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Spinning fluids in general relativity

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We derive the equations of motion for a fluid with intrinsic spin in general relativity from a variational principle. Our theory is a direct extension of the theory of spinning fluids in special relativity.

I. INTRODUCTION

The treatment of spin angular momentum in special and general relativity has received quite a bit of attention. A recent paper by Bailey[1] contains many earlier references. In this paper we consider the field equations in general relativity for a continuous medium with internal spin. It is thought that the spin of "particles," photogalaxies, turbulent eddies, or primeval black holes could play an important role in the early epochs of the Universe.[2,3]

A special-relativistic variational principle for a spinning fluid in special relativity was formulated some time ago by Halbwachs.[4] Halbwachs's variational principle gives the special-relativistic Weyssenhoff theory for a spinning fluid. Here we generalize Halbwachs's variational principle to general relativity and obtain the Einstein equations for a fluid with internal spin. It would also be possible for us to treat the Einstein-Maxwell theory in a medium with spin. This would be the general-relativistic version of the Lorentz dielectric theory and has been studied in Refs. 1–3. For simplicity we shall just treat the gravitational case.

II. HALBWACHS'S VARIATIONAL PRINCIPLE

Halbwachs[4] introduces an orthonormal tetrad of vectors $a^{\mu}{}_{i}$ which he uses in his variational principle. Here $\mu,\nu,\ldots=1,2,3,4$ label the tetrad vectors and $i,j,\ldots=1,2,3,4$ label the components. These vectors satisfy

$$a^{\mu}{}_{i}a_{\mu j}=g_{ij} \; , \tag{2.1}$$

$$a_{\mu i}a_{\nu}{}^{i}=\eta_{\mu\nu} \; , \tag{2.2}$$

where g_{ij} is the spacetime metric and $\eta_{\mu\nu}$ is the Minkowski metric. The vector $a^{4}{}_{i}$ is related to the Eulerian four-velocity of the fluid via

$$a^{4}{}_{i}=U_i/c \; , \tag{2.3}$$

whereas the spin density of the fluid is described by

$$S_{ij}=\rho\kappa(a^{1}{}_{i}a^{2}{}_{j}-a^{1}{}_{j}a^{2}{}_{i}) \; . \tag{2.4}$$

In Eq. (2.4) ρ is the conserved density of the fluid

$$(\rho U^i)_{;i}=0 \; , \tag{2.5}$$

where the semicolon denotes covariant differentiation and κ is a scalar function proportional to the magnitude of the spin of the fluid and has the dimensions of angular momentum per unit rest mass. With the choices (2.3) and (2.4), the spin vector S_i of the fluid is associated with $a^{3}{}_{i}$:

$$S_i=\frac{1}{2c}\eta_{ijkl}S^{jk}U^l=\rho\kappa a^{3}{}_{i} \; . \tag{2.6}$$

The fluid also satisfies the auxiliary condition

$$S_{ij}U^j=0 \; . \tag{2.7}$$

The angular velocity of the spin vector is given by

$$W_{ij}=\tfrac{1}{2}(\dot{a}^{\mu}{}_{i}a_{\mu j}-a^{\mu}{}_{i}\dot{a}_{\mu j}) = \dot{a}^{\mu}{}_{i}a_{\mu j} \; , \tag{2.8}$$

where the overdot denotes differentiation along the fluid flow

$$\dot{a}^{\mu}{}_{i}=a^{\mu}{}_{i;j}U^j \; . \tag{2.9}$$

The spin kinetic energy density of the fluid has the form

$$\Upsilon=\tfrac{1}{2}S_{ij}W^{ij} = +\rho\kappa a^{1}{}_{i}a^{2i}{}_{;j}U^j \; . \tag{2.10}$$

This form for the kinetic energy density was given earlier by Unal and Vigier[5] who also briefly discuss Halbwachs's variational principle. The Lagrangian density for the spinning fluid has the form

$$\begin{aligned}\mathscr{L}_f = &\sqrt{-g}\,F(\rho,s)/c - \sqrt{-g}\,\rho\kappa a^1{}_i a^{2i}{}_{;j} U^j/c \\ &+\sqrt{-g}\,\lambda_1(g_{ij}U^iU^j + c^2) \\ &+\sqrt{-g}\,\lambda_2(\rho U^i)_{;i} + \sqrt{-g}\,\lambda_3 X_{,j} U^i \\ &+\sqrt{-g}\,\lambda_4 S_{,i} U^i + \lambda^{\mu\nu}(g_{ij}a_\mu{}^i a_\nu{}^j - \eta_{\mu\nu})\ ,\end{aligned}$$

where λ_1, λ_2, λ_3, λ_4, and $\lambda^{\mu\nu}$ are Lagrange multipliers associated with the various constraints. We have generalized Halbwachs's variational principle by considering the entropy s and the Lin particle identity variable X. $F(\rho,s)$ is the energy density of the fluid. The Lagrangian density for a nonspinning fluid, defined here by $a^{1i}=a^{2i}=a^{3i}=0$, has been discussed in detail in Ref. 6 to which we refer the reader.

