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Connection between the nonlinear σ model and the Einstein equations of general relativity

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We show that the equations of general relativity contain an O(2,1) σ model. This σ-model structure emerges from a 3 + 1 decomposition of the Einstein equations which holds irrespective of the presence of symmetries in the space-time. This includes, in particular, the stationary (one Killing vector) and the Ernst (two Killing vectors) formulations of the gravitational field. From this connection with the σ model we find a new family of solutions of the Einstein equations. These solutions have (3,1) signature and one Killing vector. They are complex or real and they depend on two arbitrary functions (one holomorphic and one antiholomorphic). In particular, in the presence of two Killing vectors they give two different subclasses of solutions: one is associated with the instantons of the σ model and the other is of Taub-NUT (Newman-Unti-Tamburino) type.

I. INTRODUCTION

Nonlinear σ models, Yang-Mills theory, and, of course, general relativity are known to admit a natural geometric formulation. Our aim in this paper is to study the connection between the structure of these theories and the correspondence between their respective solutions. Here we will emphasize the connection between general relativity (GR) and the σ model. The connection with Yang-Mills theory will be studied elsewhere.[1] The Zelmanov formulation[2] of GR, based on a 3 + 1 decomposition of the field equations $R_{\mu\nu}=0$, is particularly well suited for our purposes. This decomposition is invariant under the transformation

$$\tilde{x}^0=\tilde{x}^0(x^0,x^i)\ ,$$

$$\tilde{x}^i=\tilde{x}^i(x^i)\quad (i=1,2,3)\ ,$$

i.e., it is chronometric invariant (CI). The four-dimensional Einstein equations for $g_{\mu\nu}$ are projected into a three-dimensional space of metric

$$h_{ij}=g_{ij}-g_{0i}g_{0j}/g_{00}$$

in which (CI) operators are defined. [For instance, the derivatives ${}^*\partial_i=\partial_i-(g_{0i}/g_{00})\partial_0$ replace the ordinary ones ∂_i.] We show that in the equations $R_{0\nu}=0$ lies a nonlinear σ-model structure. By expressing the projected equations $R_{00}=0$, $R_{0i}=0$ in terms of a new set of variables (two kinds of potentials V,ϕ) and in terms of the conformal metric $\gamma_{ij}=Vh_{ij}$ (instead of h_{ij} itself) an O(2,1) σ-model structure for the complex potential $\mathscr{E}=V+i\phi$ emerges. The (V,ϕ) parametrization is related to the standard $(\sigma^1,\sigma^2,\sigma^3)$ one by $V=1/(\sigma^1+\sigma^3)$, $\phi=\sigma^2/(\sigma^1+\sigma^3)$ (see Secs. II and IV). The σ field lies in a three-dimensional space with metric γ_{ij}, corresponding to CI covariant derivatives ${}^*\nabla_i$. [For Einstein equations with Euclidean signature an O(2,1) σ model appears too but for a real potential $\mathscr{E}_\pm=V\pm\psi$.] This connection is general, independent of the presence of any isometry group (i.e., of any symmetry in the space-time). In particular, in the presence of one Killing vector field, the CI derivatives ${}^*\nabla_i$ become the ordinary covariant ones ∇_i with respect to the metric γ_{ij} and it gives the 3 + 1 decomposition considered by Gibbons and Hawking in the context of gravitational instanton isometries.[3] In the presence of two Killing vectors one recovers the well-known Ernst formulation of the stationary axially symmetric gravitational field.[4] The Ernst equation $(\mathrm{Re}\,\mathscr{E})\vec{\nabla}^2\mathscr{E}-(\vec{\nabla}\mathscr{E})^2=0$ is just an O(2,1) σ model in three-dimensional flat space [in this case $\mathscr{E}$ depends only on x_3 and $\rho=({x_1}^2+{x_2}^2)^{1/2}$]. The gravitational field in the presence of two Killing vectors has been extensively studied in the literature.[4–13] The known solutions can be grouped in three different physical situations: (i) stationary axisymmetric gravitational fields,[4–13] (ii) colliding plane waves,[14–16] (iii) cylindrical waves,[17] depending on whether both

Killing vectors are space-like [cases (ii), (iii)] or one is space-like and one is time-like [case (i)]. However, we point out that all the known solutions involved in the situations (i), (ii), and (iii) above correspond to solutions $\mathscr{E}$ of the σ model not given by holomorphic functions. As was found by Belavin and Polyakov,[18] the holomorphic mappings are instantons, i.e., self-dual solutions of the O(3) σ model. Moreover, they provide all (rather than merely some possible) instanton solutions of this model.[19] Holomorphic (antiholomorphic) mappings also provide self-dual solutions for the O(2,1) σ model. We are therefore led to investigate whether these solutions are compatible with the whole set of Einstein equations (i.e.), if it is possible to find a solution of Einstein equations associated with the holomorphic mappings). We find a new family of solutions of Einstein equations having one Killing vector field associated with these mappings. This family can be complex or real with (3,1) signature. This is a Lorentzian solution which does not exist in the Euclidean [i.e., (4,0) signature] regime. It is given by

$$ds^2=-[\mathscr{E}_1(u,v)+\mathscr{E}_2(\bar{u},v)]dt^2+2dt\,dz+dx^2+dy^2\ . \tag{A}$$

