpha^2+a^2\beta^2)+a^2/2}}\frac{\{[R_2-R_1\cosh^2(\beta x+\epsilon)]^2+[2R_3\cos(\beta x+\epsilon)]^2\}^{1/2}}{\cosh(\alpha y+\delta)[\cosh(\beta x+\epsilon)]^{2+a^2}}\,. \tag{6.16}$$

The metric (6.11a) with (6.14) presents the same behavior as the metric (6.11a) and (6.11b); thus the remarks that we made about (6.11a)–(6.11b) also apply in this case. Equations (6.15) and (6.16) tell us that the pressures and the matter density propagate as a single soliton modulated by a 1-soliton gravitational wave.

## VII. DISCUSSION

The solutions to the vacuum Einstein equations for the metric (4.3) are closely related to the solution to the Einstein equations (4.2) for the same metric. This relation is a consequence of the fact that the term that represents the matter, $\Lambda$, enters only in the metric through $\omega$ in a linear way as indicated by Eqs. (4.7b) and (4.8). This fact can be used to transform any known solution to the vacuum Einstein equations for the metric (4.3) into a solution of (4.2) [with the restriction (5.3)] for the same metric. From a solution to $R_{\mu\nu}=0$ for the metric (4.3) we have $f$, $h$, and $\Sigma[f,h]$. Now, letting $\phi\to f$ and $\psi\to h$ in (5.7) we get

$$\Omega=(1+a^2)\Sigma[f,h] \ . \tag{7.1}$$

Thus, this new $\Omega$ generates a solution to Eqs. (4.2)–(5.3) for the metric (4.3). The different quantities associated with the fluid are obtained by putting $\phi\to f$ and $\psi\to h$ into the corresponding general expressions for these quantities. This solution-generating technique can be used to transform an $n$-soliton gravitational wave in an $n$-$n$-soliton solution. Solutions to the vacuum Einstein equations for the metric (4.3) can be found in Refs. 2, 13, and 14.

In Sec. VI we studied different cases wherein the localized character of the 1-soliton solution for $\phi$ and $\psi$ or $f$ and $h$ propagates to $\pi$ and $\rho=\sigma$. We also tested this property numerically using the 2-soliton solution presented in Ref. 18. For this reason solitons might be used to represent concentrations of matter.

Some physical aspects of the two-fluid model, as well as the "solution" (3.13) and (3.14), need to be better understood in order to apply the model to more realistic situations, e.g., the description of galaxies.

The methods used to obtain metric solitons depends heavily on the particular form of the metric associated with (4.3). Whether this form can be changed by performing a general change of variables, without losing the soliton solutions, has yet to be studied; specifically is the soliton character of these solutions stable under an arbitrary change of variables? An answer to this question can be given by studying the invariants of curvature associated with the particular solutions, a task that is not easy to accomplish owing to the algebraic complexity of these types of solutions.

To end this section we want to add that the Lagrangian density (3.20) is interesting by itself, since many models of field theory are particular cases of (3.20), e.g., the $\phi^4$ field theory,[1] the two-dimensional generalization of the sine-Gordon equation,[19] etc. Thus, all those models have a fluid interpretation given by (3.21)–(3.24). Also, some of them have soliton solutions, but when coupled to the Einstein equations they yield complicated coupled systems of equations too difficult to study using standard techniques, even for simple space-times like (4.3) [with $t\to T(t,r)$]. The origin of these difficulties is the appearance in those theories of a "mass term" different from zero ($G\neq 0$).


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[15]Equation (5.1a) differs from the equivalent equation of the first citation of Ref. 2 because it is written in "cylindrical" coordinates and not in characteristic coordinates. Also, we have fixed the gauge freedom choosing $\det\gamma = t^2$; this can be done without losing generality as explained in the same Ref. 2.

[16]See for instance the first citation in Ref. 12.

[17]Equations (6.8) and (6.9) differ from the solutions presented in Ref. 2 by a translation in the variable $r$. Note that (5.1) is invariant under such a translation.

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