The variables to be varied in $\mathscr{L}_f$ are U^i, ρ, $a^{\mu i}$, s, X, λ_1, λ_2, λ_3, and $\lambda^{\mu\nu}$. Since a^{3i} does not appear in the kinetic energy terms in $\mathscr{L}_f$ it may be left out of the variations without changing the final results. Also the constraint for λ^{44} is the same as that for λ_1 and these may be combined together into one term whose Lagrange multiplier we call λ_1. Thus, we have the final form of the Lagrangian density for the fluid

$$\begin{aligned}\mathscr{L}_f = &\sqrt{-g}\,F/c - \sqrt{-g}\,\rho k a^1{}_i a^{2i}{}_{;j} U^j/c + \sqrt{-g}\,\lambda_1(g_{ij}U^iU^j + c^2) + \sqrt{-g}\,\lambda_2(\rho U^i)_{;i} + \sqrt{-g}\,\lambda_3 X_{,i} U^i \\ &+\sqrt{-g}\,\lambda_4 s_{,i} U^i + \sqrt{-g}\,\lambda^{11}(g_{ij}a^{1i}a^{1j} - 1) + \sqrt{-g}\,\lambda^{22}(g_{ij}a^{2i}a^{2j} - 1) + 2\sqrt{-g}\,\lambda^{12} g_{ij} a^{1i} a^{2j} \\ &+2\sqrt{-g}\,\lambda^{14} g_{ij} a^{1i} U^j/c + 2\sqrt{-g}\,\lambda^{24} g_{ij} a^{2i} U^j/c\ . \end{aligned} \tag{2.12}$$

The spin density of the fluid S_{ij} is defined by

$$\sqrt{-g}\,S_{ij} = c\left[a^\nu{}_i \frac{\partial \mathscr{L}_f}{\partial \dot{a}^{\nu j}} - a^\nu{}_j \frac{\partial \mathscr{L}_f}{\partial \dot{a}^{\nu i}}\right], \tag{2.13}$$

which gives, when applied to (2.12),

$$S_{ij} = \rho\kappa(a^1{}_i a^2{}_j - a^1{}_j a^2{}_i)\ , \tag{2.14}$$

from which we see that our earlier definition of the spin density is consistent with the field-theory definition (2.13). The spin of a fluid particle is given by[4,7]

$$s_{ij} = \kappa(a^1{}_i a^2{}_j - a^1{}_j a^2{}_i)\ , \tag{2.15}$$

from which we see that the spin of a fluid particle is proportional to the scalar $\kappa(x)$. Later we shall prove that κ is constant along the flow for our problem.

An equivalent way to introduce spin in field theory is via the Belinfante-Rosenfeld spin tensor $\tau_{ij}{}^k$:

$$\sqrt{-g}\,\tau_{ij}{}^k = cF_{ij}^{rs}\frac{\partial \mathscr{L}_f}{\partial \psi^r{}_{;k}}\psi_s\ . \tag{2.16}$$

For the spin variables $\psi^r \to a^{2r}$,

$$F_{ij}^{rs} = \tfrac{1}{2}(\delta_j^r\delta_i^s - \delta_i^r\delta_j^s)$$

and we obtain

$$\tau_{ij}{}^k = \tfrac{1}{2}S_{ij}U^k\ ,$$

the Weyssenhoff convective form for the spin tensor. In Sec. III we carry out the variations that yield the equations of motion for a spinning fluid in general relativity.

III. FIELD EQUATIONS

In order to generalize Halbwachs's Lagrangian density $\mathscr{L}_f$ (2.12) to general relativity we only need add on the gravitational Lagrangian density $\sqrt{-g}\,R$, where R is the scalar curvature. This gives the total Lagrangian density

$$\mathscr{L} = c_1\sqrt{-g}\,R + \mathscr{L}_f\ , \tag{3.1}$$

where $c_1 = c^3/(16\pi k)$ and k is the gravitational constant.

We now outline the results of varying the action associated with (3.1) with respect to the field variables a^{1i}, a^{2i}, U^i, ρ, s, X, and g_{ij}. For more details of this calculation see Refs. 4 and 6.