It depends on two arbitrary (one holomorphic and one antiholomorphic) functions $\mathscr{E}_1(u,v)$ and $\mathscr{E}_2(\bar{u},v)$ (where $u=x+iy$, $\bar{u}=x-iy$, $v=z+it$). The Wick rotation $y=it$ $(t=-iy)$ maps (1) onto a different solution. This solution is real and of plane-wave type but with signature (2,2), i.e.,

$$ds^2=dx^2-dt^2+[\mathscr{E}_1(x-t,z+y)+\mathscr{E}_2(x+t,z+y)]dy^2-2dy\,dz\ . \tag{B}$$

In particular, (a) if $\mathscr{E}_1(u,z-\tau)=\mathscr{E}_2(\bar{u},z-\tau)\equiv\mathscr{E}$ (where we have put $t=i\tau$), (A) gives a Lorentzian real metric of Peres's type. For $\mathscr{E}$ independent of v (i.e., in the case of two Killing vectors), $\mathscr{E}_1(\mu)=\mathscr{E}_2(\bar{u})\equiv\mathscr{E}$ gives the metric associated with the holomorphic solutions of the σ model. The choice $\mathscr{E}=a$ rational function gives the solution associated with the (multi)instanton solutions of the σ model.

On the other hand, (b) if we put $\mathscr{E}_1=0$ or $\mathscr{E}_2=0$ in either the complex or real solutions (A) or (B), we obtain metrics of Taub-NUT (Newman-Unti-Tamburino) type as considered by Gibbons and Hawking.[3] Here, the connection with Taub-NUT metrics appears in the presence of two Killing vectors. Note that the metrics associated with the instantons of the σ model are neither axisymmetric, nor of Taub-NUT type.

Following Pohlmeyer's reduction for the O(3) σ model[20] one can relate the O(2,1) σ model to the sine-Gordon and to the Liouville equation (see Sec. IV). In this way, besides the holomorphic functions, several types of solutions can be found : Painlevé transcendents,[21] complex multisolitons,[22] and Liouville-type solutions. In the axisymmetric case, several solutions of Einstein equations are known, which depend on Painlevé transcendents of 3rd and 5th type.[23-25] However, for solutions that are not axisymmetric, we show that the only solutions of the σ model that are compatible with Einstein equations are the holomorphic functions. In the nonaxisymmetric case, the reduction from the O(2,1) σ model to the sine-Gordon or to the Liouville equation is not compatible with the Einstein equations.

In Ref. 27 we generalize the above solutions to the case when there is no Killing vector. In this case, the metric depends on holomorphic (and antiholomorphic) functions of both u and $\bar{u}$, v and $\bar{v}$. This provides a unifying pattern in which a wide variety of situations can be considered.

II. O(3) and O(2,1) NONLINEAR σ MODELS

As is well known, the O(3) nonlinear σ model is defined by

$$\mathscr{L}=\frac{1}{2}\sum_{a=1}^{3}(\partial_u\sigma^a)(\partial^u\sigma^a)\ , \tag{1}$$

$$\sum_{a=1}^{3}(\sigma^a)^2=1\ . \tag{2}$$

The field $\sigma=(\sigma^1,\sigma^2,\sigma^3)$ belongs to S^2. That is,

$$\mathscr{L}_\sigma=\frac{1}{2}(\partial_u\sigma^a)(\partial^u\sigma^a)+\frac{1}{2}\Lambda[(\sigma^a)^2-1]\ , \tag{3}$$

Λ being a Lagrangian multiplier.

From the Euler-Lagrange equations and the constraint (2), it follows that

$$\Lambda=\sigma^a\Box^2\sigma^a\ ,$$

i.e., the equation of motion of the O(3) σ model is

$$\Box^2\sigma^a=(\sigma^b\Box^2\sigma^b)\sigma^a\ . \tag{4}$$

For the O(2,1) σ model, Eq. (2) is replaced by

$$(\sigma^1)^2+(\sigma^2)^2-(\sigma^3)^2=1\ . \tag{5}$$

Different parametrizations can be used. By defining

$$V=\frac{1}{\sigma^1+\sigma^3}\ ,\qquad \psi=\frac{\sigma^2}{\sigma^1+\sigma^3}\ , \tag{6}$$

the Lagrangian for the O(2,1) σ model can be written as

$$\mathscr{L}_{\sigma O(2,1)}=\frac{1}{V^2}[(\nabla V)^2-(\nabla\psi)^2]\ , \tag{7}$$

and the corresponding equations of motion are

$$\nabla^2 V-\frac{\nabla V}{V}-\frac{\nabla\psi}{V}=0\ ,\qquad \nabla^2\psi-\frac{2(\nabla V)(\nabla\psi)}{V}=0\ . \tag{8}$$

By defining $\mathscr{E}_+=V+\psi$, $\mathscr{E}_-=V-\psi$, these equations read

$$\nabla^2\mathscr{E}_+-\frac{2}{(\mathscr{E}_++\mathscr{E}_-)}(\nabla\mathscr{E}_+)^2=0\ ,\qquad \nabla^2\mathscr{E}_--\frac{2}{(\mathscr{E}_++\mathscr{E}_-)}(\nabla\mathscr{E}_-)^2=0\ . \tag{9}$$

Putting $\psi=i\phi$, these can be written in the compact form

$$\nabla^2\mathscr{E}-\frac{1}{(\mathrm{Re}\mathscr{E})}(\nabla\mathscr{E})^2=0\ , \tag{10}$$

where $\mathscr{E}=V+i\phi$.