Variations with respect to a^{1i} and a^{2i} lead to the field equations

$$-\rho\kappa\dot{a}^2{}_i/c + 2\lambda^{11}a_{1i} + 2\lambda^{12}a_{2i} + 2\lambda^{14}U_i/c = 0 \tag{3.2}$$

and

$$\begin{aligned}\rho\dot{\kappa}a^1{}_i/c + \kappa\rho\dot{a}^1{}_i/c + 2\lambda^{22}a_{2i} + 2\lambda^{12}a_{1i}& \\ + 2\lambda^{24}U_i/c = 0\ .&\end{aligned} \tag{3.3}$$

Dotting (3.2) with U_i leads to

$$\lambda^{14}=-\rho\kappa\dot{a}^2{}_iU^i/(2c^2)$$
$$=\rho\kappa a^2{}_i\dot{U}^i/(2c^2)\ . \qquad (3.4)$$

The same procedure applied to (3.3) yields

$$\lambda^{24}=-\rho\kappa a^1{}_i\dot{U}^i/(2c^2)\ . \qquad (3.5)$$

Next if we dot (3.2) with a^{1i} we obtain

$$\lambda^{11}=\rho\kappa a^{1i}\dot{a}^2{}_i/(2c)\ , \qquad (3.6)$$

whereas the same procedure applied to (3.3) produces

$$\lambda^{12}=-\dot{\kappa}\rho/(2c)\ . \qquad (3.7)$$

If we now dot (3.2) with a^{2i} we obtain

$$\lambda^{12}=0\ , \qquad (3.8)$$

which with (3.7) gives

$$\dot{\kappa}=0\ . \qquad (3.9)$$

This means that the magnitude of the spin is constant along the flow. Dotting (3.3) with a^{2i} gives

$$\lambda^{22}=\kappa\rho a^1{}_i\dot{a}^2{}_i/(2c)=\lambda^{11}\ . \qquad (3.10)$$

If we next multiply (3.2) by $a^1{}_j$ and (3.3) by $a^2{}_j$, then add the resulting equations, we find

$$\rho\kappa(\dot{a}^1{}_ia^2{}_j-\dot{a}^2{}_ia^1{}_j)/c+2\lambda^{11}(a_{2i}a^2{}_j+a_{1i}a^1{}_j)-\rho\kappa a^2{}_ja^1{}_k\dot{U}^kU_i/c^3+\rho\kappa a^1{}_ja^2{}_k\dot{U}^kU_i/c^3=0\ . \qquad (3.11)$$

If we switch i and j and subtract the resulting equation from (3.11) we arrive at the equation of motion for the spin,

$$\dot{s}_{ij}+s_{jk}U_i\dot{U}^k/c^2+s_{ki}U_j\dot{U}^k/c^2=0\ . \qquad (3.12)$$

This equation expresses the fact that the spin undergoes Fermi-Walker transport along the four-velocity U^i and has also been derived in general relativity by Mathisson[8] and Papapetrou[9] using considerably different methods than those employed here.

The variation with respect to the density ρ leads to

$$\lambda_{2,k}U^k+\kappa a^1{}_i\dot{a}^{2i}/c=F_\rho/c\ , \qquad (3.13)$$

where $F_\rho=\partial F/\partial\rho$.

Variation with respect to the four-velocity leads to the expression

$$2\lambda_1U_i+\lambda_3X_{,i}+\lambda_4s_{,i}-\rho\kappa a^1{}_ka^{2k}{}_{;i}/c$$
$$+\rho s_{ik}\dot{U}^k/c^3-\lambda_{2,i}\rho=0\ . \qquad (3.14)$$

In a nonspinning fluid $a^{1i}=a^{2i}=a^{3i}=0$ and (3.14) gives the important velocity potential (Clebsch) representation of the four-velocity. For the case of spinning matter the four-velocity does not come out in a potential representation. If we dot (3.14) with U^i and employ (3.13) we find

$$\lambda_1=-\rho F_\rho/(2c^3)\ . \qquad (3.15)$$

This is the same solution for λ_1 as for a nonspinning fluid. The variation with respect to X and s lead to simple equations which are the same as in Ref. 6. Since we shall not need these equations explicitly in this paper we shall not write them out. If one were to differentiate (3.14) to obtain an expression for $\dot{U}_i$, then these equations would have to be used to eliminate λ_3 and λ_4 from the resulting equation, which would be the generalized Euler equation.

The variation with respect to the metric g_{ik} leads to the Einstein equations for a fluid with internal spin. This calculation is somewhat tedious and we give only the final result which is

$$G^{ik}=\frac{8\pi k}{c^4}\left[-\rho F_\rho U^iU^k/c^2+g^{ik}(F-\rho F_\rho)+\frac{1}{c^2}\rho U^{(i}s^{k)l}\dot{U}_l+\rho U^{(k}s^{i)j}{}_{;j}+[\rho U^{(k}]_{;j}s^{i)j}\right]\ , \qquad (3.16)$$

where the parentheses around indices imply symmetrization, $A_{(ik)}=\frac{1}{2}(A_{ik}+A_{ki})$. If we make use of the form for $F(\rho,s)$, $F=-\rho(c^2+\epsilon)$, where ϵ is the rest specific internal energy then (3.16) can be written