In terms of $\xi=(1+\mathscr{E})/(1-\mathscr{E})$, the O(2,1) σ model reads

$$(\xi\bar{\xi}-1)\nabla^2\xi=2\bar{\xi}(\nabla\xi)^2\ . \tag{11}$$

In this parametrization, the O(3) σ model [Eq. (4)] reads

$$(W\overline{W}+1)\nabla^2 W=2\overline{W}(\nabla W)^2\ . \tag{12}$$

Here, $W=(\sigma^1+i\sigma^2)/(1+\sigma^3)$ with $(\sigma^1,\sigma^2,\sigma^3)$ satisfying (2).

The Lagrangian (7) has the three-parameter group of isometries SL(2,R) realized by[3] (i) dilations

$$\tilde{V}+\tilde{\psi}=b(V+\psi)\ ,\qquad \tilde{V}-\tilde{\psi}=b(V-\psi)\ , \tag{13}$$

(ii) translations

$$\tilde{V}+\tilde{\psi}=V+\psi+a\ ,\qquad \tilde{V}-\tilde{\psi}=V-\psi-a\ , \tag{14}$$

(iii) Ehler's transformations

$$\tilde{V}+\tilde{\psi}=\frac{V+\psi+b(V^2-\psi^2)}{(1+b\psi)^2-b^2V^2}\ ,\qquad \tilde{V}-\tilde{\psi}=\frac{V-\psi-b(V^2-\psi^2)}{(1-b\psi)^2-b^2V^2}\ . \tag{15}$$

In the hyperboloid standard parametrization $(\sigma^1,\sigma^2,\sigma^3)$ defined by (5) the dilations (13) correspond to

$$\tilde{\sigma}^1+\tilde{\sigma}^3=\frac{1}{b}(\sigma^1+\sigma^3)\ ,\qquad \tilde{\sigma}^1-\tilde{\sigma}^3=b(\sigma^1-\sigma^3)\ ,\qquad \tilde{\sigma}^2=\sigma^2\ ,$$

i.e., hyperbolic rotations (of angle $\log b$) in the $(\tilde{\sigma}_1,\tilde{\sigma}_3)$ plane. The translations (14) correspond to

$$\tilde{\sigma}^1+\tilde{\sigma}^3=\sigma^1+\sigma^3\ ,\qquad \tilde{\sigma}^1-\tilde{\sigma}^3=\sigma^1-\sigma^3-a^2(\sigma^1+\sigma^3)-2a\sigma^2\ ,\qquad \tilde{\sigma}^2=\sigma^2+a(\sigma^1+\sigma^3)\ ,$$

and the Ehler's transformations give

$$\tilde{\sigma}^1+\tilde{\sigma}^3=\sigma^1+\sigma^3+b^2(\sigma^1-\sigma^3)-2b\sigma^2\ ,\qquad \tilde{\sigma}^1-\tilde{\sigma}^3=\sigma^1-\sigma^3\ ,\qquad \tilde{\sigma}^2=\sigma^2+b(\sigma^1-\sigma^3)\ .$$

III. EINSTEIN EQUATIONS IN PRESENCE OF ONE KILLING VECTOR FIELD— THE STATIONARY GRAVITATIONAL FIELD

Let $g_{\mu\nu}$ be the metric of an oriented four-dimensional Riemannian manifold $\mathscr{M}$. In the presence of a one-parameter isometry group ($K=\partial/\partial x^0$ being the Killing vector, X^0 being the group parameter), it is possible to choose a coordinate frame such that the components of $g_{\mu\nu}$ are independent of X^0. By introducing the notation

$$V=g_{00}\ ,\qquad \omega_j=\frac{g_{0j}}{g_{00}}\ ,\qquad h_{ij}=g_{ij}-\frac{g_{0i}g_{0j}}{g_{00}}\ ,$$

the squared length reads

$$ds^2=V(dx^0+\omega_j dx_j)^2-h_{ij}dx^i dx^j\ . \tag{16}$$

One can project the Einstein equations $R_{\mu\nu}=0$ into the three-dimensional space (B) of metric h_{ij}. The projected equations must be invariant under the transformation

$$\tilde{X}^0 = X^0 + \beta(x^j) ,$$
$$\tilde{X}^j = X^j ,$$

which does not affect the stationarity of the field. Under such a transformation,

$$\omega_i \to \omega_i - \frac{\partial\beta}{\partial x^j} ,$$

but V, h_{ij}, and $f_{ij} = \partial_i\omega_j - \partial_j\omega_i$ are invariant. The projected field equations contain ω only in the combination given by f_{ij}. These equations are[28]

$$\frac{R_{00}}{V} = \frac{1}{\sqrt{V}}\nabla^2(\sqrt{V}) + \frac{V}{4} f_{ij}f^{ij} , \quad (17)$$

$$R_0{}^i = -\frac{\sqrt{V}}{2} f^{ij}_{,j} - \tfrac{3}{2} f^{ij}(\sqrt{V})_{,j} , \quad (18)$$

$$R^{ij} = {}^{(3)}R^{ij} + \frac{V}{2} f^{ik}f_k{}^j - \frac{1}{\sqrt{V}}(\sqrt{V})^{i,j} , \quad (19)$$

where ${}^{(3)}R^{ij}$ is the three-dimensional Ricci tensor of B calculated with respect to the metric h_{ij}. All quantities on the right-hand side of Eqs. (17)–(19) are defined with respect to the metric h_{ij}. Indices i,j,k are lowered or raised by h_{ij} and ∇ stands for covariant differentiation with respect to h_{ij}.