$$G^{ik}=\frac{8\pi k}{c^4}\left[\rho(1+\epsilon/c^2+P/\rho c^2)U^iU^k+g^{ik}P+\frac{1}{c^2}\rho U^{(i}s^{k)l}\dot{U}_l+\rho U^{(k}s^{i)j}{}_{;j}+[\rho U^{(k}]_{;j}s^{i)j}\right]\ , \qquad (3.17)$$

where P is the pressure $P=\rho^2(\partial\epsilon/\partial\rho)_s$. The first two terms on the right-hand side of (3.17) give the energy-momentum tensor for a fluid without spin:

$$T_F^{ik}=\rho(1+\epsilon/c^2+P/\rho c^2)U^iU^k+g^{ik}P\ . \qquad (3.18)$$

The remaining terms on the right-hand side of (3.17) give the intrinsic spin contribution to the energy-momentum tensor

$$T_s{}^{ik}=\frac{1}{c^2}\rho U^{(i}s^{k)l}\dot{U}_l+\rho U^{(k}s^{i)j}{}_{;j}+[\rho U^{(k}]_{;j}s^{i)j}\ . \qquad (3.19)$$

As far as we are aware this spin contribution to the energy-momentum tensor has not been derived previously although it is a straight-forward generalization of Halbwachs's treatment of spinning fluids in special relativity. The symmetric energy-momentum tensor T^{ik},

$$T^{ik}=T_F^{ik}+T_s^{ik}\ , \tag{3.20}$$

satisfies the Bianchi identities $T^{ik}{}_{;k}=0$ which lead to the relativistic Euler equation for the fluid.

IV. CONCLUSIONS

We have given a detailed treatment of the equations for a spinning fluid in general relativity. Our procedure was to follow Halbwachs's introduction of a tetrad to represent both the spin density and four-velocity of the fluid. We were then able to formulate an Eulerian variational principle, which is correct in special relativity, to derive the form of the Einstein equations. The resulting energy-momentum tensor for a spinning fluid is given in (3.17) and has apparently not been previously derived. An alternative derivation of T^{ik} is to calculate the canonical energy-momentum tensor t_k^i:

$$\sqrt{-g}\,t_i^k=c\delta_i^k\mathscr{L}_f-c\frac{\partial\mathscr{L}_f}{\partial\psi^r_{;k}}\psi^r_{;i}\ , \tag{4.1}$$

which has the form

$$\begin{aligned}t_i^k=&\ \delta_i^kF-\delta_i^k\rho\kappa a^1{}_j\dot a^{2j}+\rho\kappa a^1{}_r a^{2r}{}_{;i}U^k\\&-c\lambda_2\rho U^k{}_{;i}-c\lambda_2U^k\rho_{,i}\\&-c\lambda_3U^kX_{,i}-c\lambda_4U^ks_{,i}\ ,\end{aligned} \tag{4.2}$$

and then to carry out the Belinfante-Rosenfeld symmetrization procedure. This again, of course, leads to the symmetric energy-momentum tensor given in (3.17).

The equations of motion of the fluid also follow from the variational principle. The spin s_{ij} undergoes a Fermi-Walker transport and the generalized Euler equations follow from the Bianchi identities or by differentiating U_i in (3.14) and using the other equations from the variation principle.

There are several suggested applications of the results of this paper. First one could study exact solutions to the Einstein equations for a spinning fluid in general relativity. In a cosmological context the galaxies would be the spinning particles of the fluid. The study of such a fluid of galaxies in various anisotropic Bianchi universes would give an indication of how the spin interacts gravitationally with the anisotropy of the spacetime. Israel[2] studied a simple model in a Bianchi type-I universe and found that spin induces a Lense-Thirring rotation of the local inertial axis relative to the directions along the spin of the fluid.

The work by Bailey and Israel (Refs. 1 – 3) is the closest to that presented in this paper. Their study does not make use of an explicit Lagrangian density, as we have done, but derives various relations which must be satisfied via Noether's theorem. Their theory is therefore valid for arbitrary Lagrangian densities. Our work, on the other hand, deals with a specific Lagrangian density that describes spinning fluids in special relativity. The work of Israel and Bailey also allows for other fields to be present in the Lagrangian density. This we can do by adding more terms to describe these fields. For example, Maxwell's theory and the generalized Lorentz dielectric model would be a first step.

Our formulation of spinning fluids in general relativity can also be used to investigate the equations of motion for a spinning fluid in the Einstein-Cartan metric-torsion theory. In a preliminary study[10] we investigated the introduction of torsion into the fluid variational principle of Ref. 6. We can now extend this work using the results of this paper to give the first variational derivation of the equations of motion for a spinning fluid in the Einstein-Cartan theory. We are presently studying this problem.

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