To fix the notation we recall that

$$f_{ij} = \sqrt{h}\, e_{ijk} f^k ,$$
$$f_i = \sqrt{h}\, e_{ijk} f^{jk} , \quad (20)$$
$$(\nabla \times A)^i = \frac{1}{2\sqrt{h}} e^{ijk}\partial_j f_k .$$

Then, Eq. (18) can be written as

$$\frac{R_0{}^i}{\sqrt{V}} = -\frac{1}{V}(\nabla \times V^{3/2} f)^i . \quad (21)$$

This allows us to define ϕ such that

$$V^{3/2} f_k = \nabla_k \phi , \quad (22)$$

i.e., Eq. (18) is equivalent to writing

$$\nabla\cdot(\nabla_k\phi / V^{3/2}) = 0 . \quad (23)$$

In terms of ϕ, Eqs. (17) and (18) read

$$\frac{R_{00}}{V} = \frac{1}{2}\nabla^2 V - \frac{1}{4V^2}\nabla_j V \nabla^j V + \frac{1}{2}\nabla_j\phi\nabla^j\phi = 0 , \quad (24)$$

$$\nabla\cdot f = \frac{1}{V}\nabla^2\phi - \frac{3}{2V^2}\nabla_j\phi\nabla^j V = 0 . \quad (25)$$

We express these equations in terms of the metric Vh_{ij} (instead of h_{ij}).

Note that

$$(\nabla^2)_{h_{ij}} = (V\nabla^2 - \tfrac{1}{2}\nabla_j V \nabla^j)_{(Vh_{ij})} ,$$
$$(\nabla_j\phi\nabla^j)_{h_{ij}} = (V\nabla_j\phi\nabla^j)_{(Vh_{ij})} .$$

We have

$$\frac{1}{2}\nabla^2 V - \frac{1}{2V}\nabla_j V\nabla^j V + \frac{1}{2V}\nabla_j\phi\nabla^j\phi = 0 , \quad (26)$$

$$\frac{1}{2}\nabla^2\phi - \frac{2}{V}\nabla_j V\nabla^j\phi = 0 , \quad (27)$$

which can be summarized as

$$\nabla^2(V + i\phi) - \frac{2}{V}[\nabla(V + i\phi)]^2 = 0 . \quad (28)$$

Then, in terms of γ_{ij}, V, and ϕ, the Einstein equations $R_{\mu\nu} = 0$ can be written as

$$\nabla^2\mathscr{E} - \frac{1}{\mathrm{Re}\mathscr{E}}(\nabla\mathscr{E})^2 = 0 , \quad (29)$$

$${}^{(3)}R^{ij} - \frac{1}{2V^2}(\nabla_i V\nabla_j V + \nabla_i\phi\nabla_j\phi) = 0 . \quad (30)$$

Here $\mathscr{E} = V + i\phi$. We see that Eq. (29) describes an O(2,1) nonlinear σ model [the same as Eq. (10)] but in a three-dimensional space of metric γ_{ij}. Equation (30) determines γ_{ij}. Equation (29) for the complex potential $\mathscr{E}$ arises from the Einstein equations with Minkowskian signature. Einstein equations with Euclidean signature also give rise to an O(2,1) σ model but for the real potential $\mathscr{E}_{\pm} = V \pm \psi$. [Euclidean signature corresponds to the Wick rotation $\tau = it, \phi = i\psi$ in the metric (16).] In this way, stationary four-dimensional real Minkowskian solutions of Einstein equations are related to three-dimensional complex solutions of the O(2,1) σ model. Real Euclidean solutions of Einstein equations are connected to real solutions of this model. Recall that V is the g_{00} coefficient of g_{uv}. ϕ does not enter directly into the metric g_{uv}. ϕ is related to ω through

$$f_k = (\nabla\times\omega)_k = \frac{\nabla_k\phi}{V^2} . \quad (31)$$

IV. EINSTEIN EQUATIONS IN THE PRESENCE OF TWO KILLING VECTORS FIELDS

In the presence of two Killing vectors $g_{\mu\nu}$ can be written in the canonical form[29,7] as

$$ds^2 = \lambda V(dx^1 + \omega_2 dx^2)^2 + \frac{\gamma_{ij}}{V} dx^i dx^j , \quad (32)$$

where

$$\gamma_{ij}dx^i dx^j = e^{2\gamma}[(dx^3)^2 - \lambda(dx^4)^2] + s^2(dx^2)^2 , \quad (33)$$

and $\lambda=\pm 1$. V, ω, γ, and s depend only on x^3 and x^4. In this case Eq. (10) reads

$$\left[\lambda\partial_3{}^2 - \partial_4{}^2 + \lambda\left[\frac{\partial_3 s}{s}\right]\partial_3 - \left[\frac{\partial_4 s}{s}\right]\partial_4\right]\mathscr{E} - \frac{1}{(\mathrm{Re}\mathscr{E})}[\lambda(\partial_3\mathscr{E})^2 - (\partial_4\mathscr{E})^2] = 0 . \quad (34)$$

From γ_{ij} as given by Eq. (33), we obtain

$$^{(3)}R_{33} = \lambda\partial_4{}^2\gamma - \partial_3{}^2\gamma + \frac{\partial_3 s}{s}\partial_3\gamma + \lambda\frac{\partial_4 s}{s}\partial_4\gamma - \frac{\partial_3{}^2 s}{s} ,$$

$$^{(3)}R_{34} = \frac{\partial_3 s}{s}\partial_4\gamma + \frac{\partial_4 s}{s}\partial_3\gamma - \frac{\partial_{3,4}{}^2 s}{s} , \quad (35)$$

$$^{(3)}R_{44} = \lambda\partial_3{}^2\gamma - \partial_4{}^2\gamma + \frac{\partial_4 s}{s}\partial_4\gamma + \lambda\frac{\partial_3 s}{s}\partial_3\gamma - \frac{\partial_4{}^3 s}{s} ,$$

$$^{(3)}R_{2j} = 0 \quad (j=2,3,4)$$

which give

$$\partial_3{}^2 s - \lambda\partial_4{}^2 s = 0 . \quad (36)$$

On the other hand, from Eqs. (30) we have

$$^{(3)}R_{33} = \frac{1}{2V^2}[(\nabla_3 V)^2 + (\nabla_3\psi)^2] ,$$

$$^{(3)}R_{34} = \frac{1}{2V^2}[\nabla_3 V\nabla_4 V + \nabla_3\psi\nabla_4\psi] , \quad (37)$$

$$^{(3)}R_{44} = \frac{1}{2V^2}[(\nabla_4\psi)^2 + (\nabla_4\psi)^2] .$$

As is known different physical situations described by the metric (32) are

(i) axially symmetric stationary fields[4–13] (for $\lambda=-1$, $x^1=t$, $x^2=\varphi$, $x^3=\rho$, $x^4=z$);

(ii) cylindrical waves[17] (for $\lambda=1$, $x^1=z$, $x^2=\varphi$, $x^3=\rho$, $x^4=t$);

(iii) colliding plane waves[14–16] (for $\lambda=1$, $x^1=\rho$, $x^2=\varphi$, $x^3=z$, $x^4=t$). For all these situations, s is taken equal either to x^3 or x^4 and the coordinates are of cylindrical type.

For situation (i), Eq. (34) gives the well-known Ernst equation, extensively treated in the current literature, i.e.,

$$\left[\partial_\rho{}^2 + \frac{1}{\rho}\partial_\rho + \partial_z{}^2\right]\mathscr{E} - \frac{1}{(\mathrm{Re}\mathscr{E})}[(\partial_\rho\mathscr{E})^2 + (\partial_z\mathscr{E})^2] = 0 . \quad (38)$$

For instance, the Kerr solution corresponds to

$$\mathscr{E} = \frac{\cosh\alpha + ia\cos\beta - m}{\cosh\alpha + ia\cos\beta + m} ,$$

α,β, being prolate spheroidal coordinates related to ρ, z by

$$\cosh\alpha = \tfrac{1}{2}\{[(z+1)^2+\rho^2]^{1/2} + [(z-1)^2+\rho^2]^{1/2}\} ,$$

$$\cos\beta = \tfrac{1}{2}\{[(z+1)^2+\rho^2]^{1/2} + [(z-1)^2+\rho^2]^{1/2}\} .$$

Here m and a stand for the mass and the angular momentum per unit mass, respectively, satisfying $m^2-a^2=1$. For $a=0$ it gives the Schwarzschild solution, $r=1+\cosh\alpha$ and $\theta=\beta$ being identified as the Schwarzschild coordinates.

It can be pointed out that all the known solutions involved in the situations (i), (ii), and (iii) mentioned above correspond to solutions of the σ model [Eq. (34)] which are not holomorphic functions. As we will see below, this is so because for the situations (i), (ii), and (ii) above, s is not a constant.

In what follows we will take $\lambda=-1$, $x^1=t$, $x^2=z$, $x^3=x$ and $x^4=y$ as Cartesian-type coordinates and we ask for solutions $\mathscr{E}=V+i\phi$ of Eq. (34) given by holomorphic functions, i.e, V and ϕ satisfying the Cauchy-Riemann equations

$$\partial_x V = \partial_y\phi , \qquad \partial_y V = -\partial_x\phi , \quad (39)$$

that is,

$$V = \tfrac{1}{2}[\mathscr{E}(u) + \mathscr{E}(\bar{u})] , \qquad \phi = \frac{1}{2i}[\mathscr{E}(u) - \mathscr{E}(\bar{u})] \quad \left[\begin{matrix} u = x+iy, \\ \bar{u} = x-iy \end{matrix}\right] . \quad (40)$$

From Eq. (34), this imposes s = const.

The problem is to see if the ansatz (40) is compatible with the other set of Einstein equations (35) and if it leads to a nontrivial metric. As is easily seen from Eqs. (26) and (27), we can generalize the ansatz (40) to include two different functions $\mathscr{E}_1$ and $\mathscr{E}_2$, namely,

$$V = \tfrac{1}{2}[\mathscr{E}_1(u) + \mathscr{E}_2(\bar{u})] , \qquad \phi = \frac{1}{2i}[\mathscr{E}_1(u) - \mathscr{E}_2(\bar{u})] , \quad (41)$$

which allow V and ϕ to be complex. Because of the holomorphy (antiholomorphy) of $\mathscr{E}_1$ ($\mathscr{E}_2$), from Eqs. (37) it follows that

$$^{(3)}R_{xx}={}^{(3)}R_{yy}=(\partial_x{}^2+\partial_y{}^2)\ln\frac{[\mathscr{E}_1(u)+\mathscr{E}_2(\bar{u})]}{2}\ , \qquad (42)$$
$$^{(3)}R_{xy}=0\ .$$

On the other hand, $s=\text{const}$, from Eq. (35), gives

$$^{(3)}R_{xx}={}^{(3)}R_{yy}=-(\partial_x{}^2+\partial_y{}^2)\gamma\ , \qquad (43)$$
$$^{(3)}R_{xy}=0\ .$$

We see that Eqs. (42) and (43) are compatible and allow us to determine γ easily as

$$2\gamma=\ln\frac{[\mathscr{E}_1(u)+\mathscr{E}_2(\bar{u})]}{2}+g_1(u)+g_2(\bar{u})\ . \qquad (44)$$

Then $^3ds^2$ [Eq. (33)] can be written as

$$\gamma_{ij}dx^idx^j=\frac{[\mathscr{E}_1(u)+\mathscr{E}_2(\bar{u})]}{2}du\,d\bar{u}+dz^2\ . \qquad (45)$$

g_1 and g_2 can be eliminated by a conformal transformation

$$e^{g_1(u)}du\to du;\quad e^{g_2(\bar{u})}d\bar{u}\to d\bar{u}\ .$$

We can determine $\omega\equiv\omega_z$ from Eqs. (31), (32), and (41). One gets

$$f_x=\frac{1}{s}\partial_y\omega=\frac{\partial_x\phi}{V^2}\ ,$$
$$f_y=\frac{1}{s}\partial_x\omega=\frac{\partial_y\phi}{V^2}\ ,$$

i.e.,

$$\partial_u\omega=-2\partial_u[1/(\mathscr{E}_1+\mathscr{E}_2)]\ ,$$
$$\partial_{\bar{u}}\omega=-2\partial_{\bar{u}}[1/(\mathscr{E}_1+\mathscr{E}_2)]\ ,$$

which gives

$$\omega=-1/V+\text{const}\ . \qquad (46)$$

By replacing V, γ_{ij}, and γ given by Eqs. (41), (45), and (44) in the canonical form (32), we obtain

$$ds^2=-[\mathscr{E}_1(x+iy)+\mathscr{E}_2(x-iy)]dt^2+2dt\,dz+dx^2+dy^2\ . \qquad (47)$$

The signature of the metric is (3,1). This is a Lorentzian complex metric. By appropriate Wick rotations of the coordinates, it is not possible to get the signature to be Euclidean. The Wick rotation $t=i\tau, z=i\mathfrak{z}$ (which in this case is equivalent to setting $t=i\tau$, $A_z\to iA_z$) does not change the signature of the metric.

On the other hand, the Wick rotation

$$t=iy\quad(y=-it)\ ,$$
$$z=i\mathfrak{z}$$

maps the solution (47) into a different solution

$$ds^2=dx^2-dt^2+[\mathscr{E}_1(x-t)+\mathscr{E}_2(x+t)]dy^2-2dy\,d\mathfrak{z}\ , \qquad (48)$$

which is real and of plane-wave type, but with signature (2,2).

In particular if

(a) $\mathscr{E}_1=0$ or $\mathscr{E}_2=0$, then $V=\pm i\phi$, the three-spatial metric γ_{ij} is flat and ω and V satisfy the relation $\nabla\times\omega=\nabla(1/V)$ [which implies in this case $\nabla^2(1/V)=0$]. This gives a metric of the Taub-NUT type as considered by Gibbons and Hawking. It can be noted that here the metric is complex with signature (3,1). In this case, there is no real metric with signature (3,1). The only possibility to have a real metric with this signature is $V=\pm\phi=$ const (the flat space-time).

A real metric of Taub-NUT type, but with signature (2,2) is obtained from the solution (48) with $\mathscr{E}_1=0$ or $\mathscr{E}_2=0$.

(b) If $\mathscr{E}_1(u)=\overline{\mathscr{E}}_2(\bar{u})$ the solution (47) is real and it is of Peres type[30] (Petrov type N).

All components of R_{ab} vanish trivially except for $R_{00}=\frac{1}{2}\nabla^2V$. Because V is harmonic, it is not bounded. Otherwise it would be a constant. Rigorously speaking $\nabla^2V=4\pi\rho_0$, where ρ_0 is a distribution concentrated at the singularities. In the vacuum case, the singularities of V are the sources of the field. The zeros of V describe event horizons. This generalizes to the case when matter is present satisfying

$$R_{\mu\nu}=\frac{8\pi k}{c^4}(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T^\gamma_\gamma)\ .$$

In this case, $\nabla^2V=4\pi\rho$ for ρ positive definite representing the density matter in the space.

In particular, the homographic function

$$\mathscr{E}=\frac{(u-a_0)\ldots(u-a_q)}{(u-b_0)\ldots(u-b_q)}$$

corresponds to the (multi)-instantons solutions of the σ model as considered by Belavin and Polyakov[20]; $a_0,\ldots,a_q$ represent the positions where the instantons are centered

$$\mathscr{E}=\frac{au+b}{cu+d},$$

with $\Delta\equiv ad-cb\neq 0$, describes a single instanton located at $\mu_0=-d$. The metric coefficients associated with it are

$$V=\frac{1-\Delta(x+d)}{(x+d)^2+y^2},$$

$$\psi=\frac{\Delta y}{(x+d)^2+y^2},$$

$$\omega_z=1+\frac{\Delta(x+d)}{(x+d)(x+d-\Delta)+y^2}.$$

The field f is given by

$$f_x=\frac{-2\Delta(x+d)y}{[(x+d)(x+d-\Delta)+y^2]^2},$$

$$f_y=\frac{\Delta[(x+d)^2-y^2]}{[(x+d)(x+d-\Delta)+y^2]^2}\quad(\nabla\cdot f=0).$$

Here we have taken $a=1$ in order to have $V_\infty=1$. Without loss of generality, we choose $c=1$, i.e., $\mathscr{E}=1-\Delta/(x+d)$. Note that this solution requires $V\neq i\phi$; $(x=-d,\ y=0)$ is a zero of the denominator. At the point $(-d,0)$ the metric g_{uv} has a true singularity.

Following Pohlmeyer's reduction for the O(3) σ model[20] one can relate the O(2,1) model to the sine-Gordon and to the Liouville equation. We can choose a basis in C^3 given by $(\sigma,\partial_u\sigma,\partial_{\bar u}\sigma)$ [we recall that $\sigma=(\sigma^1,\sigma^2,\sigma^3)$], such that

$$(\partial_u\sigma)^2=C_1,\quad (\partial_{\bar u}\sigma)^2=C_2,\quad (\partial_u\sigma)(\partial_{\bar u}\sigma)=e^{\alpha(u,\bar u)},\tag{49}$$

where C_1 and C_2 are constants and α is in general a complex function on $u,\bar u$. Here

$$\sigma(\partial_u\sigma)=\sigma(\partial_{\bar u}\sigma)=0;\quad \sigma^2=1.$$

(Recall that $\sigma\cdot\tilde\sigma=\sigma_1\tilde\sigma_1+\sigma_2\tilde\sigma_2-\sigma_3\tilde\sigma_3$.) In terms of this parametrization, the O(2,1) σ-model equation reads

$$\partial_u\partial_{\bar u}\alpha=C_1C_2e^{-\alpha}-e^{\alpha}-(\partial_u\alpha)(\partial_{\bar u}\alpha)\frac{C_1C_2}{(C_1C_2-e^{2\alpha})}.\tag{50}$$

By putting

$$e^{\alpha}=\cos\beta,\quad C_1C_2=K,$$

it can be written as

$$\partial_u\partial_{\bar u}\beta+\frac{(\partial_u\beta)(\partial_{\bar u}\beta)}{\cos\beta\sin\beta}=\frac{\cos^2\beta-K}{\sin\beta}+\frac{(\partial_u\beta)(\partial_{\bar u}\beta)\sin\beta}{\cos\beta\left[1-\cos^2\frac{\beta}{K}\right]}.\tag{51}$$

In particular,

(i) if $K\neq 0$ we can always choose $K=1$ and we get the sine-Gordon equation

$$\partial_u\partial_{\bar u}\beta+\sin\beta=0.$$

(ii) If $K=0$ (i.e., $C_1=0$ or $C_2=0$), Eq. (51) is the Liouville equation

$$\partial_u\partial_{\bar u}\alpha+e^{\alpha(u,\bar u)}=0,$$

whose general solution is given by

$$\alpha(u,\bar u)=\ln\left\{\frac{2A'(u)B'(\bar u)}{[A(u)+B(\bar u)]^2}\right\},$$

A and B being arbitrary functions.

On the other hand, from Eq. (6) we have

$$\sigma^1=\frac{1+V^2-\phi^2}{2V},\quad \sigma^2=\frac{\phi}{V},\quad \sigma^3=\frac{1+\phi^2-V^2}{2V},\tag{52}$$

i.e., in terms of the (V,ϕ) parametrization we have

$$(\partial_u\sigma)^2=\frac{(\partial_u\phi)^2-(\partial_uV)^2}{V^2},\quad (\partial_{\bar u}\sigma)^2=\frac{(\partial_{\bar u}\phi)^2-(\partial_{\bar u}V)^2}{V^2},\quad (\partial_u\sigma)(\partial_{\bar u}\sigma)=\frac{(\partial_u\phi)(\partial_{\bar u}\phi)-(\partial_uV)(\partial_{\bar u}V)}{V^2}.\tag{53}$$

For the case (ii), if $C_1=0$, $C_2=1$, from Eq. (52) and (53) it follows that

$$\partial_u\phi=\pm(\partial_uV),\quad \partial_{\bar u}\phi=(\partial_{\bar u}V)\pm g(\bar u).\tag{54}$$

Here $g(\bar u)$ must be nonzero because $(\partial_{\bar u}\sigma)^2\neq 0$. Then it is not possible to obtain Eqs. (40) (i.e, $\partial_uV=\partial_u\phi,\partial_{\bar u}V=\partial_{\bar u}\phi$) which give V,ϕ as linear combinations of holomorphic and antiholomorphic functions.

In the nonaxisymmetric case (i.e., s =const), the solutions of the sine-Gordon as well as of the Liouville equations are not compatible with the Einstein equations because they exclude V,ϕ to be holomorphic (antiholomorphic) functions.

It can be pointed out that the equation considered in Refs. 8,9, and 10 is an O(2,1) σ model, even when s is not a constant.

The above solutions (47) and (48) can be generalized in a simple way to the case when only one Killing vector is present. In this case $\mathscr{E}_1$ and $\mathscr{E}_2$ also depend on $v=x_3-t$, i.e.,

$$ds^2=-[\mathscr{E}_1(u,v)+\mathscr{E}_2(\bar{u},v)]dt^2+2dt\,dz+dx^2+dy^2 \tag{55}$$

or

$$ds^2=dx^2-dt^2+[\mathscr{E}_1(u,v)+\mathscr{E}_2(\bar{u},v)]dy^2+2dy\,dz\ . \tag{56}$$

These solutions can also be generalized to the case when there is no Killing vector to include holomorphic (antiholomorphic) functions depending on u, $\bar{u}$, v, and $\bar{v}$. This is considered in another paper.[27]

V. GENERAL CASE: NO KILLING VECTOR FIELD

We will consider here Zelmanov's formalism on chronometric invariants[2] (CI) to give a 3 + 1 decomposition of Einstein equations when there are no Killing vector fields. Let us consider the transformations

$$\tilde{x}=\tilde{x}^0(x^0,x^i)\ , \tag{57}$$

$$\tilde{x}^i=\tilde{x}^i(x^i)\ , \tag{58}$$

such that

$$\frac{d\tilde{x}^i}{dx^0}=0\ .$$

(Latin indices run through only 1,2,3.) These transformations relate all the coordinate systems which are at rest with respect to the same reference frame. CI can be considered as three-dimensional tensors invariant with respect to (57) and covariant with respect to (58), in a three-dimensional space of metric $h_{ij}=g_{ij}=g_{0i}g_{0j}/g_{00}$. The CI derivatives

$$^*\partial_0=\frac{1}{\sqrt{g_{00}}}\partial_0,\quad ^*\partial_i=\partial_i-\frac{g_{0i}}{g_{00}}\partial_0$$

replace the ordinary ones. Note that

$$^*\partial_i{}^*\partial_0-{}^*\partial_0{}^*\partial_i=E_i{}^*\partial_0\ ,$$

$$^*\partial_i{}^*\partial_k-{}^*\partial_k{}^*\partial_i=2A_{ik}{}^*\partial_0\ ,$$

where

$$A_{ik}=\tfrac{1}{2}\sqrt{V}({}^*\partial_k\omega_i-{}^*\partial_i\omega_k)\ ,$$

$$E_i=-{}^*\partial_i\ln\sqrt{V}+\frac{1}{\sqrt{V}}{}^*\partial_0(V\omega_i)\ .$$

A_{ik} and E_i generalize the homologous quantities f_{ik} and $\partial_i\ln\sqrt{V}$ of the stationary case. Let us define

$$D_{ik}=\tfrac{1}{2}{}^*\partial_0 h_{ik},\quad D={}^*\partial_0\ln\sqrt{h}\ .$$

The Einstein equations for g_{uv} can be expressed in terms of E_i, A_{ik}, D_{ik}, D, and the three-dimensional curvature $^3R_{ik}$ for h_{ik}.[20] This gives

$$\frac{R_{00}}{\sqrt{V}}={}^*\partial_0 D+D_{jl}D^{lj}+A_{jl}A^{lj}+{}^*\nabla_j E^j-E_jE^j\ , \tag{59}$$

$$\frac{R_0^i}{\sqrt{V}}={}^*\nabla_j(h^{ij}D-D^{ij}-A^{ij})+2E_jA^{ij}\ , \tag{60}$$

$$R_{ik}={}^*\partial_0 D_{ik}-(D_{ij}+A_{ij})(D_k^j+A_k^j)+DD_{ik}-D_{ij}D_k^j+3A_{ij}A_k^j+E_iE_k+\frac{1}{2}({}^*\nabla_iE_k+{}^*\nabla_kE_i)+{}^{(3)}R_{ik}\ . \tag{61}$$

These equations generalize Eqs. (17)—(19) of the stationary case. Equation (60) can be written as

$$\frac{R_0^i}{\sqrt{V}}=-{}^*\nabla_j(VA^{ij})+{}^*\nabla_j(h^{ij}D-D^{ij})+\frac{2}{\sqrt{V}}A^{ij*}\partial_0(V\omega_j)=0\ . \tag{62}$$

Here

$$^*\nabla_j(VA^{ij})=\frac{1}{2\sqrt{h}}e^{ijk*}\partial_j(VA_k)\ .$$

We can express A_k as

$$A_k=\frac{1}{V}({}^*\nabla_k\phi+{}^*\partial_0\mathscr{R}_k)\ , \tag{63}$$

which generalizes Eq. (21). The equation

$${}^*\nabla_j(h^{ij}D-D^{ij})-e^{ijk*}\nabla_j({}^*\partial_0\mathscr{R}_k)=0 \tag{64}$$

determines ${}^*\partial_0\mathscr{R}_k$. From Eqs. (62) and (63) it follows that

$${}^*\partial_0 D+D_{jl}D^{lj}+2A_jA^j+{}^*\nabla_jE^j-E_jE^j+2{}^*\nabla_j\left[\frac{A^j}{\sqrt{V}}\right]=0\ . \tag{65}$$

By expressing this equation in terms of V, ϕ, and $({}^*\partial_0\mathscr{R}_k)$ and in terms of the metric (Vh_{ik}), we obtain

$${}^*\nabla^2(V+i\phi)-\frac{1}{V}[{}^*\nabla(V+i\phi)]^2+{}^*\nabla_k({}^*\partial_0\mathscr{R}^k)+\frac{1}{2V^2}[({}^*\partial_0\mathscr{R}_k)^2+({}^*\partial_0\mathscr{R}_k)({}^*\partial^k\phi)]+\frac{1}{2}D_kD^k+\frac{1}{4}D_{ik}D^{ik}=0\ . \tag{66}$$

This equation generalizes the Ernst equation of the gravitational field to the case when there are no symmetries in the space-time. Besides the potentials (V,ϕ), this equation involves other fields $(\partial_0\mathscr{R}_k)$ and D_{ik} whose geometrical meaning is not immediately apparent. These fields vanish when stationarity is required. The first terms of Eq. (66) look like an O(2,1) σ model. However, the coupling with the other fields basically destroys its simple geometrical interpretation.

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